Affine diffusions with non-canonical state space

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Outline

1. Affine jump-diffusions
2. Affine transform formula
3. Existence of solutions to Riccati equations
4. Existence of exponential moments

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1 Affine jump-diffusions

2 Affine transform formula

3 Existence of solutions to Riccati equations

4 Existence of exponential moments

Semimartingale

X is (special) semimartingale with characteristics (B, C, ν) if

$$X_t = X_0 + B_t + X_t^c + \mathrm{id} * (\mu^X - \nu)_t$$

- B_t is drift of bounded variation
- X_t^c is continuous local martingale with $\langle X^c \rangle_t = C_t$
- μ^X is jump-measure, i.e. $\mu^X([0,t] \times A) = \sum_{s \le t} 1_A(\Delta X_s)$
- ν is compensator of μ^X

Jump-diffusion

X is jump-diffusion with local characteristics (b, c, K) if

- $\mathrm{d}B_t = b(X_t)\mathrm{d}t$
- $\mathrm{d}C_t = c(X_t)\mathrm{d}t$
- $\nu(\mathrm{d}t, \mathrm{d}z) = K(X_t, \mathrm{d}z)\mathrm{d}t$

Example (Lévy process)

- $b(x) = \mu$
- $c(x) = \sigma^2$
- $K(x, \mathrm{d}z) = \Pi(\mathrm{d}z)$

Then

$$X = X_0 + \mu t + \sigma W_t + \mathrm{id} * (\mu^X - \nu)_t$$

is Lévy process with Lévy-characteristics (μ, σ^2, Π).

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Affine jump-diffusion

Jump-diffusion X is affine with state space $\mathcal{X} \subset \mathbb{R}^p$ if we have

- Affine local characteristics: for $x \in \mathcal{X}$

$$b(x) = a^0 + \sum_{i=1}^{p} a^i x_i$$

$$c(x) = A^0 + \sum_{i=1}^{p} A^i x_i$$

$$K(x, \mathrm{d}z) = F^0(\mathrm{d}z) + \sum_{i=1}^{p} F^i(\mathrm{d}z) x_i$$

- Existence and uniqueness for all initial values $x \in \mathcal{X}$
- Stochastic invariance: $X_t \in \mathcal{X}$ for all $t \geq 0$ and all initial values $x \in \mathcal{X}$

Example affine diffusion

Example (One dimensional square root process)

- unique strong solution to SDE

$$\mathrm{d}X_t = (a^0 + aX_t)\mathrm{d}t + \sqrt{X_t}\mathrm{d}W_t, \quad X_0 \geq 0,$$

  with $a^0 \geq 0$.
- state space $\mathcal{X} = [0, \infty)$
- local characteristics

$$b(x) = a^0 + ax$$
$$c(x) = x$$
$$K(x, \mathrm{d}z) = 0$$

Canonical and other state spaces

- Canonical state space $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^{p-m}$
- Matrix-valued state space Sem^p
- Parabolic state space $\{x_1 \geq \sum_{i=2}^m x_i^2\}$
- Cone $\{x_1 \geq (\sum_{i=2}^m x_i^2)^{1/2}\}$

Conditions are needed for stochastic invariance and uniqueness

For continuous diffusions on canonical state space:

$$a_j^i \geq 0 \text{ for } i, j \leq m,\ i \neq j$$
$$a_i^0 \geq 0 \text{ for } i \leq m$$
$$A_{ij}^k = 0 \text{ for } i, j, k \leq m, \text{ unless } k = j = i$$
$$A^k = 0 \text{ for } k \geq m$$

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Affine transform formula for affine jump-diffusions

- Feynman-Kac formula gives (when applicable)

$$\mathrm{E}(\exp(u^\top X_T)|\mathcal{F}_t) = \exp(\psi_0(T-t,u) + \psi(T-t,u)^\top X_t)$$

- (ψ_0, ψ) solves generalized Riccati equations

$$\dot{\psi}_i = R_i(\psi), \quad \psi_i(0) = u_i$$

with

$$R_i(y) = y^\top a^i + \frac{1}{2} y^\top A^i y + \int_{\mathbb{R}^p} (e^{y^\top z} - 1 - y^\top z) F^i(\mathrm{d}z)$$

- Used for pricing bonds

$$\mathrm{E}(\exp(-\int_t^T r_s)|\mathcal{F}_t) \text{ with } r_s = \delta_0 + \delta^\top X_s$$

When is the affine transform formula valid?

For canonical state space $\mathcal{X} = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^{p-m}$ we have

$$\mathrm{E}(\exp(u^\top X_T)|\mathcal{F}_t) = \exp(\psi_0(T-t,u) + \psi(T-t,u)^\top X_t)$$

- [DFS03]: $u \in \mathbb{C}^m_{\leq 0} \times \mathrm{i}\mathbb{R}^{p-m}$
- [FM09]: $u \in \mathbb{C}^p$ s.t. either side exists, for continuous diffusions
- [KMK10]: $u \in \mathbb{R}^p$ s.t. right-hand side exists, under exponential moment conditions on jumps, e.g.

$$\int_{|z|\geq 1} e^{\psi^\top z} F^i(\mathrm{d}z) < \infty$$

Extending results

Theorem

For general convex state space $\mathcal{X}$ with $\mathcal{X}^\circ \neq \emptyset$ we have

$$\mathrm{E}(\exp(u^\top X_T)|\mathcal{F}_t) = \exp(\psi_0(T-t,u) + \psi(T-t,u)^\top X_t)$$

for $u \in \mathbb{C}^p$ s.t. either side exists under exponential moment conditions on jumps, e.g.

$$\int_{|z|\geq 1} e^{k^\top z} F^i(\mathrm{d}z) < \infty \quad \text{for all } k \in \mathbb{R}^p$$

Corollaries:

- $\mathrm{E}\exp(u^\top X_T) < \infty \Rightarrow \mathrm{E}\exp(u^\top X_t)$ for all $t \leq T$
- $\{u \in \mathbb{R}^p : \psi(T,u) \text{ exists}\}$ is convex for all T

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Exponential process

- Assume $\psi(T,u)$ exists, $u \in \mathbb{R}^p$
- Define $M_t := \exp(\psi_0(T-t,u) + \psi(T-t,u)^\top X_t)$
- Then $M_T = \exp(u^\top X_T)$ and

$$M_t = \mathrm{E}(\exp(u^\top X_T)|\mathcal{F}_t) \quad \text{iff} \quad M \text{ is martingale on } [0,T]$$

- Itô's formula yields

$$M = M_0 \mathcal{E}(\psi^\top \cdot X^c + (e^{\psi^\top z} - 1) * (\mu^X - \nu^X))$$

- $L = M/M_0$ is martingale on $[0,T]$ iff

$$\mathrm{E} L_T = 1$$

Change of measure

- Suppose L is martingale. Transform measure: $\mathrm{d}\mathbb{Q} = L_T \mathrm{d}\mathbb{P}$
  Then X is jump-diffusion under $\mathbb{Q}$ with characteristics

$$\begin{aligned}
\widetilde{b}(t,x) &= b(x) + c(x)\psi + \int z(e^{\psi^\top z} - 1)K(x,\mathrm{d}z) \\
\widetilde{c}(t,x) &= c(x) \\
\widetilde{K}(t,x,\mathrm{d}z) &= e^{\psi^\top z} K(x,\mathrm{d}z)
\end{aligned}$$

- Conversely, existence of this jump-diffusion implies L_t is martingale
- [KMK10] use time-inhomogeneous affine processes
- We use existence of solution to martingale problem with bounded coefficients + random time change

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From real to complex values

- Assume $U \subset \mathbb{R}^p$ open, non-empty and for $u \in U$

$$E(\exp(u^\top X_T)|\mathcal{F}_t) = \exp(\psi_0(T-t,u) + \psi(T-t,u)^\top X_t)$$

- Both sides analytic in u
- Uniqueness of holomorphic functions $\Rightarrow$ equality for $u \in U + \mathrm{i}\mathbb{R}^p$

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Outline of approach

- Assume $\mathrm{E}_x \exp(u^\top X_t) < \infty$, all $x \in \mathcal{X}$, some $u \in \mathbb{R}^p$
- To show: $\psi(t, u)$ exists
- Idea: let $D(t)$ be set of points v for which $\psi(t, v)$ exists
- $D(t)$ is open neighborhood of 0
- By previous

$$\mathrm{E}_x \exp(v^\top X_t) = \exp(\psi_0(t, v) + \psi(t, v)^\top x) \text{ for all } v \in D(t)$$

- Suppose $u \notin D(t)$
- To show: If v tends to $\partial D(t)$ then $\psi(t, v)$ explodes
- Then $\mathrm{E}_x \exp(v^\top X_t)$ also explodes and $\mathrm{E}_x \exp(u^\top X_t) = \infty$

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Example of non-explosion

- Consider Riccati ODE in $\mathbb{C}$

$$\dot{x} = x^2, \quad x(0) = u$$

- Solution $x(t, u) = u/(1 - ut)$ for $t < u^{-1}$ if $u \in \mathbb{R}_+$
- Domain of existence $D(t) = \mathbb{C} \backslash [t^{-1}, \infty)$ with boundary $\partial D(t) = [t^{-1}, \infty)$
- But $x(t, u)$ does not explode if $u \to (t^{-1}, \infty)$
- Corresponds with Riccati ODE in $\mathbb{R}^2$

$$\dot{x}_1 = x_1^2 - x_2^2$$
$$\dot{x}_2 = 2x_1x_2$$

- Corresponding diffusion matrix

$$c(x) = \begin{pmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{pmatrix} \text{ is positive semi-definite iff } x = 0$$

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Riccati equations revisited

System of Riccati equations

$$\dot{\psi}_i = R_i(\psi), \quad \psi_i(0) = u_i$$

with

$$R_i(y) = y^\top a^i + \frac{1}{2} y^\top A^i y + \int_{\mathbb{R}^p} (e^{y^\top z} - 1 - y^\top z) F^i(\mathrm{d}z)$$

is considered as inhomogeneous linear ODE

Variation of constants

Variation of constants yields

$$\psi_0(t) + \psi(t)^\top x = u^\top y_t + \int_0^t \Big(\tfrac{1}{2}\psi(s)^\top c(y_{t-s})\psi(s) + \int_{z\in\mathbb{R}^p} (e^{\psi(s)^\top z} - 1 - \psi(s)^\top z) K(y_{t-s}, \mathrm{d}z) \Big) \mathrm{d}s,$$

with $y_t = \mathrm{E}_x X_t$ and solves

$$\dot{y} = b(y), \quad y(0) = x$$

Gives enough interplay to handle explosions

Summary

For affine jump-diffusions with general convex state space we verified affine transform formula

$$E(\exp(u^\top X_T)|\mathcal{F}_t) = \exp(\psi_0(T-t,u) + \psi(T-t,u)^\top X_t)$$

for $u \in \mathbb{C}^p$ s.t. either side exists.

Remarks:

- Need exponential moments

$$\int_{|z|\geq 1} e^{k^\top z} F^i(\mathrm{d}z) < \infty \quad \text{for all } k \in \mathbb{R}^p$$

- No explicit use of geometry of state space

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