GARCH Intensity Models for Asset Price and Their Application to Option Valuation

Geon Ho Choe   Kyungsub Lee

Department of Mathematical Sciences
KAIST

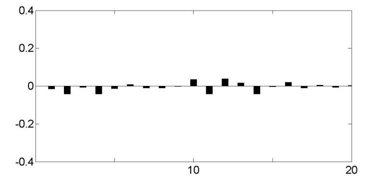
26 June 2010,
6th World Congress of the Bachelier Finance Society,
Toronto

Outline

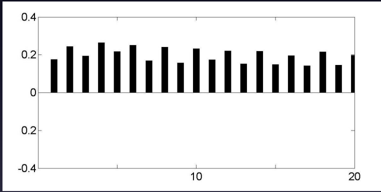
- Empirical studies
- Introduction to intensity model
- GARCH for intensity
- Estimation result
- Conclusion

S&P 500 return series (1990 - 2009)

- Absence of serial correlation (left).
-



$\mathrm{Corr}(X_t, X_{t-\ell})$

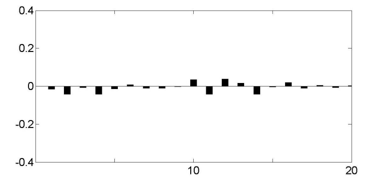


$\mathrm{Corr}(|X_t|, |X_{t-\ell}|)$

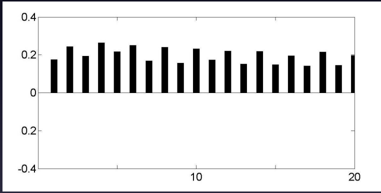
X_t : the log-return at t.

S&P 500 return series (1990 - 2009)

- Absence of serial correlation (left).
- Volatility clustering (right).



$\mathrm{Corr}(X_t, X_{t-\ell})$

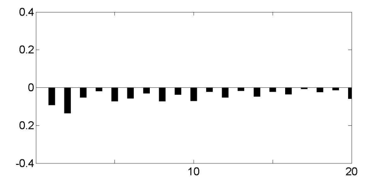


$\mathrm{Corr}(|X_t|, |X_{t-\ell}|)$

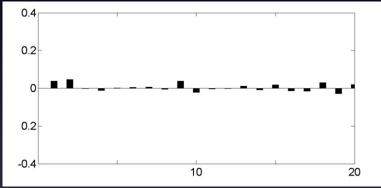
X_t : the log-return at t.

Leverage effect

- Negative response of $|X_t|$ to $X_{t-\ell}$, $\ell > 0$ (left).
-



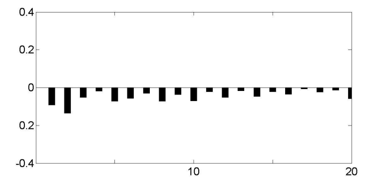
$\mathrm{Corr}(|X_t|, X_{t-\ell})$



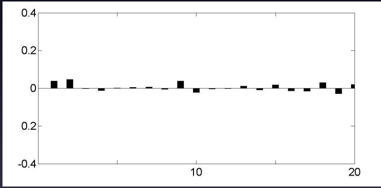
$\mathrm{Corr}(X_t, |X_{t-\ell}|)$

Leverage effect

- Negative response of $|X_t|$ to $X_{t-\ell}$, $\ell > 0$ (left).
- On the other hand, $\mathrm{Corr}(X_t, |X_{t-\ell}|)$ is negligible (right).



$\mathrm{Corr}(|X_t|, X_{t-\ell})$



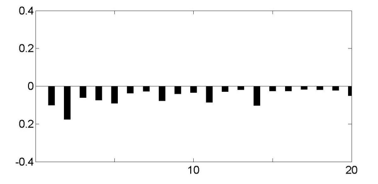
$\mathrm{Corr}(X_t, |X_{t-\ell}|)$

Return dependence

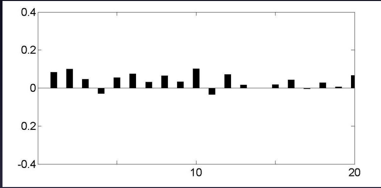
Even though $\mathrm{Corr}(X_t, X_{t-\ell})$ and $\mathrm{Corr}(X_t, |X_{t-\ell}|)$, for $\ell > 0$ are insignificant, we have non-negligible correlation between current return and past (magnitude of) return depending on the condition of $\mathrm{sign}(X_t)$.

Conditional correlation

Leverage effect captured by correlation on the condition of current return's sign :



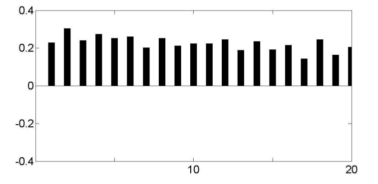
$\mathrm{Corr}(X_t, X_{t-\ell}|X_t > 0)$



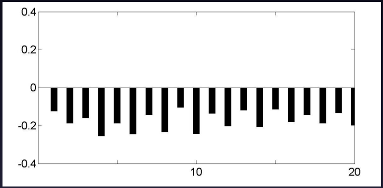
$\mathrm{Corr}(X_t, X_{t-\ell}|X_t < 0)$

Conditional correlation(2)

On the condition of current return's sign :



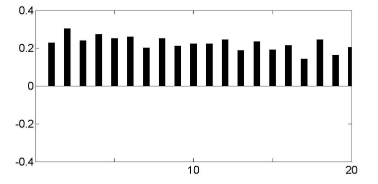
$\mathrm{Corr}(X_t, |X_{t-\ell}|\ |X_t > 0)$



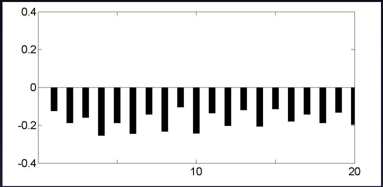
$\mathrm{Corr}(X_t, |X_{t-\ell}|\ |X_t < 0)$

Conditional correlation(2)

On the condition of current return's sign :



$\mathrm{Corr}(X_t, |X_{t-\ell}| \, |X_t > 0)$



$\mathrm{Corr}(X_t, |X_{t-\ell}| \, |X_t < 0)$

The price is less affected by the previous information $|X_{t-\ell}|$ when the price decreases than the case when the price increases.

Intensity model

The asset price $S(t)$ satisfies

$$S(t) = S(0) \exp \{\delta(N_+(t) - N_-(t))\}$$

for some constant $\delta > 0$.

Intensity model

The asset price $S(t)$ satisfies

$$S(t) = S(0) \exp \{\delta(N_+(t) - N_-(t))\}$$

for some constant $\delta > 0$.

- $(N_\pm(t) - N_\pm(t_{i-1})) \,|\mathcal{F}(t_{i-1}) \sim \mathrm{Poisson}(\lambda_\pm(t_{i-1})(t - t_{i-1}))$, $t_{i-1} \le t \le t_i$.

Intensity model

The asset price $S(t)$ satisfies

$$S(t) = S(0) \exp \{\delta(N_+(t) - N_-(t))\}$$

for some constant $\delta > 0$.

- $(N_\pm(t) - N_\pm(t_{i-1})) \,|\mathcal{F}(t_{i-1}) \sim \text{Poisson}(\lambda_\pm(t_{i-1})(t - t_{i-1}))$, $t_{i-1} \le t \le t_i$.
- $\lambda_+(t) = \lambda_+(t_{i-1})$ and $\lambda_-(t) = \lambda_-(t_{i-1})$, $t_{i-1} \le t < t_i$.

Intensity model

The asset price $S(t)$ satisfies

$$S(t) = S(0) \exp\{\delta(N_+(t) - N_-(t))\}$$

for some constant $\delta > 0$.

- $(N_\pm(t) - N_\pm(t_{i-1}))\,|\mathcal{F}(t_{i-1}) \sim \text{Poisson}(\lambda_\pm(t_{i-1})(t - t_{i-1}))$, $t_{i-1} \leq t \leq t_i$.
- $\lambda_+(t) = \lambda_+(t_{i-1})$ and $\lambda_-(t) = \lambda_-(t_{i-1})$, $t_{i-1} \leq t < t_i$.
- $N_+(t) - N_+(t_{i-1})$ and $N_-(t) - N_-(t_{i-1})$ are conditionally independent with given $\mathcal{F}(t_{i-1})$, $t_{i-1} \leq t \leq t_i$.

Conditional variance

The conditional variance of log return $X(t_i)$ is given by a linear combination of intensities. More precisely,

$$\mathrm{Var}(X(t_i)|\mathcal{F}(t_{i-1})) = \delta^2(\lambda_+(t_{i-1}) + \lambda_-(t_{i-1})).$$

Drift, correction factor and shock

Definition (Decomposition of Log-Return)

Define μ(drift), γ(mean correction), ε(shock) by

$$\begin{aligned}
\mu(t_i) &= \{(e^{\delta}-1)\lambda_+(t_{i-1}) + (e^{-\delta}-1)\lambda_-(t_{i-1})\}\Delta t \\
\gamma(t_i) &= \{(e^{\delta}-1-\delta)\lambda_+(t_{i-1}) + (e^{-\delta}-1+\delta)\lambda_-(t_{i-1})\}\Delta t \\
\varepsilon(t_i) &= X(t_i) - \mathbb{E}[X(t_i)|\mathcal{F}(t_{i-1})].
\end{aligned}$$

Then

$$X(t_i) = \mu(t_i) - \gamma(t_i) + \varepsilon(t_i).$$

Equivalent martingale measure

Definition (Radon–Nikodym derivative)

Take $\widetilde{\lambda}_+$ and $\widetilde{\lambda}_-$ such that

$$(e^{\delta}-1)\widetilde{\lambda}_+(t)+(e^{-\delta}-1)\widetilde{\lambda}_-(t)=r$$

and let

$$\begin{aligned} Z(T)=\exp\sum_{i=1}^{N}\Big\{ &\Big(\lambda_+(t)+\lambda_-(t)-\widetilde{\lambda}_+(t)-\widetilde{\lambda}_-(t)\Big)\Delta t \\ &+(N_+(t_i)-N_+(t_{i-1}))\log\frac{\widetilde{\lambda}_+(t_{i-1})}{\lambda_+(t_{i-1})} \\ &+(N_-(t_i)-N_-(t_{i-1}))\log\frac{\widetilde{\lambda}_-(t_{i-1})}{\lambda_-(t_{i-1})}\Big\}. \end{aligned}$$

Measure change

$\mathbb{Q}(A) = \int_A Z(T) \mathrm{d}\mathbb{P}.$

	$\mathbb{P}$	$\mathbb{Q}$
Intensities	$\lambda_+(t_i)$	$\tilde{\lambda}_+(t_i)$
	$\lambda_-(t_i)$	$\tilde{\lambda}_-(t_i)$
Drift	$\mu(t_i)$	$r\Delta t$
Shock	$\varepsilon(t_i)$	$\widetilde{\varepsilon}(t_i)$

Heteroscedasticity

Remark

Asset price(return) movements model using autoregressive heteroscedasticity.

$$\begin{aligned} h(t_i) &= \mathrm{Var}(X(t_i)|\mathcal{F}(t_{i-1})) \\ &= \omega + \beta h(t_{i-1}) + \alpha\varepsilon^2(t_{i-1}) \end{aligned}$$

$h(t_i)$: conditional variance
$\varepsilon(t_i)$: innovation
$\{\omega, \beta, \alpha\}$: parameters.

Autoregressive for intensity

GARCH assumption for intensities:

$$\lambda_+(t_i) = \omega_+ + \beta_+ h(t_{i-1}) + \alpha_+ \varepsilon^2(t_i)$$

$$\lambda_-(t_i) = \omega_- + \beta_- h(t_{i-1}) + \alpha_- \varepsilon^2(t_i)$$

implies

$$h(t_i) = \omega^* + \beta h(t_{i-1}) + \alpha^* \varepsilon^2(t_{i-1})$$

if $\beta_+ = \beta_-$.

Maximum likelihood estimation

The joint distribution of $X_1, \ldots X_n$ with a parameter set θ is given by

$$
\begin{aligned}
f_\theta(x_1, \ldots, x_n | \lambda_\pm(t_0)) =& f_\theta(x_1|\lambda_\pm(t_0)) f_\theta(x_2|\lambda_\pm(t_1)) \\
& \times \cdots \times f_\theta(x_n|\lambda_\pm(t_{n-1}))
\end{aligned}
$$

where

$$
\begin{aligned}
f_\theta(x_i|\lambda_\pm(t_{i-1})) = \exp\{-\lambda_+(t_{i-1}) - \lambda_-(t_{i-1})\} \left(\frac{\lambda_+(t_{i-1})}{\lambda_-(t_{i-1})}\right)^{x_i/2\delta} \\
\times I_{|x_i/\delta|}(2\sqrt{\lambda_+(t_{i-1})\lambda_-(t_{i-1})}).
\end{aligned}
$$

Goal : Find θ maximizing $f_\theta(x_1, \ldots, x_n|\lambda_\pm(t_0))$.

Intensities for estimation

GJR GARCH :

$$\lambda_{\pm}(t_i) = \omega_{\pm} + \beta_{\pm}\lambda_{\pm}(t_{i-1}) + (\alpha_{\pm} + \gamma_{\pm}I(t_i))\varepsilon^2(t_i)$$

$$\lambda_{\pm}(t_i) = \omega_{\pm} + \beta_{\pm}\lambda_{\pm}(t_{i-1}) + (\alpha_{\pm} + \gamma_{\pm}I(t_i))\varepsilon^2(t_i)$$

where

$$I(t_i) = \begin{cases} 1, & \varepsilon(t_i) < 0 \\ 0, & \varepsilon(t_i) \geq 0. \end{cases}$$

Estimates

Note that asset price and intensities are

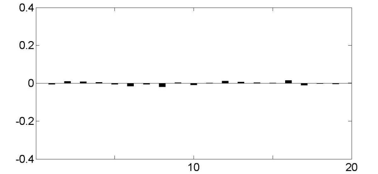
$$S(t) = S(0)\exp\left\{\delta(N_+(t) - N_-(t))\right\},$$

$$\lambda_+(t_i) = \omega_+ + \beta_+\lambda_+(t_{i-1}) + (\alpha_+ + \gamma_+ I(t_i))\varepsilon^2(t_i),$$
$$\lambda_-(t_i) = \omega_- + \beta_-\lambda_-(t_{i-1}) + (\alpha_- + \gamma_- I(t_i))\varepsilon^2(t_i).$$

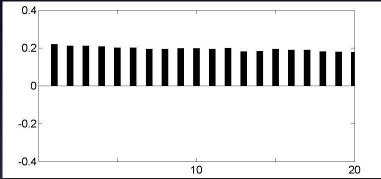
$\delta = 2.0 \times 10^{-3}$			
ω_+	8.50×10^{-2}	ω_-	7.28×10^{-2}
β_+	9.39×10^{-1}	β_-	9.42×10^{-1}
α_+	9.79×10^{2}	α_-	8.49×10^{2}
γ_+	1.09×10^{4}	γ_-	1.07×10^{4}

Simulation – GJR GARCH intensity

Absence of serial correlation (left).
Volatility clustering (right).



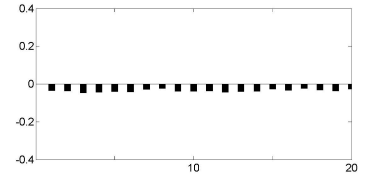
$\mathrm{Corr}(X_t, X_{t-\ell})$



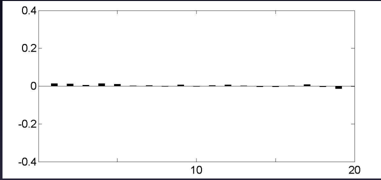
$\mathrm{Corr}(|X_t|, |X_{t-\ell}|)$

Leverage effect

Negative response of $|X_t|$ to $X_{t-\ell}$, $\ell > 0$ (left).
On the other hand, $\mathrm{Corr}(X_t, |X_{t-\ell}|)$ is negligible (right).



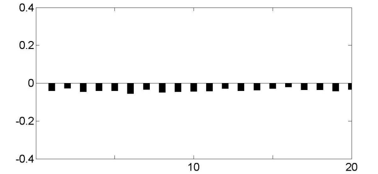
$\mathrm{Corr}(|X_t|, X_{t-\ell})$



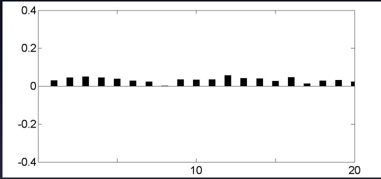
$\mathrm{Corr}(X_t, |X_{t-\ell}|)$

Conditional correlation

Leverage effect captured by correlation on the condition of current return's sign :



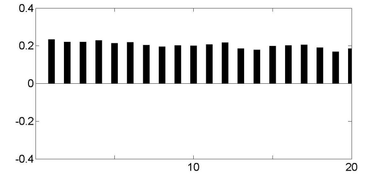
$\mathrm{Corr}(X_t, X_{t-\ell} | X_t > 0)$



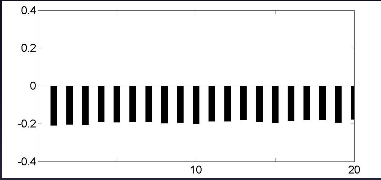
$\mathrm{Corr}(X_t, X_{t-\ell} | X_t < 0)$

Conditional correlation(2)

On the condition of current return's sign :



$\mathrm{Corr}(X_t, |X_{t-\ell}| \mid X_t > 0)$



$\mathrm{Corr}(X_t, |X_{t-\ell}| \mid X_t < 0)$

Concluding remark

- Conditional asymmetries of stock returns responding to the past information.
- Poisson intensity model as a new approach for describing asset returns.
- Linkage between GARCH and intensity model.
- Issues on measure changes and martingale methods for derivative pricing.
- Estimation results and conditional asymmetries.

Conclusion

Thank you!