UNSTABLE VOLATILITY: THE BREAK PRESERVING LOCAL LINEAR ESTIMATOR

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What is our work about?

- Aim?: **estimation of discontinuous volatility functions.**
- Discontinuities?: **abrupt structural changes.**
- Method?: **nonparametric kernel estimation.**
- Contribution?: **Break preserving local linear.**

Model

$$Y_i = m(X_i) + \sigma(X_i)\epsilon_i \quad \epsilon \sim i.i.d(0,1)$$

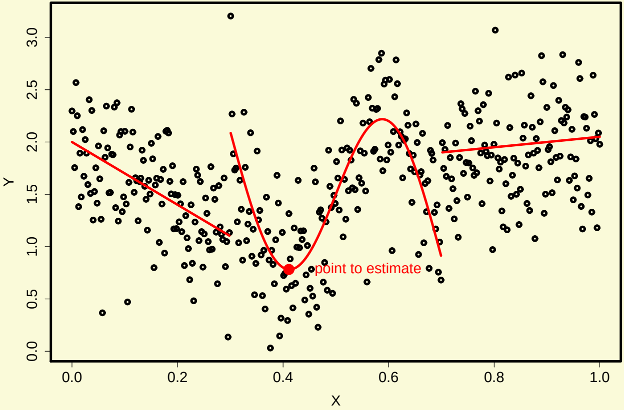
- Fixed design or random design.
- $E(\epsilon|X) = 0$, $E(\epsilon^2|X) = 1$ and $E(\epsilon^4|X) < \infty$
- $E(Y|X = x) = m(x)$
- $E((Y - m(X))^2|X = x) = \sigma^2(x)$

Previous work: drift estimator

Drift estimation

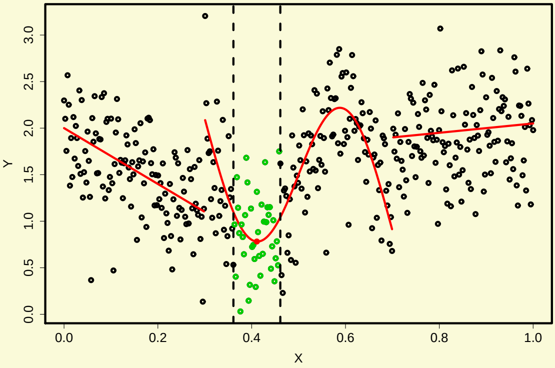
$$Y_i = m(X_i) + 0.4\epsilon_i \quad \text{with } \epsilon \sim IID(0,1)$$

Given a point x in the continuous part, estimator of $\mathbf{m(x)}$?

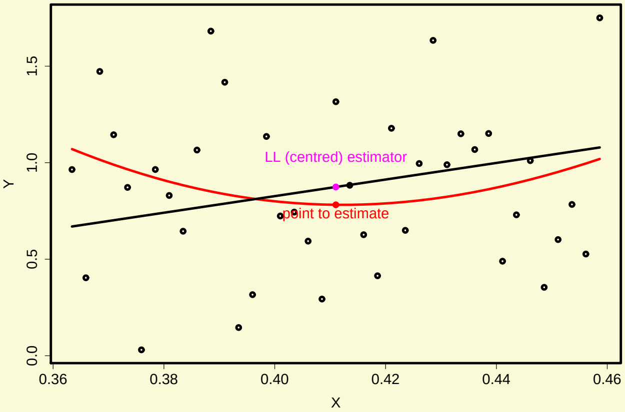


Drift estimation: centred estimator

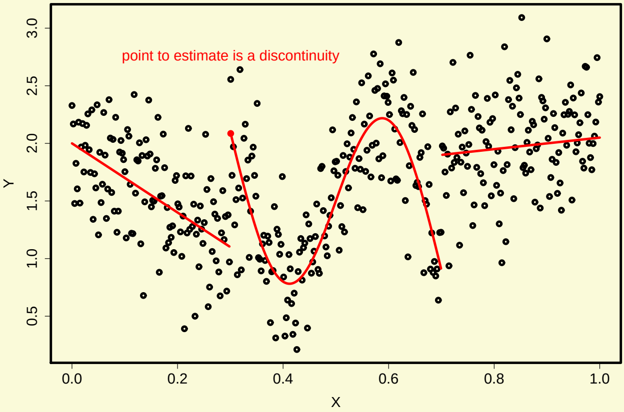
The centred estimator, $\hat{\mathbf{m}}_{\mathbf{c}}(\mathbf{x})$, is obtained as a regression using the points in a neighbourhood of x, Fan and Gijbels (1997).



Drift estimation: centred estimator



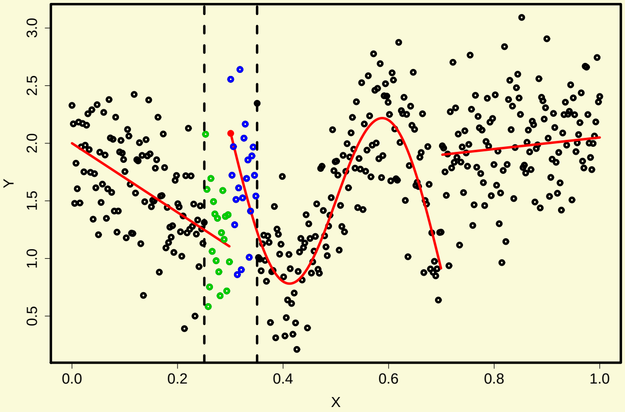
Drift estimation: What happens at discontinuities?



We expect the centred estimator to fall in the middle of the jump.

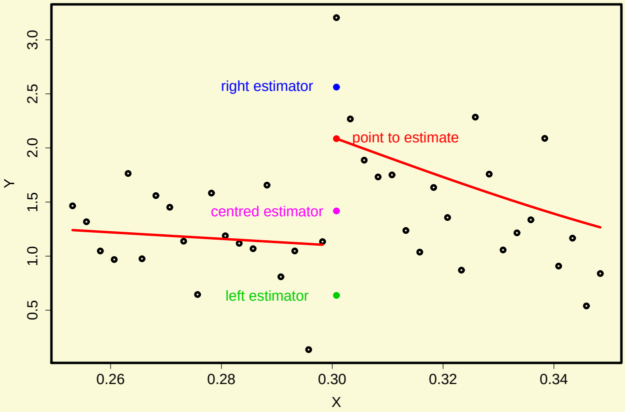
Drift estimation: What happens at discontinuities?

The asymmetric estimator: find two estimators, left and right, and choose appropriately, Qiu (2003).



Drift estimation: What happens at discontinuities?

We have three estimator, which is the best choice?



Contribution: volatility estimator

Estimation of a discontinuous volatility

$$Y_i = m(X_i) + \sigma(X_i)\epsilon_i \quad \epsilon \sim i.i.d(0,1)$$

Define $\hat{r}_i = (Y_i - \hat{m}(X_i))$. Then, $E(\hat{r}^2|X = x) = \hat{\sigma}^2(x)$.

Fan and Yao (1998):

"While the bias of $\hat{m}$ itself is of order $O(h_1^2)$, its contribution to $\hat{\sigma}^2(\cdot)$ is only of $o(h_1^2)$".

So, we expect to get a good estimate of the volatility even if the drift function is unknown.

Estimation of a discontinuous volatility

Do you think that the centred estimator (Fan and Yao, 1998) is a good choice to estimate a discontinuous volatility function?

Estimation of a discontinuous volatility

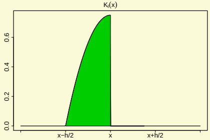
Do you think that the centred estimator (Fan and Yao, 1998) is a good choice to estimate a discontinuous volatility function?

- No, because it is not consistent at discontinuities.
- **Solution**: the break preserving local linear (BPLL) estimator.

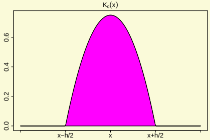
Estimation of a discontinuous volatility

$$\hat{\sigma}_k^2(x) = \hat{a}_{0,k}(x) \quad \text{and} \quad \hat{\dot{\sigma}}_k^2 = \hat{a}_{1,k}$$

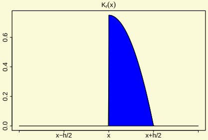
$$(\hat{a}_{0,k}(x), \hat{a}_{1,k}(x)) = \min_{(a_0,a_1)} \sum_{i=1}^{n} \left\{ \hat{r}_i^2 - \mathbf{a_{0,k}} - a_{1,k}(X_i - x) \right\}^2 K_k\left(\frac{X_i - x}{h_2}\right)$$



left (k=l)



centred (k=c)



right (k=r)

Estimation of a discontinuous volatility

The expression of the three volatility estimators:

$$\hat{\sigma}_k^2(x) = \sum_{i=1}^{n} \hat{r}_i^2 K_k\left(\frac{X_i - x}{h_2}\right) \frac{s_{k,2} - s_{k,1}(X_i - x)}{s_{k,0}s_{k,2} - s_{k,1}^2} \quad k = c, l, r$$

where

$$s_{k,j} = \sum (X_i - x)^j K_k\left(\frac{X_i - x}{h_2}\right)$$

.

- Easy to compute.
- No numerical minimisation.

Estimation of a discontinuous volatility

How well are the estimators fitted to the data set?
Weighted Residuals Mean Square.

$$WRMS_k(x) = \frac{\sum_{i=1}^{n} \left\{\hat{r}_i^2 - \mathbf{\hat{a}_{0,c}} - \hat{a}_{1,c}(X_i - x)\right\}^2 K_k\left(\frac{X_i - x}{h_2}\right)}{\sum_{i=1}^{n} K_k\left(\frac{X_i - x}{h_2}\right)}$$

Break preserving local linear

The break preserving local linear estimator:

$$\hat{\sigma}^2_{BPLL}(x) = \begin{cases} \hat{\sigma}^2_c(x) & \text{diff}(x) < u \\ \hat{\sigma}^2_l(x) & \text{diff}(x) \geq u \text{ and } WRMS_l(x) < WRMS_r(x) \\ \hat{\sigma}^2_r(x) & \text{diff}(x) \geq u \text{ and } WRMS_l(x) > WRMS_r(x) \\ \dfrac{\hat{\sigma}^2_l(x) + \hat{\sigma}^2_r(x)}{2} & \text{diff(x)} \geq u \text{ and } WRMS_l(x) = WRMS_r(x) \end{cases}$$

where $\text{diff}(x) = \max(WRMS_c(x) - WRMS_l(x), WRMS_c(x) - WRMS_r(x))$, and $0 \leq u \leq Q$ for all x and Q a constant.

How is the WRMS for each estimator?

Let $[a, b]$ be the support of X and $\{x_q\}$ for $q = 1, \ldots, m$ be the finite set of points where the volatility function is discontinuous. Then, two regions can be differentiated:

- D_1 is the region where the volatility function is continuous,

$$D_1 = \left[a + \frac{h_2}{2}, b - \frac{h_2}{2}\right] \setminus D_2$$

- D_2 contains the points of discontinuity and their neighbourhoods:

$$D_2 = \bigcup_{q=1}^{m} \left[x_q - \frac{h_2}{2}, x_q + \frac{h_2}{2}\right]$$

How is the WRMS for each estimator?

Under certain regularity conditions :
For $x \in D_1$,

$$WRMS_k(x) = \sigma^4(x)(E(\epsilon^4|X) - 1) + R_{k,1}(x)$$

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For $x \in D_1$,

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For $x \in D_2$ such that $x = x_q + \tau h_2$ with $\tau \in [0, \frac{1}{2}]$ and a jump of magnitude d,

$$\begin{aligned} WRMS_l(x) &= \sigma^4(x)(E(\epsilon^4|X) - 1) + d^2 C_{l,\tau}^2 + R_{l,2}(x) \\ WRMS_r(x) &= \sigma^4(x)(E(\epsilon^4|X) - 1) + R_{r,2}(x) \\ WRMS_c(x) &= \sigma^4(x)(E(\epsilon^4|X) - 1) + d^2 C_{c,\tau}^2 + R_{c,2}(x) \end{aligned}$$

How is the WRMS for each estimator?

Under certain regularity conditions :
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For $x \in D_2$ such that $x = x_q + \tau h_2$ with $\tau \in [-\frac{1}{2}, 0]$ and a jump of magnitude d,

$$\begin{aligned} WRMS_l(x) &= \sigma^4(x)(E(\epsilon^4|X) - 1) + R_{l,3}(x) \\ WRMS_r(x) &= \sigma^4(x)(E(\epsilon^4|X) - 1) + d^2C^2_{r,\tau} + R_{r,3}(x) \\ WRMS_c(x) &= \sigma^4(x)(E(\epsilon^4|X) - 1) + d^2C^2_{c,\tau} + R_{c,3}(x) \end{aligned}$$

MSE (continuous points)

Under certain regularity conditions and with

$$\mu_{k,j} = \int u^j K_k(u)du \text{ and } V_k = \int K_k^2(u)\left[\frac{\mu_{k,2} - \mu_{k,1}u}{\mu_{k,0}\mu_{k,2} - \mu_{k,1}^2}\right]^2 du:$$

For $x \in D_1$ (**Continuous points**),

$$\text{Bias}(\hat{\sigma}_k^2(x)) = \frac{h_2^2 \ddot{\sigma}^2(x)}{2} \frac{\mu_{k,2}^2 - \mu_{k,1}\mu_{k,3}}{\mu_{k,2}\mu_{k,0} - \mu_{k,1}^2} + o_p(h_1^2 + h_2^2 + \tfrac{1}{nh_2})$$

$$\text{Variance}(\hat{\sigma}_k^2(x)) = \frac{(E(\epsilon^4|X) - 1)\sigma^4(x)}{nh_2 f_X(x)} V_k + o_p\left(\frac{1}{nh_2}\right)$$

$$\text{MSE}(\hat{\sigma}_k^2(x)) = \text{Bias}^2 + \text{Variance}$$

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$$\text{Variance}(\hat{\sigma}_k^2(x)) = \frac{(E(\epsilon^4|X)-1)\sigma^4(x)}{nh_2 f_X(x)} V_k + o_p\left(\frac{1}{nh_2}\right)$$

If $\mathbf{h_1, h_2 \to 0}$, $\mathbf{n \to \infty}$ and $\mathbf{nh_2 \to \infty}$

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For $x \in D_1$ (**Continuous points**),

Bias$(\hat{\sigma}_k^2(x)) =$ 

Variance$(\hat{\sigma}_k^2(x)) = \frac{(E(\epsilon^4|X)-1)\sigma^4(x)}{nh_2 f_X(x)} V_k + o_p\left(\frac{1}{nh_2}\right)$

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For $x \in D_1$ (**Continuous points**),

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Variance$(\hat{\sigma}_k^2(x)) =$ 

If $\mathbf{h_1, h_2 \to 0,\ n \to \infty}$ and $\mathbf{nh_2 \to \infty}$

MSE$(\hat{\sigma}_k^2(x)) =$ 

MSE (right side of discontinuity)

For $x \in D_2$ such that $x = x_q + \tau h_2$ with $\tau \in [0, \frac{1}{2}]$ and a jump of magnitude d,

$$\text{MSE}(\hat{\sigma}_l^2(x)) = \left[d \int_{-\frac{1}{2}}^{\tau} K_l(u) \frac{\mu_{l,2} - \mu_{l,1} u}{\mu_{l,0}\mu_{l,2} - \mu_{l,1}^2} du \right]^2 + \frac{(E(\epsilon^4|X) - 1)\sigma^4(x)}{n h_2 f_X(x)} V_l + o_p(1)$$

$$\text{MSE}(\hat{\sigma}_c^2(x)) = \left[d \int_{-\frac{1}{2}}^{\tau} K_c(u) du \right]^2 + \frac{(E(\epsilon^4|X) - 1)\sigma^4(x)}{n h_2 f_X(x)} V_c + o_p(1)$$

MSE (right side of discontinuity)

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If $\mathbf{h_1, h_2 \to 0}$, $\mathbf{n \to \infty}$ and $\mathbf{n} h_2 \to \infty$

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$$\text{MSE}(\hat{\sigma}_c^2(x)) = \left[d \int_{-\frac{1}{2}}^{\tau} K_c(u) du \right]^2 +$$

If $\mathbf{h_1, h_2 \to 0}$, $\mathbf{n \to \infty}$ and $\mathbf{n} h_2 \to \infty$

Consistency

- At points of continuity: all the estimators are consistent.
- At the right of the discontinuity: only the right estimator is consistent.
- At the left of the discontinuity: only the left estimator is consistent.
- **The BPLL is consistent everywhere.**

CLT

Theorem

If $h_1, h_2 \to 0$, $n \to \infty$ and $nh_1, nh_2 \to \infty$ and under certain regularity conditions, $\sqrt{nh_2}(\sigma^2(x) - \hat{\sigma}^2_{BPLL}(x) - \beta_n(x))$ is asymptotically normal with mean 0 and variance

$$\frac{(E(\epsilon^4|X) - 1)\sigma^4(x)}{nh_2 f_X(x)} \int K_k^2(u) \left[\frac{\mu_{k,2} - \mu_{k,1}u}{\mu_{k,0}\mu_{k,2} - \mu_{k,1}^2}\right]^2 du + o_p\left(\frac{1}{nh_2}\right),$$

and bias

$$\beta_n = \frac{h_2^2 \ddot{\sigma}^2(x)}{2} \frac{\mu_{k,2}^2 - \mu_{k,1}\mu_{k,3}}{\mu_{k,2}\mu_{k,0} - \mu_{k,1}^2}$$

for $k = c, l, r$ as appropriate.

Bandwidth selection

Alternative to the plug-in bandwidth estimator:

1. The leave–one–out cross validation:

$$(h_2^{cv}, u_{cv}) = \text{arg}\min_h \sum_{i=1}^{n} \left[\hat{r}_i^2 - \hat{\sigma}_{-i}^2\right]^2$$

   where $\hat{\sigma}_{-i}^2$ is calculated without using the pair $(X_i, \hat{r}_i^2)$.

2. The leave a b–block–out cross validation for dependent data (Patton, Politis and White (2009) shows how to find the size of the block).

$$(h_2^{b}, u_{b}) = \text{arg}\min_h \sum_{i=1}^{n} \left[\hat{r}_i^2 - \hat{\sigma}_{-b_i}^2\right]^2$$

   where $\hat{\sigma}_{-b_i}^2$ is calculated without using the $2b+1$ pairs $(X_{i-b}, \hat{r}_{i-b}^2), \ldots, (X_i, \hat{r}_i^2), \ldots, (X_{i+b}, \hat{r}_{i+b}^2)$.

Ensuring positivity

The LL estimator, and therefore the BPLL estimator, is sometime negative for finite samples.
Solutions:

- Discard negative values.
- The re–weighted Nadaraya–Watson estimator (see Hall *et al.*, 1999; Cai, 2002; and Phillips and Xu, 2007). It cannot be extended to estimate discontinuous volatility functions.
- The exponential local linear (ELL) (see Ziegelmann, 2002). Computationally heavy and theoretically obscure.
- Substitute any negative values of $\hat{\sigma}_k^2(x)$ by $\hat{\sigma}_{k,ELL}^2(x)$ for $k = c, l, r$.

Experiment 1: iid variables

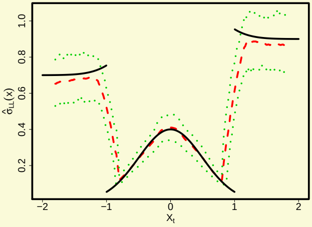
$$Y_i = m(X_i) + \sigma(X_i)\epsilon_i$$

- $\epsilon \sim IIDN(0,1)$.
- $X_i = IIDU(-2,2)$, random design.
- x are $T = 250$ equidistant values in [-1.8,1.8].
- $n = 500, 1000, 2000$, number of simulations N=200.
- ϵ_i and X_i are independent.
- Leave–one–out cross validation.
- $\sigma(x)$ has two discontinuities at $x = -1, 1$ Plot.
- Four scenarios depending on $m(x)$:
  - Scenario I: $m \equiv 0$
  - Scenarios II, III, IV: Plot.

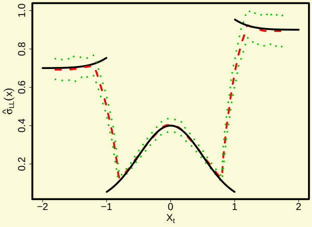
Comparison LL vs. BPLL (MISE)

Method	LL $\widehat{\text{MISE}}$	LL $\widehat{\text{MISE}}_q$	BPLL $\widehat{\text{MISE}}$	BPLL $\widehat{\text{MISE}}_q$
$n = 500$				
Scenario I	0.0089	0.0060	0.0098	0.0039
Scenario II	0.0098	0.0062	0.0120	0.0039
Scenario III	0.0093	0.0061	0.0109	0.0040
Scenario IV	0.0107	0.0067	0.0123	0.0042
$n = 1000$				
Scenario I	0.0047	0.0034	0.0037	0.0015
Scenario II	0.0044	0.0032	0.0043	0.0018
Scenario III	0.0048	0.0034	0.0045	0.0017
Scenario IV	0.0044	0.0032	0.0040	0.0016
$n = 2000$				
Scenario I	0.0021	0.0016	0.0012	0.0005
Scenario II	0.0020	0.0016	0.0012	0.0006
Scenario III	0.0020	0.0016	0.0012	0.0006
Scenario IV	0.0022	0.0016	0.0013	0.0005

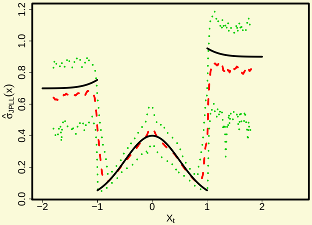
Comparison LL vs. BPLL



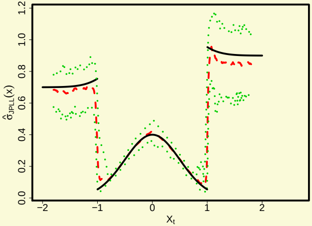
(a) LL with $n = 500$



(b) LL with $n = 2000$

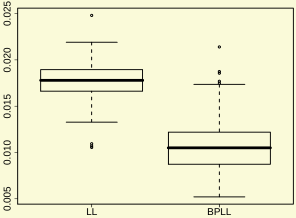


(c) BPLL with $n = 500$

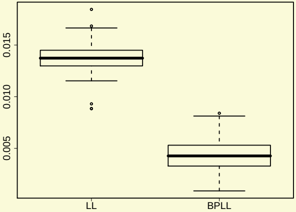


(d) BPLL with $n = 2000$

Comparison LL vs. BPLL (Error boxplot)



(a) $n = 2000$ in D_1



(b) $n = 2000$ in D_2

Experiment 2: a square root diffusion

The process is of the form:

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dB_t$$

The process was generated following the algorithm in Section 3.4 of Glasserman (2004).

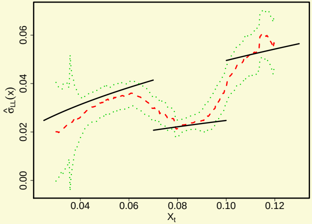
- x are $T = 250$ equidistant values in [0.03,0.12].
- $n = 500, 1000, 2000$, number of simulations $N = 400$.
- B_t and X_t are independent.
- Leave–b–block–out cross validation to obtain the bandwidth.
- The drift and diffusion are discontinuous at $x = 0.1$.

Comparison LL vs. BPLL (MISE)

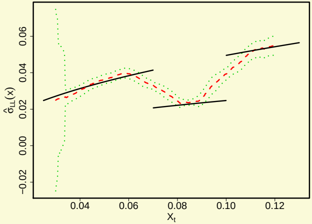
Method	LL		BPLL	
	$\widehat{\text{MISE}}$	$\widehat{\text{MISE}_q}$	$\widehat{\text{MISE}}$	$\widehat{\text{MISE}_q}$
$n = 500$	0.0241	0.0083	0.0114	0.0057
$n = 1000$	0.0120	0.0041	0.0039	0.0022
$n = 2000$	0.0059	0.0021	0.0013	0.0008

Table: MISE of LL and BPLL comparison for Experiment 2.

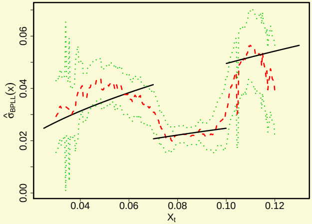
Comparison LL vs. BPLL



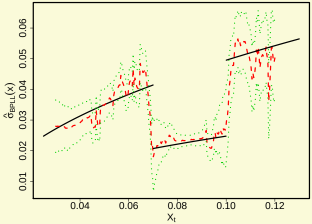
(a) LL with $n = 500$



(b) LL with $n = 2000$



(c) BPLL with $n = 500$



(d) BPLL with $n = 2000$

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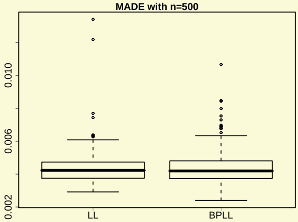
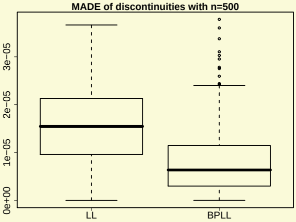
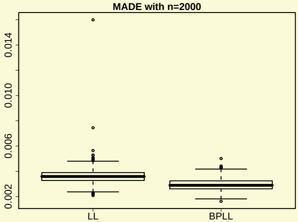
Comparison LL vs. BPLL (Boxplots)



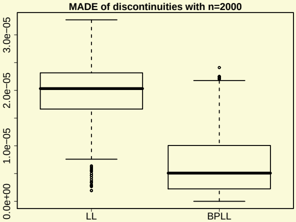
(a) $n = 500$ in D_1



(b) $n = 500$ in D_2



(a) $n = 2000$ in D_1



(b) $n = 2000$ in D_2

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Conclusions

- The break preserving estimator is consistent in the presence of discontinuities.
- It is always positive.
- It keeps some of the smooth properties of the LL in the continuous parts.

Further interest

- Application to the spot volatility of intra–day data (SPDR).
  - Y. Zu and P. Boswijk (2009). *Estimating realized spot volatility with noisy high–frequency data*.
  - P. Mykland, E. Renault and L. Zhang (2009). *Aggregated and instantaneous volatility: connections and comparisons*.
  - F. Bandi (2009). *Nonparametric identification in stochastic volatility models*.
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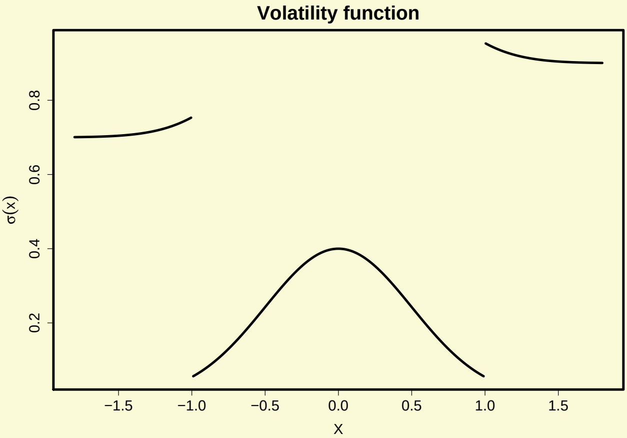
  - ...
- Application to the estimation of interest rates: changes of structure in the drift and volatility.
  - R. Stanton (1997). *A Nonparametric Model of Term Structure Dynamics and the Market Price of Interest Rate Risk*.
  - D. A. Chapman and N. Pearson (2000). *Is the Short Rate Drift Actually Nonlinear?*.
  - S. L. Heston (2007). *A model of discontinuous interest rate behavior, yield curves, and volatility*.
  - ...

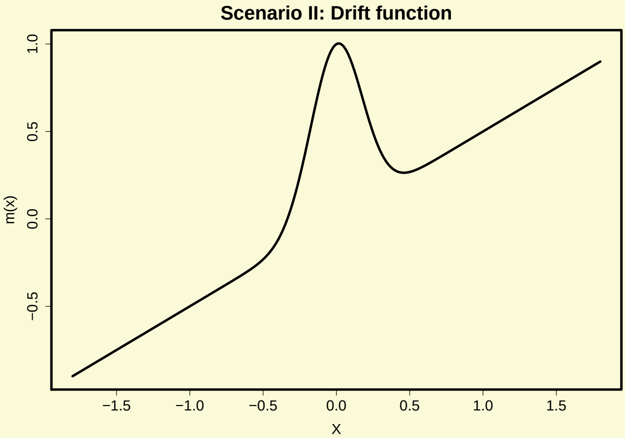
Simulated volatility function

Back



Scenario II

▸ Next Continuous function.



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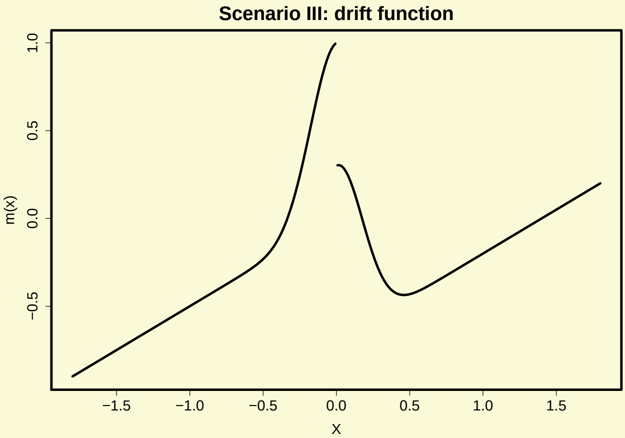
Scenario III

Next One discontinuity at $x = 0$.



Scenario IV

Back Two discontinuities at the same points than the volatility function $x = -1$ and $x = 1$.