Asymptotic Method for Singularity in Path-Dependent Option Pricing

Sang-Hyeon Park, Jeong-Hoon Kim

Dept. Math. Yonsei University

June 2010

Outline

1. Introduction
   - Perturbation Method
   - Matched asymptotics
2. Application for lookback option (S.V model)
   - Outer expansion
   - Matching for Second correction
3. Application for barrier option (S.V model)
   - Outer expansion
   - Inner expansion

Introduction : Perturbation Method

Definition (Perturbation Method)

Modification of a given problem by adding or eliminating a small multiple ε times a higher order term.

Example (Algebraic equation)

$$x^2 + \epsilon x - 1 = 0 \tag{1}$$

If $\epsilon = 0$, $x = \pm 1$. Here, we expand solution

$$x(\epsilon) = x_0 + \epsilon x_1 + \epsilon^2 x_2 .... \tag{2}$$

Then

$${x_0}^2 = 1, \quad 2x_0 x_1 = 1, \quad , {x_1}^2 + 2x_0 x_2 + x_1 = 0 \tag{3}$$

$$x(\epsilon)^1 = 1 + \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 .... \tag{4}$$

$$x(\epsilon)^2 = -1 - \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 .... \tag{5}$$

Introduction : Matched asymptotics

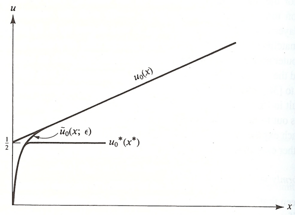
Definition (Matched asymptotics)

A singular perturbation method for boundary conditions which have some singularity.

Example (Burger's equation)

$$\epsilon u'' + u' = \frac{1}{2} \quad (6)$$

$$u(0) = 0 \quad u(1) = 1 \quad 0 < \epsilon \ll 1 \quad (7)$$



This equation u'' go to infinity near 0. So regular perturbation is broken near 0. In finance, prices of many derivatives have PDE solution, and we can't use regular perturbation method in some case.

Mean-Reversion Model

Assuming that underlying risky asset price follows a geometric Brownian motion whose volatility is driven by fast mean-reverting Ornstein Uhlenbeck SDE as follows:

$$dS_t/S_t = \mu dt + \sigma_t d{W_t}^{(0)},$$
$$\sigma_t = f(Y_t),$$
$$dY_t = \frac{1}{\epsilon}(m - Y_t)dt + \frac{\sqrt{2}\nu}{\sqrt{\epsilon}} Y_t d{W_t}^{(1)}.$$

Let $0 < \epsilon \ll 1$ and $\nu^2 = Var(Y)$ as $t \to \infty$. ρ is a correlation of ${W_t}^{(0)}$ and ${W_t}^{(1)}$
The risk-neutral price of the lookback put option is given by

$$P(t, s, s^\star, y) = E^Q[e^{-r(T-t)} H(S_T, S_T^\star, Y_T) | S_t = s, {S_t}^\star = s^\star, Y_t = y]. \quad (8)$$

Where

$${S_t}^\star = \max_{u \le t} S_u \ , \ \ H(S_t, {S_t}^\star) = {S_t}^\star - S_t.$$

Here, H denotes the payoff function (for lookback put option)

Relate PDE

Using the Feynman-Kac formula, one can transform the integral problem into the PDE problem

$$\left[\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right] P = 0, \tag{9}$$

where

$$\begin{aligned}
\mathcal{L}_0 &= (m-y)\frac{\partial}{\partial y} + \nu^2\frac{\partial^2}{\partial y^2}, \\
\mathcal{L}_1 &= \sqrt{2}\nu\Lambda(y)\frac{\partial}{\partial y} + \sqrt{2}\nu f S\rho\frac{\partial^2}{\partial s\partial y}, \\
\mathcal{L}_2 &= -r\bullet + \frac{\partial}{\partial t} + rS\frac{\partial}{\partial s} + \frac{1}{2}f^2S^2\frac{\partial^2}{\partial s^2}.
\end{aligned}$$

Here, the final and boundary conditions are given by

$$P(T,s,s^\star,y) = H(s,s^\star,y)\ , \ \ \frac{\partial P}{\partial s^\star}|_{s^\star=s} = 0, \qquad 0 \le t < T.$$

letting

$$P = P_0 + \sqrt{\epsilon}P_1 + \epsilon P_2 + \ldots.$$

Solution of PDE

Theorem (1)

The leading order $P_0(t,s,s^\star) = s^\star \widetilde{P}_0$ *is given by*

$$P_0(t,s,s^\star) = (1+\frac{\bar{\sigma}^2}{2r})sN\left(\delta_+(T-t,\frac{s}{s^\star})\right) + e^{-r(T-t)}s^\star N\left(-\delta_-(T-t,\frac{s}{s^\star})\right) - \frac{\bar{\sigma}^2}{2r}e^{-r(T-t)}(\frac{s^\star}{s})^{\frac{2r}{\bar{\sigma}^2}} sN\left(-\delta_-(T-t,\frac{s^\star}{s})\right) - s,$$

where $N(d)$ is a cumulative distribution function of standard normal and

$$\delta_\pm(T-t,s) = \frac{1}{\bar{\sigma}\sqrt{T-t}}\left(\ln s + (r \pm \frac{1}{2}\bar{\sigma}^2)(T-t)\right).$$

Solution of PDE

Theorem (2)

The first correction $P_1 = s^\star \widetilde{P}_1$ is given by

$$\begin{aligned}\widetilde{P}_1 =& -(T-t)[(A_2-A_1)(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x})\widetilde{P}_0 - A_2\frac{\partial}{\partial x}(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x})\widetilde{P}_0] \\ &- \int_t^T \frac{g(s-t)}{\sqrt{2\pi(T-s)}}\overline{\sigma} e^{\frac{(2r-\overline{\sigma}^2)^2(T-s)-(2r+\overline{\sigma}^2)^2 t}{8\overline{\sigma}^2} - \frac{(x-\frac{1}{2}(2r-\overline{\sigma}^2)^2(T-s))^2}{2\overline{\sigma}^2(T-s)}} ds \\ &+ \int_t^T \frac{g(s-t)}{\sqrt{2\pi(T-s)}}\sqrt{2(T-s)}(2r-\overline{\sigma}^2)e^{\frac{(2r-\overline{\sigma}^2)(T-s)}{8}}. \\ &\cdot\{N\left(\frac{x-\frac{1}{4}(\frac{2r}{\overline{\sigma}^2}-1)}{2\overline{\sigma}\sqrt{T-s}}\right) - N\left(\frac{-\frac{1}{4}(\frac{2r}{\overline{\sigma}^2}-1)}{2\overline{\sigma}\sqrt{T-s}}\right)\}ds,\end{aligned}$$

Where $g(t) = -(T-t)\frac{\partial}{\partial x}[(A_2-A_1)(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x})\widetilde{P}_0 - A_2\frac{\partial}{\partial x}(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x})\widetilde{P}_0]|_{x=0}$.

Matched Asymptotics

The Second Correction - In this subsection we derive the second correction $\tilde{P}_2$ using matching technique as well as Green's function method. From outer Expansion,

$$\widetilde{P}_2 = V_2(t,x,y) + I(t,x), \quad V_2(t,x,y) := \frac{1}{2}\phi(y)(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x})\widetilde{P}_0(t,x).$$

Here, $I(t,x) = I_1 + I_2$ is to be determined in this subsection.

Second Correction Solution

Theorem (3)

The solution $I_1(t,x)$ *is given by*

$$\begin{aligned} I_1 =& -(T-t)Q(t,x) \\ & - \int_t^T \frac{g(s-t)}{\sqrt{2\pi(T-s)}}\overline{\sigma}\exp\{\frac{(2r-\overline{\sigma}^2)^2(T-s)-(2r+\overline{\sigma}^2)^2 t}{8\overline{\sigma}^2} \\ & \quad - \frac{(x-\frac{1}{2}(2r-\overline{\sigma}^2)^2(T-s))^2}{2\overline{\sigma}^2(T-s)}\}ds \\ & + (2r-\overline{\sigma}^2)\int_t^T \frac{g(s-t)}{\sqrt{2\pi(T-s)}}\sqrt{2(T-s)}\exp\{\frac{(2r-\overline{\sigma}^2)(T-s)}{8}\}. \\ & \quad \cdot\{N\left(\frac{x-\frac{1}{4}(\frac{2r}{\overline{\sigma}^2}-1)}{2\overline{\sigma}\sqrt{T-s}}\right) - N\left(\frac{-\frac{1}{4}(\frac{2r}{\overline{\sigma}^2}-1)}{2\overline{\sigma}\sqrt{T-s}}\right)\}ds, \end{aligned}$$

Second Correction Solution

Theorem (4)

The solution $I_2(t,x)$ is given by

$$
\begin{aligned}
I_2(t,x) =& - <\phi> (\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x})\widetilde{P}_0(t,x) \\
& - \overline{\sigma} \int_t^T \frac{g(s-t)}{\sqrt{2\pi(T-s)}} \exp\{ \frac{(2r-\overline{\sigma}^2)^2(T-s) - (2r+\overline{\sigma}^2)^2 t}{8\overline{\sigma}^2} \\
& \quad - \frac{(x - \frac{1}{2}(2r-\overline{\sigma}^2)^2(T-s))^2}{2\overline{\sigma}^2(T-s)} \} ds \\
& + \int_t^T \frac{g(s-t)}{\sqrt{2\pi(T-s)}} \sqrt{2(T-s)}(2r-\overline{\sigma}^2) \exp\{ \frac{(2r-\overline{\sigma}^2)(T-s)}{8} \}. \\
& \quad \cdot \{ N\left(\frac{x - \frac{1}{4}(\frac{2r}{\overline{\sigma}^2} - 1)}{2\overline{\sigma}\sqrt{T-\tau}} \right) - N\left(\frac{-\frac{1}{4}(\frac{2r}{\overline{\sigma}^2} - 1)}{2\overline{\sigma}\sqrt{T-\tau}} \right) \} ds,
\end{aligned}
$$

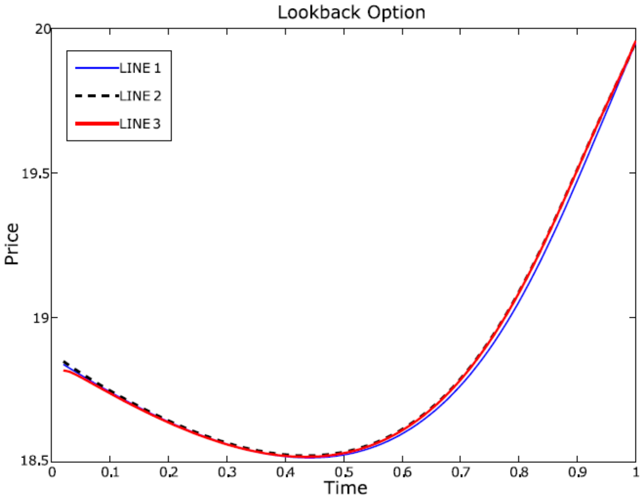


Fig. 1. LINE1 $= \widetilde{P}_0$, LINE2 $= \widetilde{P}_0 + \sqrt{\epsilon}\widetilde{P}_1$, LINE3 $= \widetilde{P}_0 + \sqrt{\epsilon}\widetilde{P}_1 + \epsilon\widetilde{P}_2$, $K = 100$, $r = 0.04$, $\sigma = 0.165$, maturity $= 1$, and $\epsilon = 0.0001$.

Barrier Option

Barrier option pricing under the stochastic volatility
We define the process $M_t = \max_{u \le t} S_u$. Then up-and-out call option price is given by

$$P(t,s,y) = E^*[e^{-r(T-t)}(S_T - K)^+ 1_{\{M_T < B\}} | S_t = s, Y_t = y] \tag{10}$$

under some risk-neutral measure. The Feynman-Kac formua tells us that barrier option price $P(t,s,y)$ satisfies the same as the PDE and the final and boundary conditions are given by

$$P(T,s,y) = (s-K)^+ \quad , \quad P(t,B,y) = 0.$$

To obtain the outer expansion, let $x = \ln s$

$$\frac{P_0(t,s,y)}{K} = \widetilde{P}(t,x,y) = \widetilde{P}_0 + \sqrt{\epsilon}\widetilde{P}_1 + \epsilon \widetilde{P}_2 + \cdots.$$

Solution of PDE

Theorem (5)

The solution of $P_0(t,s)$ is given by

$$P_0(t,s) = K\widetilde{P}_0 = s\{N(d_1) - N(d_3) - b(N(d_6) - N(d_8))\} - K\exp(-r(T-t))\{N(d_2) - N(d_4) - a(N(d_5) - N(d_7))\},$$

where $N(x)$ is a cumulative distribution function of standard normal, and

$$d_{1,2} = \frac{\ln(\frac{s}{K}) + (r \pm \frac{1}{2}\overline{\sigma}^2)(T-t)}{\overline{\sigma}^2\sqrt{T-t}} \quad , \quad d_{3,4} = \frac{\ln(\frac{s}{B}) + (r \pm \frac{1}{2}\overline{\sigma}^2)(T-t)}{\overline{\sigma}^2\sqrt{T-t}}$$

$$d_{5,6} = \frac{\ln(\frac{s}{B}) - (r \mp \frac{1}{2}\overline{\sigma}^2)(T-t)}{\overline{\sigma}^2\sqrt{T-t}} \quad , \quad d_{7,8} = \frac{\ln(\frac{sK}{B^2}) - (r \mp \frac{1}{2}\overline{\sigma}^2)(T-t)}{\overline{\sigma}^2\sqrt{T-t}}$$

$$a = (\frac{B}{s})^{-1+\frac{2r}{\sigma^2}}, \quad b = (\frac{B}{s})^{1+\frac{2r}{\sigma^2}}$$

Solution of PDE

Theorem (6)

The solution of $\widetilde{P}_1(t,x)$ is given by

$$\widetilde{P}_1(t,x) = \frac{\ln(B/K)-x}{\overline{\sigma}\sqrt{2\pi}} \int_t^T \frac{1}{(s-t)^{\frac{3}{2}}} e^{-\frac{(\ln(B/K)-x)^2}{2\overline{\sigma}^2(s-t)} - \frac{1}{8}\overline{\sigma}^2(\alpha^2+1)^2(s-t)}.$$

$$\cdot (T-s)[\frac{1}{\sqrt{2}}(\rho\nu <\phi^{'} f> -\nu <\phi^{'}\Lambda>)(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x})P_0(s,x)$$

$$-\frac{1}{\sqrt{2}}\rho\nu <\phi^{'} f> \frac{\partial}{\partial x}(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x})P_0(s,x)]|_{(x=\ln(B/K)-)} \cdot ds$$

$$-(T-t)[\frac{1}{\sqrt{2}}(\rho\nu <\phi^{'} f> -\nu <\phi^{'}\Lambda>)(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x})P_0(t,x)$$

$$-\frac{1}{\sqrt{2}}\rho\nu <\phi^{'} f> \frac{\partial}{\partial x}(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x})P_0(t,x)].$$

Inner Expansion

In barrier option case, we also face a critical issue that both Delta and Gamma are broken at barrier near the expiry. For this problem we also take the inner expansion and use matching asymptotic technique as in the lookback option case. We define the new independent variables τ and z by

$$\tau = \frac{T-t}{\epsilon^2} \quad , \quad z = \frac{\ln(B/K) - x}{\epsilon} \tag{11}$$

respectively.

Inner PDE

Then our pricing function is denoted by $P^*(\tau, z, y)$ whose PDE is given by

$$\left[\frac{1}{\epsilon^2}\mathcal{L}_0^* + \frac{1}{\epsilon\sqrt{\epsilon}}\mathcal{L}_1^* + \frac{1}{\epsilon}\mathcal{L}_2^* + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_3^* + \mathcal{L}_4^*\right] P^* = 0, \tag{12}$$

where

$$\begin{aligned}
\mathcal{L}_0^* &= -\frac{\partial}{\partial \tau} + \frac{1}{2} f^2(y) \frac{\partial^2}{\partial z^2} \quad , \quad \mathcal{L}_1^* = \sqrt{2}\nu\rho f(y) \frac{\partial^2}{\partial z \partial y}, \\
\mathcal{L}_2^* &= (m - y)\frac{\partial}{\partial y} + \nu^2 \frac{\partial^2}{\partial y^2} + (-r + \frac{1}{2} f^2(y))\frac{\partial}{\partial z} \\
\mathcal{L}_3^* &= \sqrt{2}\nu\Lambda(y)\frac{\partial}{\partial y} \quad , \quad \mathcal{L}_4^* = -r.
\end{aligned}$$

Inner Solution

We expand $P^*(\tau, z, y)$ as

$$P^*(\tau, z, y) = P_0^* + \sqrt{\epsilon}P_1^* + \epsilon P_2^* + \cdots$$

Theorem (7)

The solutions P_0^* and P_1^* are given by

$$P_0^*(\tau, z, y) = \int_0^\infty (\frac{B}{K}e^{\epsilon z} - 1)^+ G(\tau, z - \xi, f)d\xi,$$

$$P_1^*(\tau, z, y) = \int_0^\tau \int_0^\infty \mathcal{L}_1^* P_0^*(s, \xi, y)G(s, z - \xi, f)d\xi ds,$$

respectively, where

$$G(\tau, z - \xi, f) = \frac{1}{f(y)\sqrt{2\pi\tau}}(e^{-\frac{(z-\xi)^2}{2f^2(y)\tau}} - e^{-\frac{(z+\xi)^2}{2f^2(y)\tau}}).$$

Matching

To match $\widetilde{P}_0 + \sqrt{\epsilon}\widetilde{P}_1$ and $P_0^* + \sqrt{\epsilon}P_1^*$, we use Van-Dyke's matching rule. As ϵ goes to zero, the inner limit of outer solution is obviously zero and the outer limit of inner solution is also zero by dominated convergence theorem. So, our composite approximation of up-and-out call option is given by

$$P \sim \widetilde{P}_0 + P_0^* + \sqrt{\epsilon}(\widetilde{P}_1 + P_1^*) + \cdots$$

asymptotically as ϵ goes to zero.

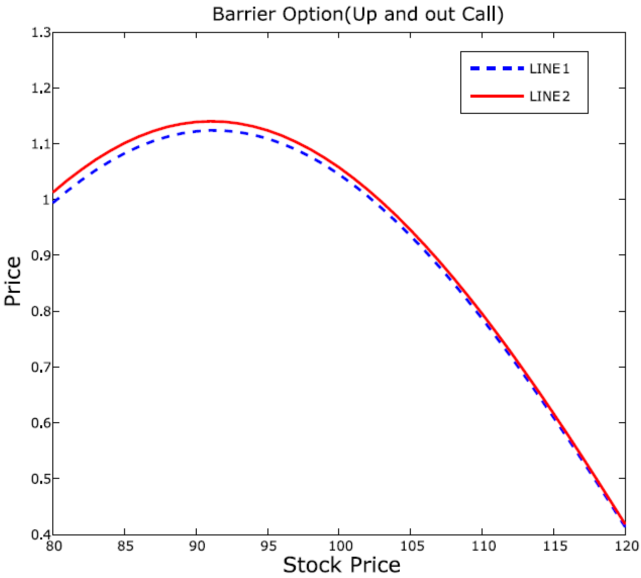


Fig. 2. LINE1 $=\widetilde{P}_0+P_0^\star$, LINE2 $=\widetilde{P}_0 P_0^\star+\sqrt{\epsilon}(\widetilde{P}_1+P_1^\star)$, $K=100$, $r=0.05$, $\sigma=0.35$, maturity $=1$, barrier $=130$ and $\epsilon=0.005$.

Conclusion

Lookback and barrier options have a large Delta and Gamma near the expiry. It may create a big error in pricing those options. To compensate this problem, we have applied matched asymptotics to the path-dependent options (lookback and barrier options) based on a fast mean-reverting stochastic volatility model. Also, it may be useful for other financial engineering problems wherever Greeks are involved, which would be an interesting extension of our work in this paper.