Perpetual Cancellable Call Option

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Outline

Game Options: Safety for the Short Side

- The current financial crisis has highlighted the importance of adequately hedging risk and limiting downside losses.
- *Hedging*:
  - Modeling fluctuations in value under changes in market factors (X, σ, etc) and constructing offsetting positions in tradable assets.
- Another way is to build extra features, such as cancellation, into the derivative specifications.

Game Option

- Consider an American-style derivative with a cancellation feature given to the writer of the contract.
- At any point during the life of the contract, the writer can force the holder to take the current payoff plus a small additional amount as compensation for terminating the contract.

Game Option

- Consider an American-style derivative with a cancellation feature given to the writer of the contract.
- At any point during the life of the contract, the writer can force the holder to take the current payoff plus a small additional amount as compensation for terminating the contract.
- We refer to this as a *Game option*.

Game Option

- Contract: Seller A Buyer B
- B can *exercise* at any time t.

$$A \xrightarrow{Y_t} B$$

- A can *cancel* at any time t.

$$A \xrightarrow{Y_t+\delta_t} B$$

- Two optimal stopping problems: Optimal Exercise Time and Optimal Cancellation Time.
- What is the fair price V that B should pay to A for the contract? What are the optimal exercise and cancellation times for this game option?

Valuation: Complete Markets

- In this setting, valuation corresponds to solving a zero-sum optimal stopping game between two players.

- For cancellation policy τ and exercise policy σ the payoff of the claim is

$$R(\sigma, \tau) := (Y_\tau + \delta_\tau) 1_{\{\tau < \sigma\}} + Y_\sigma 1_{\{\sigma \leq \tau\}}$$

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- Kifer (2000) shows the fair price is

$$\begin{aligned} V_t &= \inf_{\tau \in \mathcal{S}_{t,T}} \sup_{\sigma \in \mathcal{S}_{t,T}} \mathbb{E}[e^{-r(\sigma \wedge \tau - t)} R(\sigma, \tau) | \mathcal{F}_t] \\ &= \sup_{\sigma \in \mathcal{S}_{t,T}} \inf_{\tau \in \mathcal{S}_{t,T}} \mathbb{E}[e^{-r(\sigma \wedge \tau - t)} R(\sigma, \tau) | \mathcal{F}_t] \end{aligned}$$

- Basic Price Bound: $Y_t \leq V_t \leq Y_t + \delta_t$.

Optimal Policies

- What are the 'optimal' σ, τ stopping times?

i.e. What policies achieve the infimum and supremum?

$$\sigma_t^* := \inf\{s \geq t : V_s = Y_s\}$$
$$\tau_t^* := \inf\{s \geq t : V_s = Y_s + \delta_s\}$$

- These stopping times are also 'optimal' exercise dates.
  - The holder waits until the value drops to the exercise value.
  - The writer waits until the value reaches the cancellation value.

Perpetual Cancellable Call Option

- Let the risky asset X satisfy the following risk-neutralized evolution

$$\mathrm{d}X_t = (r - d)X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}W_t$$

- Suppose $T = \infty$ and consider the following:

$$Y_t = (X_t - K)^+;\ \delta_t = \delta > 0$$

- We call this a *Perpetual Cancellable Call Option* or simply a *δ-penalty call option*.

Valuation of δ-penalty Put Option

Completed by Kyprianou (2004):

- Value function identified explicitly.
- Optimal Stopping times (for δ small):

$$\sigma^* := \inf\{t \geq 0 : X_t = k^*\}$$
$$\tau^* := \inf\{t \geq 0 : X_t = K\}$$

Does the valuation of a Call Option with dividend $d > 0$ follow symmetrically to this result?

Optimal Policies for Perpetual American Call?

Figure: Possible Exercise and Cancellation Barriers

Valuation: $r \leq d$

Conjecture: Value function satisfies for $x \in (0, K)$,

$$\begin{aligned} \mathcal{L}V - rV &= 0 \\ V(K) &= \delta, \ \lim_{x \downarrow 0} V(x) = 0 \end{aligned}$$

and for $x \in (K, k^*)$,

$$\begin{aligned} \mathcal{L}V - rV &= 0 \\ V(K) = \delta, V(k^*) &= (k^* - K)^+, V_x(k^*) = 1, \end{aligned}$$

where

$$\mathcal{L} := (r - d)x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\sigma^2 x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

Valuation: $r \leq d$

Proof: Let $v(x)$ be the proposed value function.

$$
\begin{aligned}
v(x) \quad &\leq \quad \inf_{\tau \in \mathcal{S}_{0,\infty}} \mathbb{E}_x[e^{-r(\tau \wedge \sigma_{k*})} v(X_{\tau \wedge \sigma_{k*}})] \\
&\leq \quad \inf_{\tau \in \mathcal{S}_{0,\infty}} \mathbb{E}_x[e^{-r(\tau \wedge \sigma_{k*})}((X_{\sigma_{k*}} - K)^+ 1_{\{\sigma_{k*} \leq \tau\}} \\
&\quad + \quad ((X_\tau - K)^+ + \delta) 1_{\{\tau < \sigma_{k*}\}})] \\
&\leq \quad \sup_{\sigma \in \mathcal{S}_{0,\infty}} \inf_{\tau \in \mathcal{S}_{0,\infty}} \mathbb{E}_x[e^{-r(\tau \wedge \sigma)}((X_\sigma - K)^+ 1_{\{\sigma \leq \tau\}} \\
&\quad + \quad ((X_\tau - K)^+ + \delta) 1_{\{\tau < \sigma\}})] \\
&\leq \quad \sup_{\sigma \in \mathcal{S}_{0,\infty}} \mathbb{E}_x[e^{-r(\tau_K \wedge \sigma)}((X_\sigma - K)^+ 1_{\{\sigma \leq \tau_K\}} \\
&\quad + \quad ((X_{\tau_K} - K)^+ + \delta) 1_{\{\tau_K < \sigma\}})] \\
&\leq \quad \sup_{\sigma \in \mathcal{S}_{0,\infty}} \mathbb{E}_x[e^{-r(\tau_K \wedge \sigma)} v(X_{\tau_K \wedge \sigma})] \\
&\leq \quad v(x)
\end{aligned}
$$

Value function: $r \leq d$

Conclusion:

$$V(x) = \begin{cases} x - K & \text{if } x \in [k^*, \infty) \\ g(x) & \text{if } x \in (K, k^*) \\ \delta \left(\frac{x}{K}\right)^{\frac{\lambda}{\sigma} - \kappa} & \text{if } x \in (0, K] \end{cases}$$

$$g(x) := (k^* - K)\left(\frac{k^*}{x}\right)^{\kappa} \frac{\left(\frac{K}{x}\right)^{-\frac{\lambda}{\sigma}} - \left(\frac{K}{x}\right)^{\frac{\lambda}{\sigma}}}{\left(\frac{k^*}{K}\right)^{\frac{\lambda}{\sigma}} - \left(\frac{k^*}{K}\right)^{-\frac{\lambda}{\sigma}}} + \delta \left(\frac{K}{x}\right)^{\kappa} \frac{\left(\frac{k^*}{x}\right)^{\frac{\lambda}{\sigma}} - \left(\frac{k^*}{x}\right)^{-\frac{\lambda}{\sigma}}}{\left(\frac{k^*}{K}\right)^{\frac{\lambda}{\sigma}} - \left(\frac{k^*}{K}\right)^{-\frac{\lambda}{\sigma}}}$$

Value function $r \leq d$

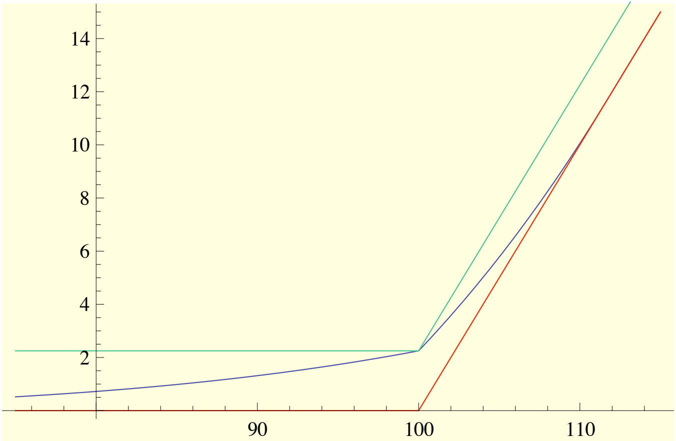

Figure: Convex value function.

Valuation: $r > d$

Conjecture: Why not same as before?

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Conjecture: Why not same as before?

- $v(x)$ violates basic inequality

$$(x - K)^+ \leq v(x) \leq (x - K)^+ + \delta$$

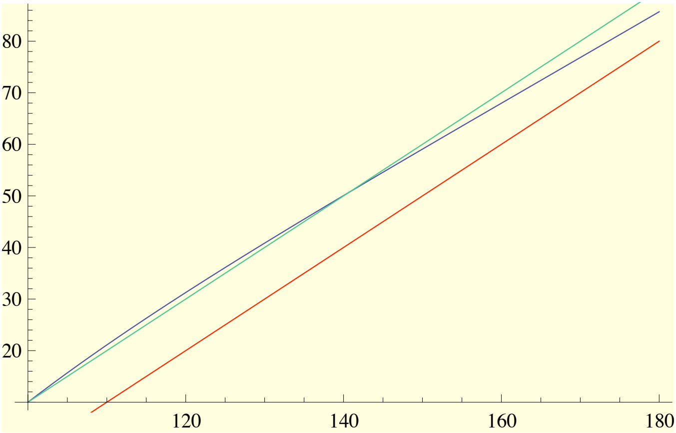

Figure: $v(x)$ (dark blue) violates upper bound.

Valuation: $r > d$

New Conjecture: Value function satisfies

for $x \in (0, K)$,

$$\begin{aligned} \mathcal{L}V - rV &= 0 \\ V(K) &= \delta, \lim_{x \downarrow 0} V(x) = 0 \end{aligned}$$

and for $x \in (h^*, k^*)$,

$$\begin{aligned} \mathcal{L}V - rV &= 0, \\ V(h^*) &= (h^* - K)^+ + \delta, V_x(h^*) = 1, \\ V(k^*) &= (k^* - K)^+, V_x(k^*) = 1 \end{aligned}$$

Valuation: $r > d$

Proof:

$$
\begin{aligned}
v(x) \quad & \geq \quad \sup_{\sigma \in \mathcal{S}_{0,\infty}} \mathbb{E}_x[e^{-r(\sigma \wedge \tau_{[K,h^*]})}\{((X_{\tau_{[K,h^*]}} - K)^+ + \delta)1_{\{\tau_{[K,h^*]} < \sigma\}} \\
& + \quad (X_\sigma - K)^+ 1_{\{\sigma \leq \tau_{[K,h^*]}\}}\}] \\
& \geq \quad \inf_{\tau \in \mathcal{S}_{0,\infty}} \sup_{\sigma \in \mathcal{S}_{0,\infty}} \mathbb{E}_x[e^{-r(\sigma \wedge \tau)}\{((X_\tau - K)^+ + \delta)1_{\{\tau < \sigma\}} \\
& + \quad (X_\sigma - K)^+ 1_{\{\sigma \leq \tau\}}\}] \\
& \geq \quad \sup_{\sigma \in \mathcal{S}_{0,\infty}} \inf_{\tau \in \mathcal{S}_{0,\infty}} \mathbb{E}_x[e^{-r(\sigma \wedge \tau)}\{((X_\tau - K)^+ + \delta)1_{\{\tau < \sigma\}} \\
& + \quad (X_\sigma - K)^+ 1_{\{\sigma \leq \tau\}}\}] \\
& \geq \quad v(x)
\end{aligned}
$$

where $\tau_{[K,k^*]} := \inf\{t \geq 0 : K \leq X_t \leq h^*].$

Value function: $r > d$

Conclusion:

$$V(x) = \begin{cases} x - K & \text{if } x \in [k^*, \infty) \\ (k^* - K)^+ \mathbb{E}_x[e^{-r\sigma_{k^*}} 1_{\{\sigma_{k^*} \le \tau_{[K,h^*]}\}}] & \\ \quad +((h^* - K)^+ + \delta)\mathbb{E}_x[e^{-r\tau_{[K,h^*]}} 1_{\{\tau_{[K,h^*]} < \sigma_{k^*}\}}] & \text{if } x \in (h^*, k^*) \\ (x - K) + \delta & \text{if } x \in [K, h^*] \\ \delta\ \mathbb{E}_x[e^{-r\tau_{[K,h^*]}}] & \text{if } x \in (0, K) \end{cases}$$

where

$$\begin{aligned} \tau_{[K,k^*]} &:= \inf\{t \ge 0 : K \le X_t \le h^*] \\ \sigma_{k^*} &:= \inf\{t \ge 0 : X_t \ge k^*\} \end{aligned}$$

Value function $r > d$

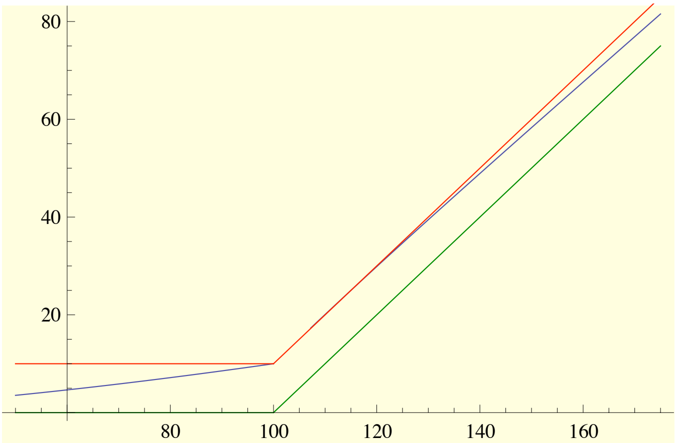

Figure: Non-convex value function.

Some Implications

- Game Options with convex underlying payoffs are *not necessarily convex*.
- Subsequently, game option prices are *not* always increasing in the volatility parameter σ.

i.e., Vega can be negative,

$$\frac{\partial V(x)}{\partial \sigma} < 0, \text{ for some } x \text{ values.}$$

Thank You!

Thank you very much for your attention!

Some References

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