Esscher Transforms and Consumption-Based Models

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Outline of the presentation:

- Introduction and literature review

- Conditional Esscher transform and equilibrium measures

- Conditional Esscher-Girsanov transforms

- Numerical results and conclusions

$ 1. Introduction and literature review

- Consider a discrete-time economy with time index set $\mathcal{T} = \{t | t = 0, 1, 2, \cdots, T\}$

- Let $M := \{M_t\}_{t \in \mathcal{T}}$ be a non-negative, $\mathcal{F}$-adapted stochastic discount factor (SDF) process defined on $(\Omega, \mathcal{F}, \mathcal{P})$

- The price at time t, Π_t, of a payoff at $t+1$, X_{t+1}:

$$\Pi_t = E^P \left[M_{t+1} X_{t+1} | \mathcal{F}_t \right]$$

- c_t denotes consumption at t

- $u(c_t)$ is the utility function

- ρ is the impatient factor of the economic agent per unit of time

- The first-order condition for an optimal consumption and portfolio choice:

$$M_{t+1} = e^{-\rho} \frac{u'(c_{t+1})}{u'(c_t)}$$

- The log return process, y_t, under the physical measure P:

$$
\begin{aligned}
y_t &= r_t + \lambda_t \sigma_t - \kappa^P_{\varepsilon_t}(\sigma_t) + \sigma_t \varepsilon_t \\
\varepsilon_t &\sim F(0,1) \\
\sigma_t &= a(\sigma_{t-1}, \sigma_{t-2}, ... \sigma_1, \varepsilon_{t-1}, \varepsilon_{t-2}, ..., \varepsilon_1) \ .
\end{aligned}
$$

- $\kappa^P_{\varepsilon_t}(z) := \ln M^P_{\varepsilon_t}(z) = \ln E^P[e^{z \varepsilon_t} | \mathcal{F}_{t-1}] < \infty, \quad |z| < h$

- If Q is an equivalent martingale pricing measure, it can be related to the SDF by the following Radon-Nikodym derivative:

$$
\left. \frac{dQ}{dP} \right|_{\mathcal{F}_T} := \exp \left(\sum_{t=1}^{T} r_t \right) \prod_{t=1}^{T} M_t \ .
$$

- Rubinstein (1976) and Brennan (1979) - risk-neutral valuation relationship (RNVR) - joint normality of asset returns and marginal utility of consumption.

- Camara (1999) - extended the RNVR when the aggregate consumption and the random payoff of the underlying asset follow a bivariate three parameter lognormal distribution - closed form solutions for European call options under this setting

- Camara (2003) - proposed a new set of equilibrium RNVRs - aggregate wealth and underlying asset have transformed normal distributions

Elliot and Madan (1998) - extended Girsanov principle

- The Radon-Nikodym derivative:

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_T} := \prod_{t=1}^{T} \frac{f_{\varepsilon_t}(\varepsilon_t + \lambda_t)}{f_{\varepsilon_t}(\varepsilon_t)}$$

- f_{ε_t} - the $\mathcal{F}_{t-1}$-conditional pdf of y_t under P

- Consistent with hedging strategies that minimize the risk-adjusted cost of hedging

- Schroder (2004) - relates this transformation to equilibrium pricing measures for some special cases of distributions of the return process

§3. Conditional Esscher transform and equilibrium measures

- **Definition 1** *Let θ_t be an $\mathcal{F}_t$ predictable process, for each $t \in \mathcal{T}\backslash\{0\}$). The probability measure Q^{ess} defined by:*

$$\left.\frac{dQ^{ess}}{dP}\right|_{\mathcal{F}_T} = \prod_{t=1}^{T} \frac{e^{\theta_t y_t}}{M^P_{y_t|\mathcal{F}_{t-1}}(\theta_t)}$$

is called the conditional Esscher transformed measure of P, generated by the return processes y and the family θ of Esscher parameters, with respect to the filtration F.

- the Esscher parameter θ_t is the solution of the following equation:

$$M^P_{y_t|\mathcal{F}_{t-1}}(1 + \theta_t) = e^{rt}M^P_{y_t|\mathcal{F}_{t-1}}(\theta_t)\ .$$

Proposition 1 *Suppose the following conditions hold:*

(a) The utility function u is of exponential form:

$$u(c_t) = \frac{1 - e^{-Rc_t}}{R}$$

(b) for any $t \in \mathcal{T}$, the change in aggregate consumption satisfies the following, time-series, regression model under P:

$$\Delta c_t = \alpha_t + \beta_t y_t + \gamma_t \eta_t , \quad t \in \mathcal{T} \backslash \{0\}$$

where $\eta_t \sim F(0,1)$ iid and independent of y_t

Then an equivalent martingale measure based on the equilibrium SDF coincides with that from the conditional Esscher transform.

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- $\theta_t = -R\beta_t$, $\quad t \in \mathcal{T}\backslash\{0\}$

- The Esscher parameter is proportional to the "beta" risk in the changes of aggregate consumption explained by the return of the risky asset.

- R is the coefficient of absolute risk aversion

- The market price of risk:

$$\lambda_t = \frac{\kappa_{\varepsilon_t}^P(\sigma_t) + \kappa_{\varepsilon_t}^P(-R\beta_t\sigma_t) - \kappa_{\varepsilon_t}^P((1 - R\beta_t)\sigma_t)}{\sigma_t}$$

Proposition 2 *Suppose the following conditions hold:*

(a) The utility function u is of isoelastic form:

$$u(c_t) = \frac{c_t^{1-R} - 1}{1 - R}$$

(b) for any $t \in \mathcal{T}$, the change in logarithm aggregate consumption satisfies the following, time-series, regression model under P:

$$\Delta \ln c_t = \alpha_t + \beta_t y_t + \gamma_t \eta_t , \quad t \in \mathcal{T} \backslash \{0\} .$$

where $\eta_t \sim F(0,1)$ iid and independent of y_t

Then an equivalent martingale measure based on the equilibrium SDF coincides with that from the conditional Esscher transform.

- **Consistency with consumption CAPM based on recursive utility - Epstein and Zin (1989, 1991)**

- The level of utility at any time point t be:

$$U_t = W(c_t, E[U_{t+1}^{\varrho} | \mathcal{F}_t]^{1/\varrho}), \quad 0 \neq \varrho < 1 \;.$$

- ϱ is the risk aversion parameter and the degree of risk aversion increases as ϱ does; W is the "aggregator" function having the following form:

$$W(c, z) = [(1 - e^{-\rho})c^{\phi} + e^{-\rho}z^{\phi}]^{1/\phi}, \quad 0 \neq \phi < 1$$

- c and z are both positive; ϕ is a parameter which reflects the inter-temporal substitution; ρ is the impatience factor

- $\varrho = 1 - R$, where R is the constant coefficient of relative risk aversion

- The maximization problem leads to:

$$E^P\left[e^{-\rho\varsigma}\left(\frac{c_t}{c_{t-1}}\right)^{\varsigma(\phi-1)}\left(\frac{S_{mt}}{S_{mt-1}}\right)^{\varsigma-1}\frac{S_t}{S_{t-1}}\bigg|\mathcal{F}_{t-1}\right] = 1, \quad \forall\, t \in \mathcal{T}\ .$$

where S_{mt} is the price index of the market portfolio at time t and $\varsigma = \varrho/\phi$

- The equilibrium SDF:

$$M_t = e^{-\rho\varsigma}\left(\frac{c_t}{c_{t-1}}\right)^{\varsigma(\phi-1)}\left(\frac{S_{mt}}{S_{mt-1}}\right)^{\varsigma-1}, \quad \forall\, t \in \mathcal{T}\ .$$

Proposition 3 *For any $t \in \mathcal{T}$, we assume that the changes in the logarithm of the aggregate consumption and the return on the market portfolio, $y_{mt} := \ln(S_{mt}/S_{mt-1})$, satisfy the following regression model under P:*

$$(\Delta \ln c_t, y_{mt})^T = \underline{\alpha}_t + \underline{\beta}_t y_t + diag(\underline{\eta}_t) \underline{\gamma}_t .$$

Here $\underline{\alpha}_t := (\alpha_{1t}, \alpha_{2t})^T$, $\underline{\beta}_t := (\beta_{1t}, \beta_{2t})^T$, $\underline{\gamma}_t := (\gamma_{1t}, \gamma_{2t})^T$ are 2-dimensional $\mathcal{F}$-predictable processes and $\underline{\eta}_t := (\eta_{1t}, \eta_{2t})^T$ is a 2-dimensional vector of i.i.d. random variables with $\eta_{1t}, \eta_{2t} \sim F(0,1)$ independent of y_t for any $t \in \mathcal{T}$ and also independent of each other. Here $diag(\eta_t)$ is the diagonal matrix formed with the elements of η_t.

Then an equivalent martingale measure based on the above equilibrium SDF is consistent with that from the conditional Esscher transform.

- The Esscher parameter is given by:

$$\theta_t = \varsigma(\phi - 1)\beta_{1t} + (\varsigma - 1)\beta_{2t} \ .$$

- θ_t - weighted average between a fraction of the "beta" risk in the changes of logarithm aggregate consumption explained by the return of the risky asset and the negative "beta" risk of the return on the market portfolio explained by the return on the risky asset

- If $\varsigma = 1$, then $\theta_t = -Rcov(\Delta \log c_t, y_t)/\sigma_t^2$ - same as standard CAPM

- If $\varsigma = 0$, then $\theta_t = cov(y_{mt}, y_t))/\sigma_t^2$

§4. Conditional Esscher-Girsanov transform and its generalization

- The conditional Esscher-Girsanov transform was introduced by Goovaerts and Laeven (2008)

- Define the function $\Psi(\varepsilon_t)$:
$$\Psi(\varepsilon_t) = \Phi^{-1}(F(\varepsilon_t))$$

- Φ is the cdf of a standard normal random variable

- $\Psi(\varepsilon_t)$ is a standard normal random variable under P conditional on $\mathcal{F}_{t-1}$.

Definition 2 (Conditional Esscher-Girsanov transform) *Let $\delta :=$ $\{\delta_t\}_{t \in \mathcal{T}}$ be an F-predictable process and let h be a positive constant. The probability measure Q^{essg} defined by:*

$$\left. \frac{dQ^{essg}}{dP} \right|_{\mathcal{F}_T} = \prod_{t=1}^{T} e^{h\delta_t \sigma_t \Psi(\varepsilon_t) - \frac{1}{2}h^2 \delta_t^2 \sigma_t^2}$$

is called the conditional Esscher-Girsanov transformed measure of P with respect to the filtration $\mathcal{F}_t$ for parameter h and penalty process δ.

1. The reason of having 2 parameters is to separate the risk aversion parameter from the risk premium

2. If h $= 1$, the Radon-Nikodym derivative resembles a conditional version of Wang transform.

Proposition 4 *Under Q^{essg} the risk-neutralized dynamics of the asset return process are given by:*

$$
\begin{aligned}
y_t &= m_t + \sigma_t \Psi^{-1}(\xi_t + h\delta_t\sigma_t), \quad \xi_t \sim N(0,1) \\
\sigma_t &= a(\sigma_{t-1}, \sigma_{t-2}, ..., \Psi^{-1}(\xi_{t-1} + h\delta_{t-1}\sigma_{t-1}), \Psi^{-1}(\xi_{t-2} + h\delta_{t-2}\sigma_{t-2}), ...)
\end{aligned}
$$

where ξ_t is a sequence of i.i.d. standard Gaussian random variables under Q^{essg}, and h and δ_t satisfy:

$$
M^P_{(\varepsilon_t, \Psi(\varepsilon_t))|\mathcal{F}_{t-1}}(\sigma_t, h\delta_t\sigma_t) = e^{r_t - m_t + \frac{1}{2}h^2\delta_t^2\sigma_t^2}
$$

Note: This is similar to the risk neutralized dynamic obtained by Duan (1999) using his generalized local RNVR

Proposition 5 *Let $\Xi(c_t)$ be the inverse Gaussian transformation of c_t through its conditional c.d.f. (ie. $\Xi(c_t)$ is standard normally distributed conditional on $\mathcal{F}_{t-1}$). If $\Psi(\varepsilon_t)$ and $\Delta\Xi(c_t)$ have a conditionally bivariate normal distribution with respect to $\mathcal{F}_{t-1}$ under P and the marginal utility function is of the form:*

$$u'(c_t) = e^{-R\Xi(c_t)}, \quad R > 0$$

then the risk neutral measure constructed based on the standard equilibrium SDF is consistent with the conditional Esscher-Girsanov transformation.

1. R is no longer the Arrow-Pratt measure of absolute risk aversion

2. It follows that $h = -R$ and $\delta_t = cov(\Delta\Xi(c_t), \Psi(\varepsilon_t)|\mathcal{F}_{t-1})/\sigma_t$

- If we assume $m_t = r_t + \lambda_t \sigma_t - \kappa_{\varepsilon t}^P(\sigma_t)$, then the market price of risk:

$$\lambda_t = \frac{\kappa_{\varepsilon t}^P(\sigma_t) + \frac{1}{2}R^2\delta_t^2\sigma_t^2 - \kappa_{(\varepsilon t, \Psi(\varepsilon t))|\mathcal{F}_{t-1}}^P(\sigma_t, -R\delta_t\sigma_t)}{\sigma_t} \ .$$

- The unit risk premium is decomposed into the sum of a risk free component and a term which contains the risk preference parameter

- If the innovations are normally distributed, then $\lambda_t = R\delta_t\sigma_t$ - same result as Esscher transform

- The choice of the inverse Gaussian transform can be extended to other distortion functions

- Let G be the cdf of a standardized random variable having a finite mgf

- Define the following transformation:

$$\Psi^{(G)}(\varepsilon_t) = G^{-1}(F(\varepsilon_t))$$

- $\Psi^{(G)}(\varepsilon_t)$ has an $\mathcal{F}_t$-conditionally standard G distributed under P

- When $G = \Phi$ - Esscher-Girsanov transform

Definition 3 (Generalized Conditional Esscher-Girsanov transform

Let $\delta^{(G)} := \{\delta_t^{(G)}\}_{t \in \mathcal{T}}$ be an F-predictable process and let h be a positive constant. The family of probability measures indexed by G, $Q^{(G)}$, defined by:

$$\left.\frac{dQ^{(G)}}{dP}\right|_{\mathcal{F}_T} = \prod_{t=1}^{T} \frac{e^{h\delta_t^{(G)}\sigma_t \Psi^{(G)}(\varepsilon_t)}}{M_{\Psi^{(G)}(\varepsilon_t)|\mathcal{F}_{t-1}}^{P}(h\delta_t^{(G)}\sigma_t)}$$

is called the generalized conditional Esscher-Girsanov transformed measure of P with respect to the filtration $\mathcal{F}_t$ with parameter h and penalty process $\delta_t^{(G)}$.

Note : The parameters h and $\delta_t^{(G)}$ satisfy the martingale equation:

$$M_{(\varepsilon_t, \Psi^{(G)}(\varepsilon_t))|\mathcal{F}_{t-1}}^{P}(\sigma_t, h\delta_t^{(G)}\sigma_t) = e^{r_t - m_t} M_{\Psi^{(G)}(\varepsilon_t)|\mathcal{F}_{t-1}}^{P}(h\delta_t^{(G)}\sigma_t)$$

Proposition 6 *Let $\Xi^{(G)}(c_t)$ be the G-transformation of c_t through its conditional cdf (i.e. $\Xi^{(G)}(c_t)$ is standard G-distributed conditional on $\mathcal{F}_{t-1}$). If there exist three $\mathcal{F}_t$-predictable processes, $\alpha_t^{(G)}$, $\beta_t^{(G)}$, and $\gamma_t^{(G)}$ and a sequence of random variables η_t independent of ε_t such that $\eta_t \sim G(0,1)$ and:*

$$\Delta\Xi^{(G)}(c_t) = \alpha_t^{(G)} + \beta_t^{(G)}\Psi^{(G)}(\varepsilon_t) + \gamma_t^{(G)}\eta_t$$

then the risk neutral measure constructed based on the standard equilibrium SDF with marginal utility function satisfying:

$$u'(c_t) = e^{-R\Xi^{(G)}(c_t)}, \quad R > 0$$

is consistent with the generalized conditional Esscher-Girsanov transform.

Note: The transformation parameters are $h = -R$ and $\delta_t^{(G)} = \beta_t^{(G)}/\sigma_t = cov(\Delta\Xi^{(G)}(c_t), \Psi^{(G)}(y_t)|\mathcal{F}_{t-1})/\sigma_t$

§5. Numerical results and conclusions

- The return dynamic under P:

$$
\begin{aligned}
y_t &= r + \lambda \sigma_t + \sigma_t \varepsilon_t \\
\varepsilon_t &\sim z(\alpha, \beta, \delta, \mu) \\
\sigma_t^2 &= a_0 + a_1 \sigma_{t-1}^2 \varepsilon_{t-1}^2 + \gamma \mathrm{I}(\varepsilon_{t-1} < 0) \sigma_{t-1}^2 \varepsilon_{t-1}^2 + b_1 \sigma_{t-1}^2
\end{aligned}
$$

- the return pdf and cumulant generating functions:

$$
\begin{aligned}
f_{\varepsilon_t}(x, \alpha, \beta, \delta, \mu) &= \frac{1}{\delta B(\alpha, \beta)} \cdot \frac{(\exp[(x - \mu)/\delta])^\alpha}{(1 + \exp[(x - \mu)/\delta])^{\alpha + \beta}} \\
\kappa_{\varepsilon_t}(u) &= \log \frac{B(\alpha + \delta u, \beta - \delta u)}{B(\alpha, \beta)} + \mu u
\end{aligned}
$$

- S&P 500 returns data from January 3rd, 1988 to January 6th, 2004

- Estimate the model parameters by MLE

- Use Monte Carlo simulation (50,000 paths) to compute option prices for MCMM, Esscher transform and Esscher-Girsanov (h=1) transform

- Call options (STRICKNET INC) sampled every Wednesday at closing prices from January 7th, 2004 to December 29th, 2004.

- 1582 European call options, average bid-ask spread is \$1.13 and average Call price is \$18.36.

- Model performance is tested using root mean squared error (RMSE):

$$RMSE = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (C_i^{market} - \hat{C}_i^{model})^2}$$

Number of Call option contracts

Mo	$DTM < 40$	$40 < DTM < 80$	$80 < DTM < 180$	$DTM > 180$	All
$[0.8, 0.9)$	8	1	3	2	14
$[0.9, 0.95)$	31	5	3	4	43
$[0.95, 0.975)$	39	6	2	2	49
$[0.975, 0.99)$	88	19	8	9	124
$[0.99, 1.01)$	259	76	37	21	393
$[1.01, 1.025)$	201	38	17	6	262
$[1.025, 1.05)$	232	57	32	4	325
$[1.05, 1.1)$	166	83	50	24	323
$[1.1, 1.2]$	9	7	23	10	49
All	1,033	292	175	82	1,582

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Average Call option prices

Mo	$DTM < 40$	$40 < DTM < 80$	$80 < DTM < 180$	$DTM > 180$	All
$[0.8, 0.9)$	165.00	177.80	173.93	152.65	166.06
$[0.9, 0.95)$	82.65	88.22	106.30	119.53	88.38
$[0.95, 0.975)$	46.54	52.75	67.40	92.85	50.04
$[0.975, 0.99)$	25.08	40.99	61.25	75.06	33.48
$[0.99, 1.01)$	13.47	30.88	47.69	65.37	22.83
$[1.01, 1.025)$	5.59	19.80	38.21	55.50	10.91
$[1.025, 1.05)$	2.72	11.82	28.79	47.33	7.43
$[1.05, 1.1)$	0.67	4.27	14.93	26.54	5.73
$[1.1, 1.2]$	0.13	0.37	4.62	11.05	4.50
All	12.84	20.01	32.31	52.28	18.36

Panel A. RMSE for z-TGARCH with MCMM

Mo	$DTM < 40$	$40 < DTM < 80$	$80 < DTM < 180$	$DTM > 180$	All
$[0.8, 0.9)$	1.11	1.53	4.83	2.27	2.57
$[0.9, 0.95)$	1.53	2.63	4.44	3.86	2.29
$[0.95, 0.975)$	2.34	4.11	6.27	1.69	2.86
$[0.975, 0.99)$	1.80	4.50	5.03	7.07	3.27
$[0.99, 1.01)$	1.98	3.81	5.48	6.69	3.26
$[1.01, 1.025)$	1.47	3.65	5.29	5.94	2.49
$[1.025, 1.05)$	1.03	2.48	5.13	5.21	2.18
$[1.05, 1.1)$	0.52	1.37	2.92	4.07	1.78
$[1.1, 1.2]$	0.12	0.17	0.83	2.12	1.12
All	1.50	3.00	4.33	5.21	2.57

Panel B. RMSE for z-TGARCH with Esscher-Girsanov (h=1) transform

Mo	$DTM < 40$	$40 < DTM < 80$	$80 < DTM < 180$	$DTM > 180$	All
$[0.8, 0.9)$	1.13	2.39	3.93	1.43	2.18
$[0.9, 0.95)$	1.52	2.37	3.94	3.09	2.07
$[0.95, 0.975)$	2.29	3.74	5.37	1.74	2.68
$[0.975, 0.99)$	1.72	4.42	4.64	6.28	3.06
$[0.99, 1.01)$	1.93	3.57	4.80	5.75	2.97
$[1.01, 1.025)$	1.45	3.46	4.73	5.22	2.33
$[1.025, 1.05)$	1.02	2.30	4.53	5.01	1.98
$[1.05, 1.1)$	0.51	1.26	2.55	3.28	1.53
$[1.1, 1.2]$	0.12	0.17	0.60	1.79	0.91
All	1.46	2.83	3.82	4.48	2.35

Panel C. RMSE for z-TGARCH with Esscher transform

Mo	$DTM < 40$	$40 < DTM < 80$	$80 < DTM < 180$	$DTM > 180$	All
$[0.8, 0.9)$	0.98	1.11	3.97	1.99	2.14
$[0.9, 0.95)$	1.50	2.60	3.79	2.93	2.05
$[0.95, 0.975)$	2.22	3.77	4.70	2.12	2.60
$[0.975, 0.99)$	1.70	4.18	4.09	5.74	2.86
$[0.99, 1.01)$	1.89	3.44	4.49	5.24	2.83
$[1.01, 1.025)$	1.42	3.40	4.38	4.38	2.22
$[1.025, 1.05)$	1.01	2.27	4.37	4.11	1.93
$[1.05, 1.1)$	0.51	1.27	2.33	3.06	1.44
$[1.1, 1.2]$	0.12	0.16	0.61	1.66	0.86
All	1.44	2.76	3.57	4.07	2.24

- Shows a consistency between Esscher-type transforms and stochastic discount factors constructed under equilibrium conditions in a general discrete time framework

- For conditional Esscher transform we identified risk neutral measures based on the standard Euler equation and recursive utility

- We derive new equilibrium SDFs consistent with the Esscher-Girsanov principle and its generalization

$\sim$ Thank you !$\sim$

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