Interest rate modelling: How important is arbitrage–free evolution?

Siobhán Devin[1] Bernard Hanzon[2] Thomas Ribarits[3]

[1]European Central Bank

[2]University College Cork, Ireland

[3]European Investment Bank

Overview

§1 Nelson–Siegel (NS) models:

- Daily yield curve estimation; forecasting.

§2 No arbitrage interest rate models:

- Heath–Jarrow–Morton.

§3 Contribution: $HJM = NS + Adj = NS_{proj} + {}_{Adj}$

- ${}_{Adj} < Adj$, ${}_{Adj}$ is small.

Some notation

Zero–coupon bonds (ZCB):

- A ZCB is a contract that guarantees its holder the payment of one unit of currency at time maturity.
- $P(t,x)$ is the value of the bond at time t which matures in x years; $P(t,0) = 1$.
- A ZCB price is a discount factor.

Common interest rates:

- Continuously compounded yield: $y(t,x) = -\frac{\log P(t,x)}{x}$.
- Short rate: $\lim_{x\to 0^+} y(t,x) = r(t)$.
- Forward rate: $F(t,x,x+\epsilon) = \frac{1}{P(t+x,\epsilon)} \frac{P(t,x+\epsilon)-P(t,x)}{\epsilon}$.
- Instantaneous forward rate: $f(t,x) = \lim_{\epsilon\to 0^+} F(t,x,x+\epsilon) = -\frac{\partial \log P(t,x)}{\partial x}, \quad r(t) = f(t,t)$.
- Relationship between f and y: $y(t,x) = \frac{1}{x}\int_0^x f(t,s)\,ds$.

§1 Nelson–Siegel (NS) models

§1 Nelson–Siegel (NS) models:

- Daily yield curve estimation; forecasting.

Yield curve estimation.

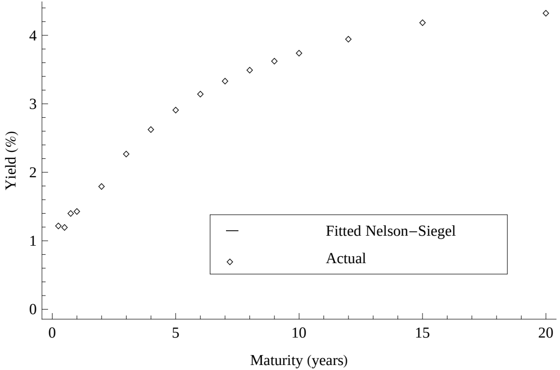

Figure: The EUR ZERO DEPO/SWAP curve as of $24/06/2009$.

Yield curve estimation.

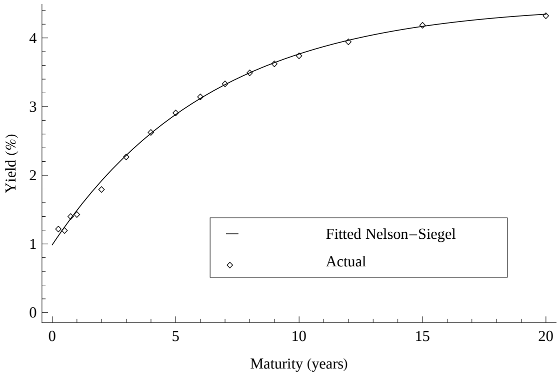

Figure: The EUR ZERO DEPO/SWAP curve as of 24/06/2009.

Yield curve estimation.

- Nelson–Siegel curves (and their extensions) are used by banks (eg central/investment) to estimate the shape of the yield curve.
- This estimation is justified by principal component analysis: low number of dimensions describes the curve with high accuracy.

Nelson–Siegel yield curve:

$$y(x) = L + S\left(\frac{1-e^{-\lambda x}}{\lambda x}\right) + C\left(\frac{1-e^{-\lambda x}}{\lambda x} - e^{-\lambda x}\right)$$

- y denotes the Nelson–Siegel yield curve.
- λ, L, S and C are estimated using yield data.

Yield curve estimation.

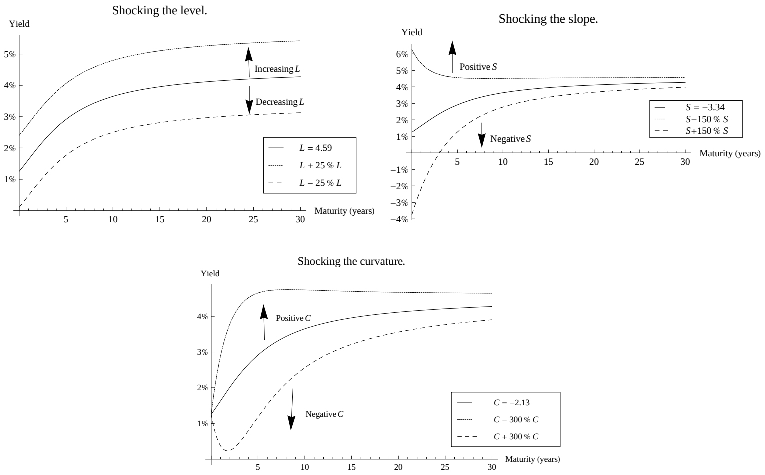

Figure: Influence of shocks on the factor loadings of the Nelson–Siegel yield curve.

Forecasting the term structure of interest rates.

Nelson–Siegel yield curve forecasting model:

$$y(t,x) = L(t) + S(t)\left(\frac{1-e^{-\lambda x}}{\lambda x}\right) + C(t)\left(\frac{1-e^{-\lambda x}}{\lambda x} - e^{-\lambda x}\right)$$

Advantages:

- ✔ Simple implementation.
- ✔ Easy to interpret.
- ✔ Can replicate observed yield curve shapes.
- ✔ Can produce more accurate one year forecasts than competitor models (Diebold and Li 2007).

A drawback?

Nelson–Siegel models are not arbitrage–free (Filipović 1999).

No arbitrage models

§2 No arbitrage interest rate models:

- Heath–Jarrow–Morton.

The HJM framework

The HJM framework:

$$df(t,x) = \alpha(f,t,x)\,dt + \sigma(f,t,x)\,dW(t),$$
$$f(0,x) = f^{o}(x),$$

where

$$\alpha(f,t,x) = \frac{\partial f(t,x)}{\partial x} + \sigma(f,t,x)\int_0^x \sigma(f,t,s)\,ds$$

A concrete model is fully specified once f^{o} and σ are given.

The HJM framework

Why use the HJM framework?

- Most short rate models can be derived within this framework.
- Automatic calibration: initial curve is a model input.
- Arbitrage–free pricing.

Interesting points:

- In practice one uses 2–3 driving Brownian motions ("factors").
- Despite this most HJM models are infinite–dimensional.
- Choice of volatility (not number of factors) determines complexity.
- A HJM model will be finite–dimensional if the volatility is an exponential polynomial function ie

$$EP(x) = \sum_{i=1}^{n} p_{\lambda_i}(x) e^{-\lambda_i x},$$

where p_{λ_i} is a polynomial associated with λ_i, (Björk, 2003).

The HJM framework

$$df(t,x) = \alpha(f,t,x)\,dt + \sigma(f,t,x)\,dW(t),$$
$$f(0,x) = f^o(0,x).$$

Possible volatility choices:

- Hull–White: $\sigma(f,t,x) = \sigma e^{-ax}$, (Ho–Lee: $\sigma(f,t,x) = \sigma$).
- Nelson–Siegel: $\sigma(f,t,x) = a + (b + cx)e^{-dx}$.
- Curve–dependent: $\sigma(f,t,x) = f(t,x)[a + (b + cx)e^{-dx}]$.

Note: Curve–dependent volatility is similar to a continuous–time version of the BGM/LIBOR market model.

Research contribution

§3a Theoretical Contribution: $HJM = NS + Adj = NS_{proj} + _{Adj}$

A specific HJM model

Consider the following HJM model:

$$df(t,x) = \left(\frac{\partial f}{\partial x} + C(\sigma, x)\right) dt + \sigma_{11}\, dB_1(t) + (\sigma_{21} + \sigma_{22} e^{-\lambda x} + \sigma_{23} x e^{-\lambda x})\, dB_2(t),$$

$$f^{o}(0,x) = f^{\mathrm{NS}}(x).$$

A specific HJM model

Consider the following HJM model:

$$df(t,x) = \left(\frac{\partial f}{\partial x} + C(\sigma,x)\right) dt + \sigma_{11}\, dB_1(t)$$
$$+ (\sigma_{21} + \sigma_{22} e^{-\lambda x} + \sigma_{23} x e^{-\lambda x})\, dB_2(t),$$
$$f^{\mathrm{o}}(0,x) = f^{\mathrm{NS}}(x).$$

(Björk 2003): f has a finite dimensional representation (FDR) since there is a finite-dimensional manifold $\mathcal{G}$ such that

- $f^o \in \mathcal{G}$
- drift and volatility are in the tangent space of $\mathcal{G}$.

A specific HJM model

Consider the following HJM model:

$$df(t,x) = \left(\frac{\partial f}{\partial x} + C(\sigma, x)\right) dt + \sigma_{11}\, dB_1(t) + (\sigma_{21} + \sigma_{22} e^{-\lambda x} + \sigma_{23} x e^{-\lambda x})\, dB_2(t),$$

$$f^{\mathrm{o}}(0,x) = f^{\mathrm{NS}}(x).$$

(Björk 2003): f has a finite dimensional representation (FDR) since there is a finite-dimensional manifold $\mathcal{G}$ such that

- $f^o \in \mathcal{G}$
- drift and volatility are in the tangent space of $\mathcal{G}$.

For our model $\mathcal{G} = span\{\bar{B}(x)\}$

- $\bar{B}(x) = (1, e^{-\lambda x}, xe^{-\lambda x}, x, x^2 e^{-\lambda x}, e^{-2\lambda x}, xe^{-2\lambda x}, x^2 e^{-2\lambda x})$

A method to construct the FDR

- f has an FDR given by $f(t,x) = \bar{B}(x).z(t)$ where

$$dz(t) = (Az(t) + b)\,dt + \Sigma\,dW(t), \quad z(0) = z_0.$$

- A, b , Σ and z_0 are determined from
  - $\bar{B}(x).z_0 = f^o(x)$
  - $\bar{B}(x).b = C(\sigma, x) = (B(x)\sigma)\int_0^x (B(s)\sigma)^T\,ds$
  - $\bar{B}(x)Az(t) = \frac{\partial f}{\partial x} = \frac{d\bar{B}(x)}{dx} z(t)$
  - $\bar{B}(x)\Sigma = B(x)\sigma$
- $B(x) = (1, e^{-\lambda x}, xe^{-\lambda x})$

- Method of proof: comparison of coefficients.
- Easily generalised to exponential–polynomial functions.

Our specific HJM model

Our specific HJM model has the following finite–dimensional representation:

$$f(t,x) = z_1(t) + z_2(t)e^{-\lambda x} + z_3(t)xe^{-\lambda x} + z_4(t)x + z_5(t)x^2e^{-\lambda x} + z_6(t)e^{-2\lambda x} + z_7(t)xe^{-\lambda x} + z_8(t)x^2e^{-2\lambda x}.$$

Interesting points:

- *Only* z_1, z_2 and z_3 are stochastic.
- A *specific* choice of initial curve will result in $z_4, \ldots, z_8$ being constant.
  (This is closely related with work by Christensen, Diebold and Rudebusch (2007) on extended NS curves).
- This model has counter–intuitive terms.

Our specific HJM model

How important is the Adjustment in the HJM model?

Previous approach: Statistical

- Coroneo, Nyholm, Vidova–Koleva (ECB working paper 2007).
- The estimated parameters of a NS model are not statistically different from those of an arbitrage–free model.

Our approach: Analytical

- We quantify the distance between forward curves,
- We analyse the differences in interest rate derivative prices.

Our Nelson-Siegel model: NS_{proj}

- $NS_{proj}(t,x) = \hat{z}_1(t) + \hat{z}_2(t)e^{-\lambda x} + \hat{z}_3(t)xe^{-\lambda x}$
- $\hat{z}(t) = (\hat{A}\hat{z}(t) + \hat{b})\,dt + \hat{\Sigma}\,dW(t)$, $z(0) = z_0$ where
    - $B(x).\hat{z}_0 = f^{NS}(x)$
    - $B(x).\hat{b} = \mathcal{P}[(B(x)\sigma)\int_0^x (B(s)\sigma)^T\,ds]$
    - $B(x)\hat{A}\hat{z}(t) = \frac{\partial f}{\partial x}^{\mathrm{NS_{proj}}}$
    - $B(x)\hat{\Sigma} = \mathcal{P}[B(x)\sigma]$

Projection formula: Projection of v onto Span (B_1, B_2, B_3):

$$\mathcal{P} : L^2 \to \text{Span } B(x) : v \mapsto \sum_{i=1}^{3}\sum_{j=1}^{3}(R^{-1})_{ij} < v, B_j > B_i(x),$$

$R_{ij} = \int B_i(s)B_j(s)\,ds$

Our Nelson-Siegel model: NS_{proj}

- $NS_{proj}(t,x) = \hat{z}_1(t) + \hat{z}_2(t)e^{-\lambda x} + \hat{z}_3(t)xe^{-\lambda x}$
- $\hat{z}(t) = (\hat{A}\hat{z}(t) + \hat{b})\,dt + \hat{\Sigma}\,dW(t)$, $z(0) = z_0$ where
  - $B(x).\hat{z}_0 = f^{NS}(x)$
  - $B(x).\hat{b} = \mathcal{P}[(B(x)\sigma)\int_0^x (B(s)\sigma)^T\,ds]$
  - $B(x)\hat{A}\hat{z}(t) = \frac{\partial f}{\partial x}^{\mathrm{NS_{proj}}}$
  - $B(x)\hat{\Sigma} = \mathcal{P}[B(x)\sigma]$

$$HJM = NS + Adj = NS_{proj} + Adj$$
$$Adj < Adj$$

- Same approach can be used for infinite–dimensional HJM.

Research contribution

§3b Applied Contribution: $HJM = NS + Adj = NS_{proj} + {}_{Adj}$

- ${}_{Adj} < Adj$, ${}_{Adj}$ is small.

An application

Recall the HJM model:

$$df(t,x) = \left(\frac{\partial f}{\partial x} + C(\sigma, x) \right) dt + \sigma_{11}\, dB_1(t)$$
$$+ (\sigma_{21} + \sigma_{22} e^{-\lambda x} + \sigma_{23} x e^{-\lambda x})\, dB_2(t),$$
$$f^o(0,x) = f^{NS}(x).$$

An application

Recall the HJM model:

$$df(t,x) = \left(\frac{\partial f}{\partial x} + C(\sigma, x)\right) dt + \sigma_{11}\, dB_1(t) \\ + (\sigma_{21} + \sigma_{22} e^{-\lambda x} + \sigma_{23} x e^{-\lambda x})\, dB_2(t),$$

$$f^o(0,x) = f^{NS}(x).$$

We can rewrite this model as:

$$dY(t,x) = \mu(t,x)\, dt + S_1(x)\, dB_1(s) + S_2(x)\, dB_2(s),$$

where $Y(t,x) = \log P(t,x)$,

$$S_1(x) = \sigma_{11} x,$$

$$S_2(x) = \frac{e^{-x\lambda}\left(-1 + e^{x\lambda}\right)(\lambda\sigma_{22} + \sigma_{23})}{\lambda^2} + x\left(\sigma_{21} - \frac{e^{-x\lambda}\sigma_{23}}{\lambda}\right).$$

HJM parameter estimation

Our data set consists of daily observations of depo/swap yields with maturities ranging from 3 months to 20 years.

HJM parameter estimation

Our data set consists of daily observations of depo/swap yields with maturities ranging from 3 months to 20 years.

To estimate the volatility we use **Principal Component Analysis (PCA)**.

HJM parameter estimation

Our data set consists of daily observations of depo/swap yields with maturities ranging from 3 months to 20 years.

To estimate the volatility we use **Principal Component Analysis (PCA)**.

$$\begin{pmatrix} \Delta Y(t,1) \\ \vdots \\ \Delta Y(t,20) \end{pmatrix} \simeq \begin{pmatrix} \mu_1(1) \\ \vdots \\ \mu_1(20) \end{pmatrix} + \begin{pmatrix} S_1(1) & S_2(1) \\ \vdots & \vdots \\ S_1(20) & S_2(20) \end{pmatrix} * \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

HJM parameter estimation

Our data set consists of daily observations of depo/swap yields with maturities ranging from 3 months to 20 years.

To estimate the volatility we use **Principal Component Analysis (PCA)**.

$$\begin{pmatrix} \Delta Y(t,1) \\ \vdots \\ \Delta Y(t,20) \end{pmatrix} \simeq \begin{pmatrix} \mu_1(1) \\ \vdots \\ \mu_1(20) \end{pmatrix} + \begin{pmatrix} S_1(1) & S_2(1) \\ \vdots & \vdots \\ S_1(20) & S_2(20) \end{pmatrix} * \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

By applying PCA to our data set

- we found that approximately 98% of the variance in the yields is captured by the **first two principal components**.

HJM parameter estimation

Our data set consists of daily observations of depo/swap yields with maturities ranging from 3 months to 20 years.

To estimate the volatility we use **Principal Component Analysis (PCA)**.

$$\begin{pmatrix} \Delta Y(t,1) \\ \vdots \\ \Delta Y(t,20) \end{pmatrix} \simeq \begin{pmatrix} \mu_1(1) \\ \vdots \\ \mu_1(20) \end{pmatrix} + \begin{pmatrix} S_1(1) & S_2(1) \\ \vdots & \vdots \\ S_1(20) & S_2(20) \end{pmatrix} * \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

By applying PCA to our data set

- we found that approximately 98% of the variance in the yields is captured by the **first two principal components**.
- we determined the volatility associated with each factor.

Parameter estimation

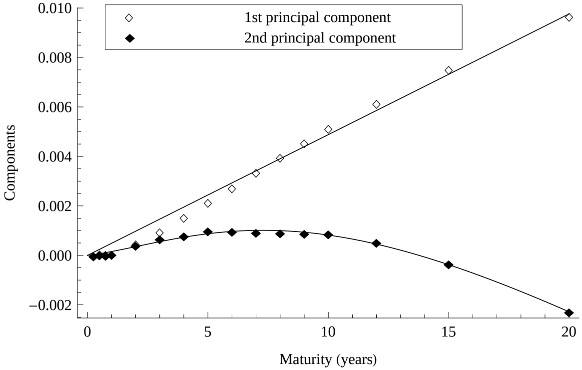

Figure: First and second principal component and fitted curves.

- First component fitted using $S_1(x) = \sigma_{11} x$.
- Second component fitted using

$$S_2(x) = \frac{e^{-x\lambda}\left(-1+e^{x\lambda}\right)(\lambda\sigma_{22}+\sigma_{23})}{\lambda^2} + x\left(\sigma_{21} - \frac{e^{-x\lambda}\sigma_{23}}{\lambda}\right).$$

Graphical analysis

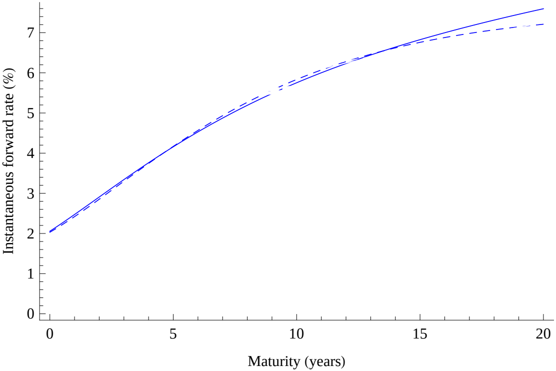

Figure: Some possible curve shapes generated by HJM and NS models after simulation for 5 years.

Graphical analysis

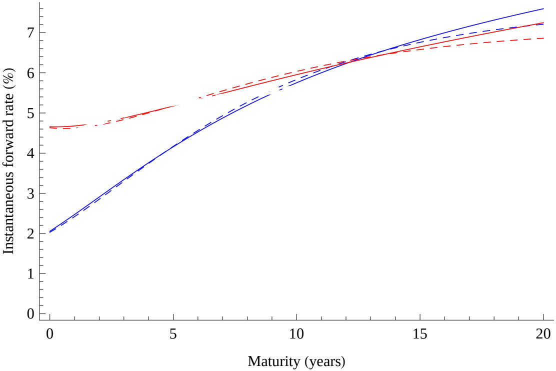

Figure: Some possible curve shapes generated by HJM and NS models after simulation for 5 years.

Graphical analysis

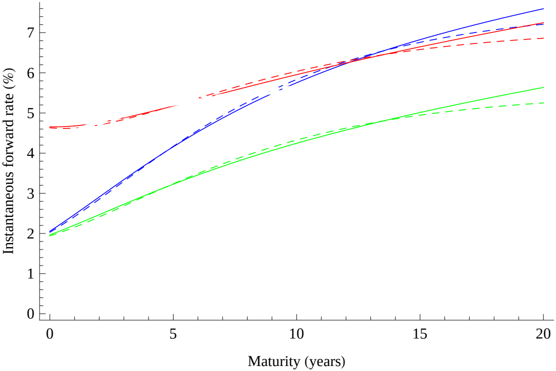

Figure: Some possible curve shapes generated by HJM and NS models after simulation for 5 years.

Graphical analysis

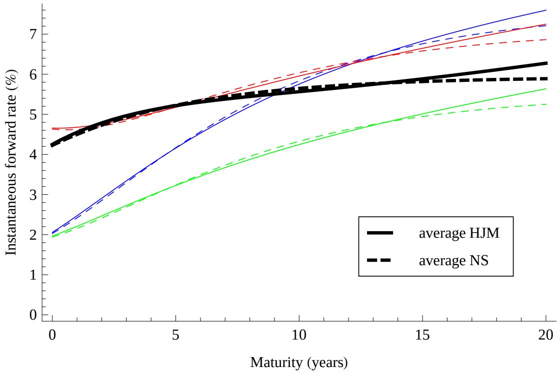

Figure: Some possible curve shapes generated by HJM and NS models after simulation for 5 years.

Note: The average curve for *any* future time can be calculated analytically at time 0.

Graphical analysis

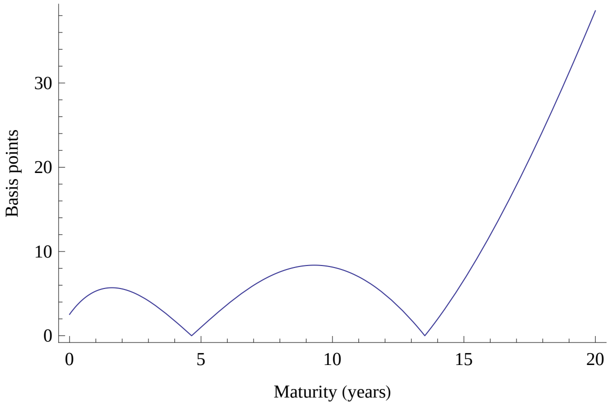

Figure: Difference in the curves after five years.

Note: This difference remains the same for each realisation.

Analysis of simulated prices

Theoretical 'European call option' prices on a 20 year bond:

T_0 (years)	5	10	15
Strike	0.565	0.686	0.865
$\Pi^{HJM}(T_0)$	0.0193	0.0146	0.00567
$\Pi^{NS_{proj}}(T_0)$	0.0192	0.0144	0.00562
% difference	0.47%	1.18%	0.89%

- T_0 denotes option maturity; Π denotes price.
- The strike is the at–the–money forward price of the bond $P(T_0, 20 - T_0)$.

Analysis of simulated prices

Theoretical 'Capped floating rate note' prices:

Cap	2%	3%	4%
Π^{HJM}	0.372	0.349	0.256
$\Pi^{NS_{proj}}$	0.371	0.256	0.163
% difference (of nominal)	0.045%	0.043%	0.033%

- Maturity of 20 years; nominal of 1; annual interest rate payment.
- Differences of 1–2% of nominal are common.

Case Studies

Case Study 1: Cap/Floor

- Nominal: EUR 180 million; Maturity: 30/6/2014.
- Receive capped and floored 3 month EURIBOR + spread:

Payout = (Nominal/4)*Max[0,Min[5%,ir+0.3875%]]

- **Valuation:**

Model	Valuation (EUR)	% difference (of nominal)
Numerix (1F HW)	$-2,045,140$	
HJM	$-2,085,124$	0.022%
NS_{proj}	$-2,085,449$	0.024%

Case Studies

Case Study 2: Curve Steepener

- Nominal: EUR $4,258,000$; Maturity: $30/5/2015$.
- Pay 'Curve steepener payoff' semi–annually:

Payout = (Nominal/2)*
Max["10 year swap"-"2 year swap",0]

- **Valuation:**

Model	Valuation (EUR)	% difference (of nominal)
Numerix (3F BGM)	$345,186$	
HJM	$295,401$	1.2%
NS_{proj}	$296,251$	1.15%

Contribution

1. $HJM = NS + Adj$

- initial curve affects shape of Adj, Adj contains counter–intuitive terms.

2. $HJM = NS_{proj} + \mathit{Adj}$, $\quad \mathit{Adj} < Adj$.

3 Simulation and Case Studies: $HJM \simeq NS_{proj}$

- for forward curve shapes, bond options, capped FRNs.
- Numerix (3F–BGM) $\approx$ 2F–HJM $\approx NS_{proj}$

Thank you

Research supported by:

- *STAREBEI* (Stages de Recherche á la BEI).
- *The Embark Initiative* operated by the Irish Research Council for Science, Technology and Engineering.
- *The Edgeworth Centre for Financial Mathematics*.

Disclaimer: This work expresses solely the views of the authors and does not necessarily represent the opinion of the ECB or EIB.