Optimal Stopping with Prospect Preference

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Based on joint work with Xun Yu Zhou

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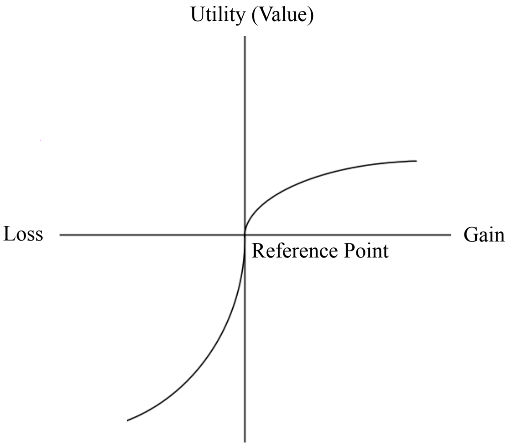
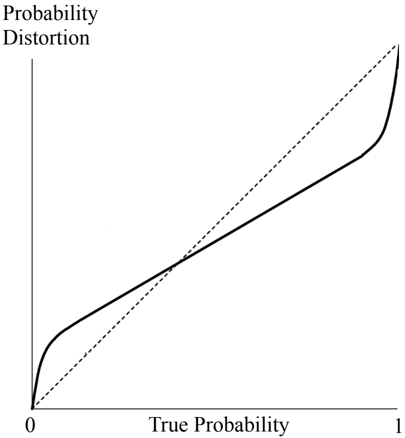
  - Distorted probabilities instead of the true probabilities

S-shaped Utility Function



Probability Distortion Function



Optimal Stopping with Prospect Preference

$$\sup_{\tau\in\mathcal{T}} J(\tau) =: \int_0^\infty w(\mathbf{P}(U(P_\tau) > x))\,\mathrm{d}x$$

- *$\mathcal{T}$ is the set of finite stopping times*
- *The asset P follows $\mathrm{d}P_t = (\mu - r)P_t\,\mathrm{d}t + \sigma P_t\,\mathrm{d}B_t$*
- *Both the utility function U and the probability distortion w are nonnegative increasing functions.*

Special Case (No Probability Distortion)

$$J(\tau) = \int_0^\infty \mathbf{P}(U(P_\tau) > x)\,\mathrm{d}x = \mathbf{E}\left[U(P_\tau)\right]$$

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  - Martingale: optional sampling theorem, change of time, change of measure
  - Markovian: dynamic programming and HJB equation (variational inequality)

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  - Take distribution/quantile of P_τ as decision variable
  - Instead of finding the optimal selling time, find the distribution of the optimal selling price
  - How to recover τ^* from the distribution of S_{τ^*}? Skorokhod embedding! A well studied and challenging probabilistic problem

Introduce Martingale

Problem (Main Problem)

$$\begin{aligned}\sup_{\tau\in\mathcal{T}} J(\tau) &= \sup_{\tau\in\mathcal{T}} \int_0^\infty w(\mathbf{P}(U(P_\tau) > x))\,\mathrm{d}x \\ &= \sup_{\tau\in\mathcal{T}} \int_0^\infty w(\mathbf{P}(u(S_\tau) > x))\,\mathrm{d}x,\end{aligned}$$

where $\beta = 1 - \frac{2(\mu-r)}{\sigma^2}$, $\quad S_t = P_t^\beta$, $\quad u(x) = U(x^{1/\beta})$.

Remark

The new asset S *follows* $\mathrm{d}S_t = \sigma\beta S_t\,\mathrm{d}B_t$, *a martingale!*
The asset P *follows* $\mathrm{d}P_t = (\mu - r)P_t\,\mathrm{d}t + \sigma P_t\,\mathrm{d}B_t$

An Example of Utility Function Transformation

Log utility function $U(x) = \ln(x+1)$, $u(x) = \ln(x^{1/\beta}+1)$

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- A "not so bad" asset, $0 < \mu - r < \sigma^2/2$: $u(x)$ is a strictly increasing S-shaped function
- A "bad" asset, $\mu \leqslant r$: $u(x)$ is a strictly increasing concave function

Distribution/Quantile Formulation

Distribution formulation:

$$\begin{aligned} J(\tau) &= \int_0^\infty w(\mathbf{P}(u(S_\tau) > x))\,\mathrm{d}x \\ &= \int_0^\infty w(\mathbf{P}(u(S_\tau) > u(y)))\,\mathrm{d}u(y) \\ &= \int_0^\infty w(\mathbf{P}(S_\tau > y))u'(y)\,\mathrm{d}y \\ &= \int_0^\infty w\left(1 - F(y)\right)u'(y)\,\mathrm{d}y \\ &=: J_D(F), \end{aligned}$$

where F is in $\mathcal{D}$, the distribution set of S_τ.

Distribution/Quantile Formulation (Cont'd)

Quantile formulation:

$$\begin{aligned} J(\tau) &= \int_0^\infty w\left(1 - F(y)\right) u'(y)\, \mathrm{d}y \\ &= \int_0^\infty u(y)\, \mathrm{d}w\left(1 - F(y)\right) \\ &= \int_0^\infty u(y) w'\left(1 - F(y)\right) \mathrm{d}F(y) \\ &= \int_0^1 u\left(G(x)\right) w'(1 - x)\, \mathrm{d}x \\ &=: J_Q(G), \end{aligned}$$

where G is in $\mathcal{G}$, the quantile set of S_τ.

Distribution/Quantile Set

- What is the distribution/quantile set of S_τ?

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  - Skorokhod embedding problem (Skorokhod 1961):

    for a given distribution m and a Brownian motion B, one looks for a stopping time τ such that the distribution of B_τ is m.

Distribution/Quantile Set (Cont'd)

Lemma

Equivalent expressions for the distribution and quantile sets:

$$\begin{aligned} \mathcal{D} &= \left\{ F \;\middle|\; \int_0^\infty (1 - F(x))\,\mathrm{d}x \leqslant S_0,\; F(0) = 0 \right\}, \\ \mathcal{G} &= \left\{ G \;\middle|\; \int_0^1 G(x)dx \leqslant S_0, \quad G > 0 \right\}. \end{aligned}$$

Equivalent Main Problems

- Distribution formulation:

$$\begin{aligned} \sup_{F\in\mathcal{D}} J_D(F) \quad &=: \quad \int_0^\infty w\left(1-F(x)\right) u'(x)\,\mathrm{d}x, \\ \text{s.t.} \quad & \quad \int_0^\infty (1-F(x))\,\mathrm{d}x \leqslant S_0,\ F(0)=0. \end{aligned}$$

- Quantile formulation:

$$\begin{aligned} \sup_{G\in\mathcal{G}} J_Q(G) \quad &=: \quad \int_0^1 u\left(G(x)\right) w'(1-x)\,\mathrm{d}x, \\ \text{s.t.} \quad & \quad \int_0^1 G(x)dx \leqslant S_0, \quad G>0. \end{aligned}$$

Increasing Convex Utility/Distribution

Theorem

In the main problem, if the utility function u or the probability distortion function w is convex, then the optimal solution takes the form of

$$G(x) = a\mathbf{1}_{(0,c]}(x) + b\mathbf{1}_{(c,1)}(x).$$

Remark

Stop loss and stop gain strategy

Increasing Convex Utility/Distribution (Cont'd)

Remark

Idea: maximize a convex function on a convex set

Special Case

There is no distortion

Increasing Concave Utility + Concave Distortion

Theorem

In the main problem, if both the utility function u and the probability distortion function w are concave, then the optimal solution takes the form of

$$G(x) = (u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right).$$

Remark

Idea: Lagrange method,

$$J_Q^{\lambda}(G) = \int_0^1 \left[u\left(G(x)\right) w'(1-x) - \lambda(G(x) - S_0)\right] \mathrm{d}x,$$

then pointwise maximize the above objective

Increasing Concave Utility + Anti S-shaped Distortion

Theorem

In the main problem, if the utility function u is concave and the probability distortion function w is anti S-shaped, then the optimal solution takes the form of

$$G(x) = a\mathbf{1}_{(0,c)}(x) + (u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\mathbf{1}_{(c,1)}(x).$$

Increasing Concave Utility + Anti S-shaped Distortion (Cont'd)

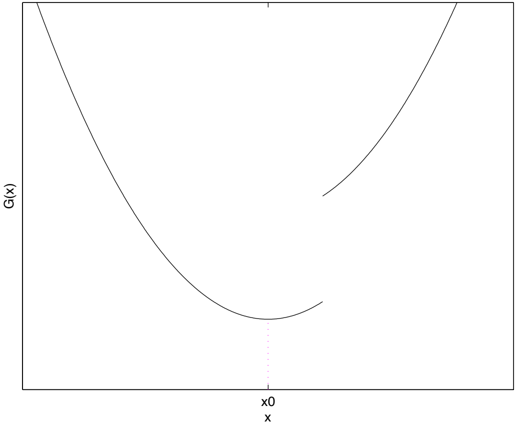
Remark

Idea: using Lagrange method,

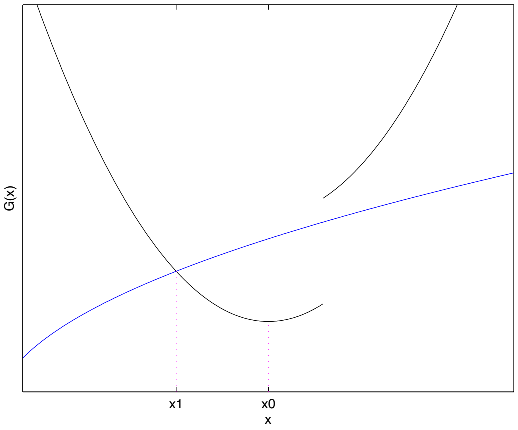
$$J_Q^{\lambda}(G) = \int_0^1 \left[u\left(G(x)\right) w'(1-x) - \lambda(G(x) - S_0)\right] \mathrm{d}x,$$

then do pointwise maximization. But the pointwise solution is not an increasing function! Modify the decreasing part

Method (Cont'd)



Method (Cont'd)



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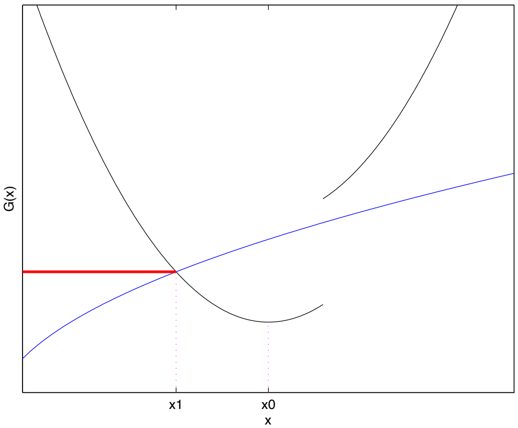

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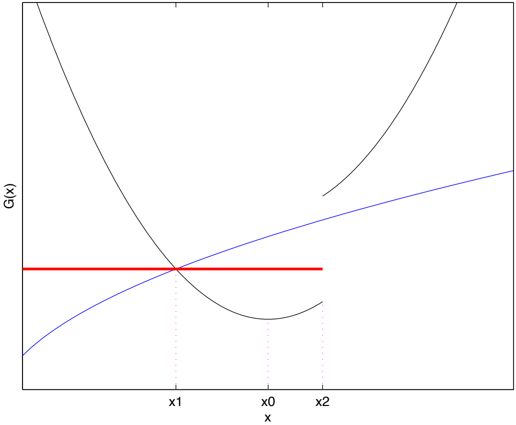

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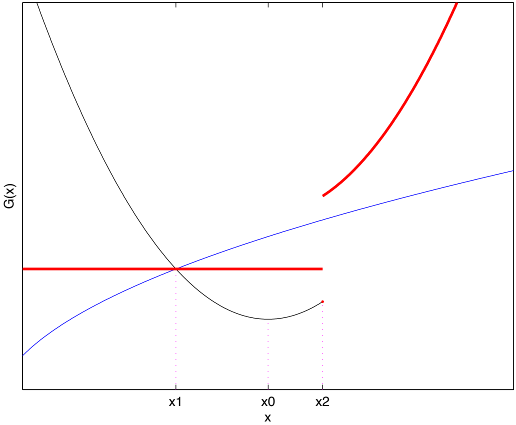

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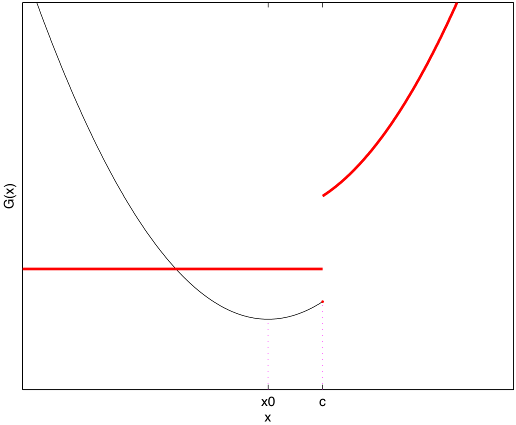


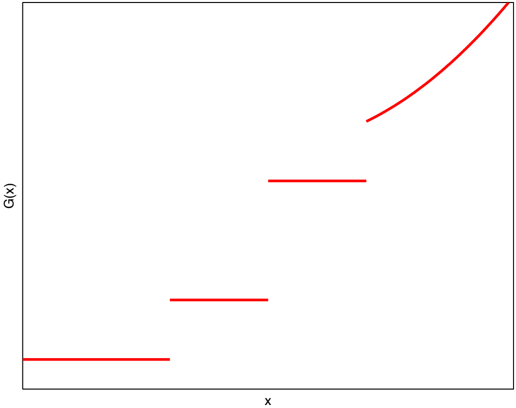
S-shaped Utility + anti S-shaped Distortion

Theorem

In the main problem, if the utility function u is S-shaped and the probability distortion function w is anti S-shaped, then the optimal solution takes the form of

$$\begin{aligned} G(x) &= a_1 \mathbf{1}_{(0,c_1]}(x) + a_2 \mathbf{1}_{(c_1,c_2]}(x) \\ &\quad + \left(a_3 \vee (u')_l^{-1} \left(\frac{\lambda}{w'(1-x)} \right) \right) \mathbf{1}_{(c_3,1)}(x). \end{aligned}$$

Solution (Cont'd)



Recover the Optimal Stopping Time

- We have got the distributions of the optimal selling prices in various cases. We can use the distributions to recover the optimal stopping times by using Skorokhod technique again.

Economic Interpretation

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- Buy and hold strategy (good asset case)

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- More general strategies

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- Mathematical difficulties arising form the probability distortion
- Distribution/Quantile formulation and Skorokhod embedding
- Economically sensible strategies derived

Possible Extensions

- Finite time horizon

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- Discounting factor involved

Thank you!