On a Heath-Jarrow-Morton approach for stock markets

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Introduction

- Consider a market with a canonical reference asset and some derivatives based on it.
- If we model the canonical reference asset in detail under the martingale measure then the prices of the derivatives are given by conditional expectation.
- If the derivatives are traded liquidly then the model prices may contradict the prices observed on the real market.
- First way out: Calibration
- 2nd way out: Model the derivatives directly
- Some references: Schönbucher (1999), Jacod & Protter (2006), Schweizer & Wissel (2008, 2009), Carmona (2007), Carmona & Nadtochiy (2009)

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The philosophy behind HJM

Summary of the situation

- There is a cononical reference asset
- There are some derivtaives based on it
- We want to model the derivatives
- It is hard to model the derivatives direclty, because they have complicated dependencies on each other
- Find a reparametrisation to get rid of the complicated dependencies (codebook)
- Model the codebook-process directly, such that all the derivatives are martingales under the martingale measure. (no arbitrage condition)
- How does the canonical reference asset look like? (consistency condition)

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How to find a good reparemtrisation? – the simple models

- Choose a class of models (simple models) such that
  - The simple models do not allow for arbitrage.
  - The class has a simple parameter space $\mathcal{C}$
  - There is a (simple) one to one function Φ which mapes a given parameter and a given price of the underlying to the price of the derivative.
- Forget about the simple models but keep the parametrisation Φ and use the parameter space $\mathcal{C}$ as the codebook space.

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Example I: Heath et. al (1992)

- Canonical reference asset: Money market account

$$dK(t) = K(t)r(t)dt$$

- Liquid derivatives: Bonds $B(t,T)$ with $B(T,T) = 1$.
- Simple model: $dK(t) = K(t)r(t)dt$ with deterministic short rate $r(t)$
  - In this setup: $B(t,T) = \exp(-\int_t^T r(s)ds)$
  - Conversely: $r(T) = -\partial_T \log(B(t,T))$
- This inspires the codebook $f(t,T) = -\partial_T \log(B(t,T))$
- Model the dynamics $df(t,T) = \alpha(t,T)dt + \beta(t,T)dW(t)$
- Drift condition: $\alpha(t,T) = \beta(t,T) \int_0^T \beta(t,s)ds$
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Example II: Carmona & Nadtochiy (2009)

- Canonical reference asset: Stock $S(t)$
- Liquid derivatives: European Calls $C(t, T, K)$ with maturity T and strike K.
- Simple model: Purely discontinuous time-inhomogenous exponential Lévy processes
  - In this setup: $C(t, T, K)$ can be obtained from the Lévy density $K(t, u)$ via a function Φ.
  - Conversely: $K(t, u)$ can be obtained from the call option prices.
- This inspires the codebook $X(t, T, u) = \Phi^{-1}(C(t, T, K))$
- Model the dynamics $X(t, T, u) = \alpha(t, T, u)dt + \beta(t, T, u)dW(t)$
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Setting Lévy in motion.

- Canonical reference asset: Stock $S(t) = \exp(X(t))$ with return X.
- Liquid derivatives: Calls $C(t, T, K)$ with maturity T and strike K.

- Simple model: exponential time-inhomogenous Lévy processes

$$E(e^{iuX_t}) = \exp\left(\int_0^t \psi^X(s,u)ds\right)$$

where $\psi^X(s,u)$ is given by a generalised Lévy-Khintchine formula.

  - In this setup: $C(t,T,K)$ can be obtained by Fourier technics from ψ^X by a formula Φ^{-1} which can be found in (Belomestny.Reiss 99)

$$\begin{aligned} \mathcal{O}(t,T,x) &= \mathcal{F}\left\{u \to \frac{1-\exp\left(\int_0^t \psi^X(s,u)ds\right)}{u^2+iu}\right\}(x), \\ C(t,T,K) &= (S(t)-K)^+ + K\mathcal{O}\left(t,T,\log\left(\frac{K}{S(t)}\right)\right). \end{aligned}$$

  - Conversely: $\psi^X(T,u)$ can be obtained from the call option prices

$$\begin{aligned} \mathcal{O}(t,T,x) &:= e^{-(x+X(t))}C\left(t,T,e^{x+X(t)}\right) - (e^{-x}-1)^+ \\ \psi^X(T,u) &= \partial_T \log\left(1-(u^2+iu)\mathcal{F}\{x \to \mathcal{O}(t,T,x)\}(u)\right) \end{aligned}$$

- This inspires the codebook $\Psi(t, T, u) := \Phi^{-1}(K \mapsto C(t, T, K))(u)$
- Model the dynamics $\Psi(t, T, u) = \alpha(t, T, u)dt + \beta(t, T, u)dL(t)$
- Drift condition: $\alpha(t, T, u) = \partial_T \psi^L \left(\int_t^T \beta(t, r, u) dr \right)$
- Consistency condition: $\psi^X(t, u) = \Psi(t, t, u)$

An example

A deterministic example

- We consider the situation: $S(t) = exp(X(t))$
- $d\Psi(t, T, u) = \alpha(t, T, u)dt + \beta(t, T, u)dL(t)$ for an increasing Lévy process L
- $\beta(t, T, u) = \frac{u^2 - iu}{2} e^{\lambda(T-t)}$ for some $\lambda \in \mathbb{R}_+$
- The conditions imply

$$\begin{aligned} dX(t) &= dM(t) - v(t)dt + \sqrt{v(t)}dW(t) \\ dv(t) &= -\lambda v(t)dt + dL(t) \end{aligned}$$

for some time inhomogeneous Lévy process M.
It is some kind of Barndorff-Nielsen & Shephard (2001) stochastic volatility model.

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