A Delay Financial Model with Stochastic Volatility; Martingale Method

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Astract

We extend a delayed geometric Brownian model by adding the stochastic volatility which is assumed to have fast mean reversion. By the martingale approach and singular perturbation method, we develop a theory for option pricing under this extended model.

Keywords: Black-Scholes, delay, stochastic volatility, martingale, option pricing, asymptotics.

The assumptions of Black-Scholes model

The assumptions of Black-Scholes model for equity market

- It is possible to borrow and lend cash at a known constant risk-free interest rate
- The price follows a geometric Brownian motion with constant drift and volatility
- There are no transaction costs
- The stock does not pay a dividend
- All securities are perfectly divisible (i.e. it is possible to buy any fraction of a share)
- There are no restrictions on short selling

Motivation for stochastic volatility. What Could Cause the Smile?

Causes of volatility smile/skew

- Crash protection/ Fear of crashes
- Transactions costs
- Local volatility
- leverage effect
- CEV models
- Stochastic volatility
- jumps/crashes

Motivation for delay model, Article' s

"Chartists believe that future prices depend on past movement of the asset price and attempt to forecast future price levels based on past patterns of price dynamics."

1. Contagion effects in a chartist-fundamentalist model with time delays
   - *Ghassan Dibeh, PHYSICA A : Statistical Mechanics and its Applications,* **382***, 52-57, 2007*

2. Speculative dynamics in a time-delay model of asset prices
   - *Ghassan Dibeh, PHYSICA A : Statistical Mechanics and its Applications,* **355***, 199-208, 2005*

"The insider knows that both the drift and the volatility of the stock price process are influenced by certain events that happened before the trading period started"

1. A stochastic delay financial model
   - *Georage Stoica, American Mathematical Society,* **133***, 1837-1841, 2004*

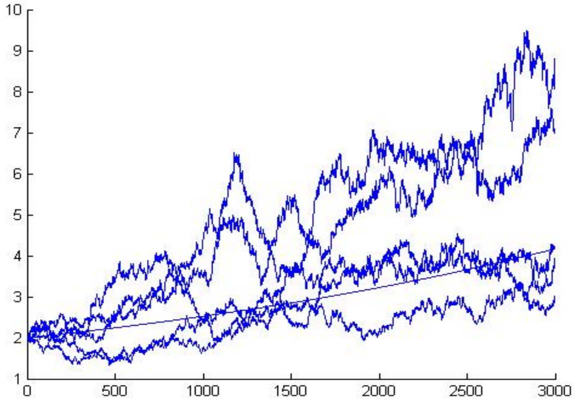
Model

DSV model

$$dX_t = \mu\xi(X_{t-a})X_t dt + f(Y_t)\eta(X_{t-b})X_t dW_t$$
$$X_t = \psi(t), \quad t \in [-L, 0], \text{ where } L = \max\{ a, b \}$$
$$dY_t = \alpha(m - Y_t)dt + \beta d\widehat{Z}_t,$$

where $f(y)$ is a sufficiently smooth function, ξ is a arbitrary function, η is an arbitrary non-zero function, and $\widehat{Z}_t$ is a Brownian motion correlated with W_t such that $d\widehat{Z}_t = \rho dW_t + \sqrt{1-\rho^2}dZ_t$, where W_t are Z_t are independent Brownian motions.

The path of DSV model



$\xi(x) = x^{0.2},\ \eta(x) = x^{0.001}$

Theorem 1

There exists a unique and positive solution for the DSV equation on $t \in [0, T]$ by $(k+1)$-step computations as follows :

$$\begin{aligned} X_t &= X_{k\ell}\exp\left(\mu\int_{k\ell}^{t}\xi(X_{s-a}) - \frac{1}{2}f^2(Y_s)\eta^2(X_{s-b})ds \right. \\ &\quad \left. + \int_{k\ell}^{t}\eta(X_{s-b})dW_s\right) \end{aligned}$$

for the positive integer k with $t \in [k\ell \wedge T, (k+1)\ell \wedge T]$ where $\ell = \min\{a, b\}$.

some notations for defining equivalent martingale measure

By Girsanov theorem, we can guarantee the existence of an equivalent martingale measure. Define W_t^* and Z_t^* as follows:

$$W_t^* = W_t + \int_0^t \frac{\mu\xi(X_{s-a}) - r}{f(Y_s)\eta(X_{s-b})} ds$$

$$Z_t^* = Z_t + \int_0^t \gamma_s ds$$

where γ_t is an adapted process to be determined

Equivalent martingale measure $\mathbb{Q}$

By Girsanov theorem, we have an equivalent martingale measure $\mathbb{Q}$ given by the Radon-Nikodym derivative

$\mathbb{Q}$

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^t \left[\left(\frac{\mu\xi(X_{s-a})-r}{f(Y_s)\eta(X_{s-b})}\right)^2 + \gamma_s^2\right] ds - \int_0^t \frac{\mu\xi(X_{s-a})-r}{f(Y_s)\eta(X_{s-b})} dW_s - \int_0^t \gamma_s dZ_s\right)$$

SDDE under equivalent martingale measure $\mathbb{Q}$

Model

Under $\mathbb{Q}$

$$
\begin{aligned}
dX_t &= r\xi(X_{t-a})X_t dt + f(Y_t)\eta(X_{t-b})X_t dW_t^* \\
X_t &= \psi(t), \quad t \in [-L, 0], \ \text{where } L = \max\{a, b\} \\
dY_t &= [\alpha(m - Y_t) - \beta\Lambda(Y_t, X_{t-a}, X_{t-b}, X_t)]\, dt \\
&\qquad + \beta d\widehat{Z}_t^*
\end{aligned}
$$

where $\widehat{Z}_t^* = \rho W_t^* + \sqrt{1-\rho^2} Z_t^*$

$$\Lambda(Y_t, X_{t-a}, X_{t-b}, X_t) = \rho \frac{\mu\xi(X_{t-a}) - r}{f(Y_t)\eta(X_{t-b})} + \sqrt{1-\rho^2}\gamma_t$$

preparation for asymptotic method

Now, we assume to have fast mean reversion. So, we introduce ε as the inverse of the rate of mean reversion α :

$$\varepsilon = \frac{1}{\alpha}$$

And, the long-run distribution of the OU process Y_t is assumed to have the moderate variance $\nu^2 = \frac{\beta^2}{2\alpha} = \mathcal{O}(1)$ so that

$$\beta = \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}$$

In term of the small parameter ε, the DSV model becomes

The DSV model

$$dX_t^\varepsilon = r\xi(X_{t-a}^\varepsilon)X_t^\varepsilon dt + f(Y_t^\varepsilon)\eta(X_{t-b}^\varepsilon)X_t^\varepsilon dW_t^*$$

$$X_t^\varepsilon = \psi(t), \quad t \in [-L, 0], \text{ where } L = \{a, b\}$$

$$dY_t^\varepsilon = \left(\frac{1}{\varepsilon}(m - Y_t^\varepsilon) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(Y_t^\varepsilon, X_{t-a}^\varepsilon, X_{t-b}^\varepsilon, X_t^\varepsilon)\right) dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} d\widehat{Z}_t^*$$

where $\Lambda(Y_t^\varepsilon, X_{t-a}^\varepsilon, X_{t-b}^\varepsilon, X_t^\varepsilon) = \rho\dfrac{\mu\xi(X_{t-a}^\varepsilon) - r}{f(Y_t^\varepsilon)\eta(X_{t-b}^\varepsilon)} + \sqrt{1-\rho^2}\gamma_t$

option price

Suppose no-arbitrage opportunity. The option price $P^\varepsilon(t)$ at time t of a derivative with terminal payoff function h is given by

$$P^\varepsilon(t) = \mathbb{E}^*\{e^{-r(T-t)}h(X_T^\varepsilon)|\mathcal{F}_t\}$$

where the conditional expectation is taken under the equivalent martingale measure $\mathbb{Q}$, and $\mathcal{F}_t$ is a filtration with respect to the past of $(X_t^\varepsilon, Y_t^\varepsilon)$

Goal : Approximation

To find Q^{ε} such that $P^{\varepsilon}(t) = Q^{\varepsilon}(t, X_t^{\varepsilon}) + \mathcal{O}(\varepsilon)$

$e^{-rt}P^{\varepsilon}(t)$: martingale

$e^{-rt}P^{\varepsilon}(t)$: martingale

The discounted price M_t^{ε} defined by

$$M_t^{\varepsilon} = e^{-rt}P^{\varepsilon}(t) = \mathbb{E}^*\{e^{-rT}h(X_T^{\varepsilon})|\mathcal{F}_t\}$$

is martingale with a terminal value given by

$$M_T^{\varepsilon} = e^{-rT}h(X_T^{\varepsilon})$$

A motivated theorem for finding $Q^{\varepsilon}(t, X_t)$

Theorem 2

Let $Q^{\varepsilon}(t, x)$ be a two-variable function with the following conditions :

(*i*) $Q^{\varepsilon}(t, x)$ satisfies $Q^{\varepsilon}(T, x) = h(x)$ at the final time T

(*ii*) $e^{-rt}Q^{\varepsilon}(t, X_t^{\varepsilon})$ can be decomposed as

$$e^{-rt}Q^{\varepsilon}(t, X_t^{\varepsilon}) = \widetilde{M}_t^{\varepsilon} + R_t^{\varepsilon}$$

where $\widetilde{M}^{\varepsilon}$ is a martingale and R_t^{ε} is of order ε

Then $P^{\varepsilon}(t) = Q^{\varepsilon}(t, X_t^{\varepsilon}) + \mathcal{O}(\varepsilon)$

Theorem 2,,, continued

Proof

Let $N_t^\varepsilon = e^{-rt} Q^\varepsilon(t, X_t^\varepsilon)$
Then, from the condition (i) and (ii),

$$M_T^\varepsilon = N_T^\varepsilon \text{ and } N_t^\varepsilon = \widetilde{M}_t^\varepsilon + R_t^\varepsilon$$

By taking a conditional expectation with respect to $\mathcal{F}_t$ on both sides of the equality $N_t^\varepsilon = \widetilde{M}_t^\varepsilon + R_t^\varepsilon$, we have

$$\begin{aligned} M_t^\varepsilon &= \mathbb{E}^*\{M_T^\varepsilon | \mathcal{F}_t\} \\ &= \mathbb{E}^*\{N_T^\varepsilon | \mathcal{F}_t\} \end{aligned}$$

Theorem 2,,, continued

proof,,,continued

$$
\begin{aligned}
&= \mathbb{E}^*\{\widetilde{M}_T^\varepsilon + R_t^\varepsilon | \mathcal{F}_t\} \\
&= \widetilde{M}_t^\varepsilon + \mathbb{E}^*\{R_T^\varepsilon | \mathcal{F}_t\} \\
&= N_t^\varepsilon + \mathbb{E}^*\{R_T^\varepsilon | \mathcal{F}_t\} - R_t^\varepsilon \\
&= N_t^\varepsilon + \mathcal{O}(\varepsilon)
\end{aligned}
$$

Therefore, by multiplying e^{rt} on both sides, we obtain

$$P^\varepsilon(t) = Q^\varepsilon(t, X_t^\varepsilon) + \mathcal{O}(\varepsilon)$$

Now, we assume that one can choose γ_t such that Λ in the DSV model becomes a function depending upon only of Y_t^ε. That is,

$$dY_t^\varepsilon = \left(\frac{1}{\varepsilon}(m - Y_t^\varepsilon) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(Y_t^\varepsilon) \right) dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} d\widehat{Z}_t^*$$

Then we obtain the infinitesimal generator $\varepsilon^{-1}\mathcal{L}_Y^\varepsilon$ of Y_t^ε where

$$\mathcal{L}_Y^\varepsilon = \nu^2 \frac{\partial^2}{\partial y^2} + (m - y - \nu\sqrt{2\varepsilon}\Lambda(y)) \frac{\partial}{\partial y}$$

Assume that $\Lambda(y)$ is bounded. Then Y^ε has a unique invariant distribution given by the probability density Φ_ε :

$$\Phi_\varepsilon(y) = J_\varepsilon \exp\left(-\frac{(y-m)^2}{2\nu^2} - \frac{\sqrt{2\varepsilon}}{\nu}\widetilde{\Lambda}(y)\right)$$

where $\widetilde{\Lambda}$ is an antiderivative of Λ that is at most linear at infinity and J_ε is a normalization constant depending on ε

θ_α

Define time-shift operators θ_α by

$$(\theta_\alpha g)(X_t^\varepsilon) = g(X_{t-\alpha}^\varepsilon)$$

for any measurable function g and any positive number α

Now, we apply the Itô-formula to N_t^ε to obtain

$$\begin{aligned} dN_t^\varepsilon &= d(e^{-rt}Q^\varepsilon(t,X_t^\varepsilon)) \\ &= e^{-rt}\left(\frac{\partial}{\partial t}Q^\varepsilon(t,X_t^\varepsilon) + \frac{1}{2}f^2(Y_t^\varepsilon)\eta^2(X_{t-b}^\varepsilon)(X_t^\varepsilon)^2\frac{\partial^2}{\partial x^2}Q^\varepsilon(t,X_t^\varepsilon)\right. \\ &\quad \left. + r\xi(X_t^\varepsilon)X_{t-a}^\varepsilon\frac{\partial}{\partial x}Q^\varepsilon(t,X_t^\varepsilon) - rQ^\varepsilon(t,X_t^\varepsilon)\right)dt \\ &\quad + e^{-rt}f(X_t^\varepsilon)\eta(X_{t-b})X_t^\varepsilon\frac{\partial Q^\varepsilon}{\partial x}(t,X_t^\varepsilon)dW_t^* \end{aligned}$$

f_α, ξ_α & η_α

Define a function f_α on the OU process Y_t^ε for any positive number α as following :

$$f_\alpha(Y_t^\varepsilon) = (\theta_\alpha f)(Y_t^\varepsilon) = f(Y_{t-\alpha}^\varepsilon) \quad \text{i.e}, f_\alpha = \theta_\alpha f$$

And, define functions ξ_α & η_α on the DSV process X_t^ε for any positive number α as following :

$$\xi_\alpha(X_t^\varepsilon) = (\theta_\alpha \xi)(X_t^\varepsilon) = \xi(X_{t-\alpha}^\varepsilon) \quad \text{i.e}, \xi_\alpha = \theta_\alpha \xi$$
$$\eta_\alpha(X_t^\varepsilon) = (\theta_\alpha \eta)(X_t^\varepsilon) = \eta(X_{t-\alpha}^\varepsilon) \quad \text{i.e}, \eta_\alpha = \theta_\alpha \eta$$

DSV operator

For convenience, we define an operator $\mathcal{L}_{DSV}(\overline{\sigma}_\varepsilon)$ as follows :

$$\mathcal{L}_{DSV}(\overline{\sigma}_\varepsilon) = \frac{\partial}{\partial t} + \frac{1}{2}\overline{\sigma}_\varepsilon^2 \eta_b^2(x) x^2 \frac{\partial^2}{\partial x^2} + r\xi_a(x) x \frac{\partial}{\partial x} - r\cdot$$

where $i(x) = x$. Then dN_t^ε becomes

$$dN_t^\varepsilon = e^{-rt}\left(\mathcal{L}_{DSV}(\overline{\sigma}_\varepsilon) + \frac{1}{2}(f^2(Y_t^\varepsilon) - \overline{\sigma}_\varepsilon^2)\eta^2(X_{t-b}^\varepsilon)(X_t^\varepsilon)^2\right)$$

$$\times \frac{\partial^2 Q^\varepsilon}{\partial x^2}(t, X_t^\varepsilon) dt$$

$$+ e^{-rt} f(X_t^\varepsilon)\eta(X_{t-b})X_t^\varepsilon \frac{\partial Q^\varepsilon}{\partial x}(t, X_t^\varepsilon) dW_t^* \qquad (1)$$

Some definitions

We will find a function Q^{ε} satisfying the condition (*ii*) assumed in Theorem 2. Before doing that, we need some definitions as follows :

Firstly, $\widetilde{P}_0^{\varepsilon}$

Define $\widetilde{P}_0^{\varepsilon}$ as :

$$\text{the solution } \widetilde{P}_0^{\varepsilon} \text{ of } \mathcal{L}_{DSV}(\overline{\sigma}_{\varepsilon})\widetilde{P}_0^{\varepsilon} = 0$$

with the terminal condition $\widetilde{P}_0^{\varepsilon}(T, x) = h(x)$

We will call $\mathcal{L}_{DSV}(\overline{\sigma}_{\varepsilon})\widetilde{P}_0^{\varepsilon} = 0$ as "Delayed Stochastic Volatility Equation (DSVE)"

Secondly, *V* and *U*

Define a 2-variable function V and U follows :

$$V(t,x) = \frac{\sqrt{\varepsilon}\nu\rho}{\sqrt{2}}\langle f\phi'\rangle_{\varepsilon}\eta_b^3(x)x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2\widetilde{P}_0^{\varepsilon}}{\partial x^2}\right)(t,x)$$

$$U(t,x) = \sqrt{\varepsilon}\nu\rho\sqrt{2}\langle f_b\phi'\rangle_{\varepsilon}\eta_b(x)\eta_b'(x)i_b(x)\eta_{2b}(x)x^2\frac{\partial^2\widetilde{P}_0^{\varepsilon}}{\partial x^2}(t,x)$$

Thirdly, $\widetilde{Q}_1^{\varepsilon}$

Define $\widetilde{Q}_1^{\varepsilon}$ as below :

$$\text{the solution } \widetilde{Q}_1^{\varepsilon} \textit{ of } \mathcal{L}_{DSV}(\overline{\sigma}_{\varepsilon})\widetilde{Q}_1^{\varepsilon} = V + U \tag{2}$$

That is, $\mathcal{L}_{DSV}(\overline{\sigma}_{\varepsilon})\widetilde{Q}_1^{\varepsilon}(t, X_t^{\varepsilon})$

$$= \frac{\sqrt{\varepsilon}\nu\rho}{\sqrt{2}}\langle f\phi'\rangle_{\varepsilon}\eta^3(X_{t-b}^{\varepsilon})X_t^{\varepsilon}\frac{\partial}{\partial x}\left(x^2\frac{\partial^2\widetilde{P}_0^{\varepsilon}}{\partial x^2}\right)(t, X_t^{\varepsilon})$$

$$+\sqrt{\varepsilon}\nu\rho\sqrt{2}\langle f_b\phi'\rangle_{\varepsilon}\eta(X_{t-b}^{\varepsilon})\eta'(X_{t-b}^{\varepsilon})\eta(X_{t-2b}^{\varepsilon})$$

$$\times(X_t^{\varepsilon})^2X_{t-b}^{\varepsilon}\frac{\partial^2\widetilde{P}_0^{\varepsilon}}{\partial x^2}(t, X_t^{\varepsilon})$$

the choice of Q^ε

It's time to choose Q^ε satisfying the conditions assumed in Theorem 2. We define Q^ε as

$$Q^\varepsilon = \widetilde{P}_0^\varepsilon + \widetilde{Q}_1^\varepsilon \tag{3}$$

It remains to show the chosen Q^ε satisfies the desired conditions.

From now, we will confirm it. For that, we need some properties. The following lemmas are helpful for the proof.

Some Lemmas

Define ϕ as the solution of

$$\mathcal{L}_Y^\varepsilon \phi(y) = f^2(y) - \langle f^2 \rangle_\varepsilon$$

Lemma 1

Let f be a sufficiently smooth function
Then $\int_0^t \left(f^2(Y_s^\varepsilon) - \overline{\sigma}_\varepsilon^2 \right) ds = \mathcal{O}(\sqrt{\varepsilon})$

Lemma 2

Lemma 2

Let f and g be a sufficiently smooth function.
Then

$$\int_0^t e^{-rs}\left(f(Y_s^\varepsilon)^2-\overline{\sigma}_\varepsilon^2\right)\eta^2(X_{s-b}^\varepsilon)(X_s^\varepsilon)^2\frac{\partial^2\widetilde{P}_0^\varepsilon}{\partial x^2}(s,X_s^\varepsilon)ds$$

$$=\sqrt{\varepsilon}(\overline{B}_t^\varepsilon+\overline{M}_t^\varepsilon)+\mathcal{O}(\varepsilon)$$

Lemma 2,,, continued

where $\overline{B}_t^\varepsilon$ is a systemic bias given by

$$\overline{B}_t^\varepsilon = \sqrt{2}\nu\rho\int_0^t e^{-rs} f(Y_s^\varepsilon)\phi'(Y_s^\varepsilon)\eta^3(X_{s-b}^\varepsilon)X_s^\varepsilon \frac{\partial}{\partial x}(x^2\frac{\partial^2 \widetilde{P}_0^\varepsilon}{\partial x^2})ds$$

$$+2\sqrt{2}\nu\rho\int_0^t e^{-rs} f_b(Y_s^\varepsilon)\phi'(Y_s^\varepsilon)\eta(X_{s-b}^\varepsilon)\eta'(X_{s-b}^\varepsilon)\eta(X_{s-2b}^\varepsilon)$$

$$\times(X_s^\varepsilon)^2 X_{s-b}^\varepsilon \frac{\partial^2 \widetilde{P}_0^\varepsilon}{\partial x^2}ds \qquad (4)$$

and $\overline{M}_t^\varepsilon$ is a martingale given by

$$\overline{M}_t^\varepsilon = \sqrt{2}\sqrt{\nu}\int_0^t e^{-rs}\phi'(Y_s^\varepsilon)\eta^2(X_{s-b}^\varepsilon)(X_s^\varepsilon)^2\frac{\partial^2 \widetilde{P}_0^\varepsilon}{\partial x^2}d\widehat{Z}_s^* \qquad (5)$$

Lemma 2,,, continued

Proof

The following facts hold. (detailed proofs are omitted)

1. $(f^2(Y_s^\varepsilon) - \langle f^2\rangle_\varepsilon)ds = \varepsilon d\phi(Y_s^\varepsilon) - \nu\sqrt{2\varepsilon}\phi'(Y_s^\varepsilon)d\widehat{Z}_s^*$
2. $\int_0^t e^{-rs}\left(f(Y_s^\varepsilon)^2 - \overline{\sigma}_\varepsilon^2\right)g^2(X_{s-b}^\varepsilon)(X_s^\varepsilon)^2\frac{\partial^2\widetilde{P}_0^\varepsilon}{\partial x^2}(s, X_s^\varepsilon)ds$

$$= \varepsilon\int_0^t e^{-rs}g^2(X_{s-b}^\varepsilon)(X_s^\varepsilon)^2\frac{\partial^2\widetilde{P}_0^\varepsilon}{\partial x^2}d\phi(Y_s^\varepsilon)$$

$$-\nu\sqrt{2\varepsilon}\int_0^t e^{-rs}g^2(X_{s-b}^\varepsilon)\phi'(Y_s^\varepsilon)(X_s^\varepsilon)^2\frac{\partial^2\widetilde{P}_0^\varepsilon}{\partial x^2}(s, X_s^\varepsilon)d\widehat{Z}_s^*$$

Lemma 2,,, continued

proof,,, continued

$$3.\ \varepsilon \int_0^t e^{-st} g^2(X_{s-b}^\varepsilon)(X_s^\varepsilon)^2 \frac{\partial^2 Q^\varepsilon}{\partial x^2}(t, X_s^\varepsilon) d\phi(Y_s^\varepsilon) = \overline{B}_t^\varepsilon + \mathcal{O}(\varepsilon)$$

Putting these facts together yields Lemma 2.

From the SDDE (1), we obtain N_t^ε as follows :

$$\begin{aligned} N_t^\varepsilon &= N_0^\varepsilon \\ &+ \int_0^t e^{-rt}\mathcal{L}_{DSV}(\overline{\sigma}_\varepsilon)Q^\varepsilon ds \\ &+ \frac{1}{2}\int_0^t e^{-rs}(f^2(Y_s^\varepsilon) - \overline{\sigma}_\varepsilon^2)\eta^2(X_{s-b}^\varepsilon)(X_s^\varepsilon)^2\frac{\partial^2 Q^\varepsilon}{\partial x^2}ds \\ &+ \int_0^t e^{-rs}\frac{\partial Q^\varepsilon}{\partial x}f(Y_s^\varepsilon)\eta(X_{s-b}^\varepsilon)X_s^\varepsilon dW_s^* \qquad (6) \end{aligned}$$

By the definition (3) and the definition of $\widetilde{P}_0^\varepsilon$,

$$\mathcal{L}_{DSV}(\overline{\sigma}_\varepsilon)Q^\varepsilon = \mathcal{L}_{DSV}(\overline{\sigma}_\varepsilon)\widetilde{Q}^\varepsilon$$

Then (6) becomes :

$$\begin{aligned} N_t^\varepsilon &= N_0^\varepsilon \\ &\quad + \int_0^t e^{-rt}\mathcal{L}_{DSV}(\overline{\sigma}_\varepsilon)\widetilde{Q}^\varepsilon ds \\ &\quad + \frac{1}{2}\int_0^t e^{-rs}(f^2(Y_s^\varepsilon) - \overline{\sigma}_\varepsilon^2)\eta^2(X_{s-b}^\varepsilon)(X_s^\varepsilon)^2 \frac{\partial^2 \widetilde{Q}_1^\varepsilon}{\partial x^2} ds \\ &\quad + \frac{1}{2}\int_0^t e^{-rs}(f^2(Y_s^\varepsilon) - \overline{\sigma}_\varepsilon^2)\eta^2(X_{s-b}^\varepsilon)(X_s^\varepsilon)^2 \frac{\partial^2 \widetilde{P}_0^\varepsilon}{\partial x^2} ds \\ &\quad + \int_0^t e^{-rs}\frac{\partial Q^\varepsilon}{\partial x} f(Y_s^\varepsilon)\eta(X_{s-b}^\varepsilon)X_s^\varepsilon dW_s^* \end{aligned}$$

Also, by Lemma 2, the above N_t^ε become :

$$\begin{aligned} N_t^\varepsilon &= N_0^\varepsilon \\ &\quad + \int_0^t e^{-rs}\mathcal{L}_{DSV}(\overline{\sigma}_\varepsilon)\widetilde{Q}_1^\varepsilon ds \\ &\quad + \frac{\sqrt{\varepsilon}}{2}(\overline{B}_t^\varepsilon + \overline{M}_t^\varepsilon) + R_1(\varepsilon) \\ &\quad + \int_0^t e^{-rs}\frac{\partial Q^\varepsilon}{\partial x} f(Y_s^\varepsilon)\eta(X_{s-b}^\varepsilon)X_s^\varepsilon dW_s^* \end{aligned}$$

where $R_1(\varepsilon) = \mathcal{O}(\varepsilon)$

By the definitions (2) and (4), we obtain N_t^ε as follows :

$$
\begin{aligned}
N_t^\varepsilon &= N_0^\varepsilon \\
&+ \frac{\sqrt{\varepsilon}\nu\rho}{\sqrt{2}} \int_0^t e^{-rs}\left(f(Y_s^\varepsilon)\phi'(Y_s^\varepsilon) - \langle f\phi'\rangle_\varepsilon\right)\eta^3(X_{s-b}^\varepsilon) \\
&\qquad \times X_s^\varepsilon \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 \widetilde{P}_0^\varepsilon}{\partial x^2}\right) ds \\
&+ \sqrt{\varepsilon 2}\nu\rho \int_0^t e^{-rs}\left(f_b(Y_s^\varepsilon)\phi'(Y_s^\varepsilon) - \langle f_b\phi'\rangle_\varepsilon\right)\eta^3(X_{s-b}^\varepsilon) \\
&\qquad \times X_s^\varepsilon \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 \widetilde{P}_0^\varepsilon}{\partial x^2}\right) ds \\
&+ \frac{\sqrt{\varepsilon}}{2}\overline{M}_t^\varepsilon + R_2(\varepsilon) + \int_0^t e^{-rs}\frac{\partial Q^\varepsilon}{\partial x} f(Y_s^\varepsilon)\eta(X_{s-b}^\varepsilon) X_s^\varepsilon dW_s^*
\end{aligned} \tag{7}
$$

where $R_2(\varepsilon) = \mathcal{O}(\varepsilon)$

Here, as in Lemma 1, the second term and the third term of (7) are included in $\mathcal{O}(\varepsilon)$ can be shown to be of order ε. So, we have

$$\begin{aligned} N_t^\varepsilon &= N_0^\varepsilon \\ &\quad + \frac{\sqrt{\varepsilon}}{2}\overline{M}_t^\varepsilon + R_3(\varepsilon) \\ &\quad + \int_0^t e^{-rs}\frac{\partial Q^\varepsilon}{\partial x} f(Y_s^\varepsilon)\eta(X_{s-b}^\varepsilon)X_s^\varepsilon dW_s^* \end{aligned} \tag{8}$$

where $R_3(\varepsilon) = \mathcal{O}(\varepsilon)$

Define $\widetilde{M}_t^\varepsilon$ & R_t^ε

$\widetilde{M}_t^\varepsilon$

$$\widetilde{M}_t^\varepsilon = N_0^\varepsilon + \frac{\sqrt{\varepsilon}}{2}\overline{M}_t^\varepsilon + \int_0^t e^{-rs}\frac{\partial Q^\varepsilon}{\partial x} f(Y_s^\varepsilon)\eta(X_{s-b}^\varepsilon)X_s^\varepsilon dW_s^*$$

R_t^ε

$$R_t^\varepsilon = R_3(\varepsilon)$$

Here, $\overline{M}_t^\varepsilon$ is a martingale and , $\int_0^t e^{-rs}\frac{\partial Q^\varepsilon}{\partial x} f(Y_s^\varepsilon)\eta(X_{s-b}^\varepsilon)X_s^\varepsilon dW_s^*$ is a martingale by Martingale Representation Theorem. Hence, $\widetilde{M}_t^\varepsilon$ is also a martingale.

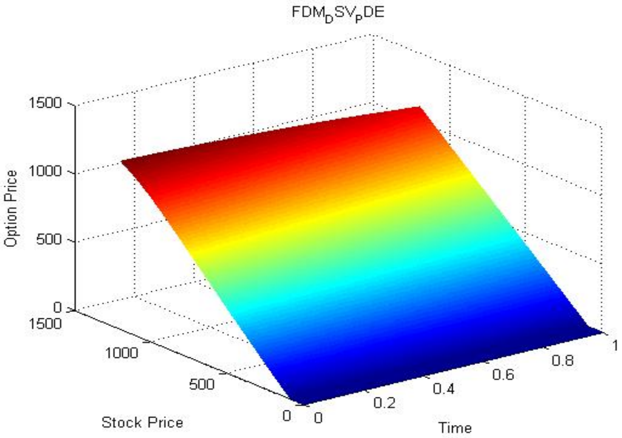
Then, from (8)

$$e^{-rt}Q^\varepsilon = N_t^\varepsilon = \widetilde{M}_t^\varepsilon + R_t^\varepsilon$$

where $\widetilde{M}^\varepsilon$ is a martingale and R_t^ε is of order ε.
So that we can confirm that the Q^ε of our choice satisfies the conditions (*i*) and (*ii*) in Theorem 2.
Therefore, by Theorem 2,

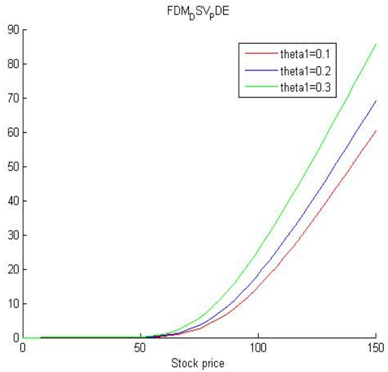
$$P_t^\varepsilon = Q^\varepsilon(t, X_t^\varepsilon) + \mathcal{O}(\varepsilon)$$

Leading order term, $\widetilde{P}$

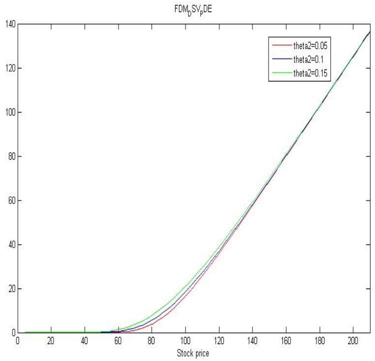


strike price=100, $\xi(x) = x^{0.2}$, $\eta(x) = x^{0.001}$

Leading order term, $\widetilde{P}$

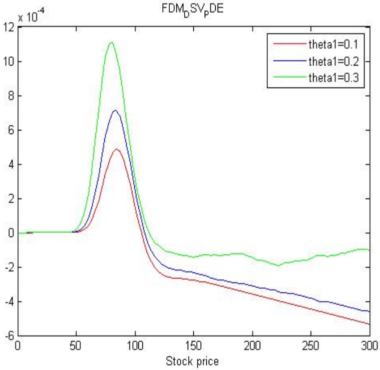


strike price=100, $\xi(x) = x^{\theta_1}$, $\eta(x) = x^{0.1}$

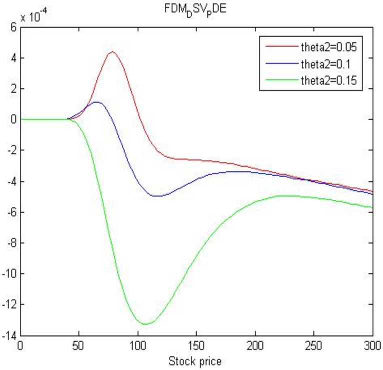


strike price=100, $\xi(x) = x^{0.2}$, $\eta(x) = x^{\theta_2}$

Correction term, $\widetilde{Q}_1^{\varepsilon}$

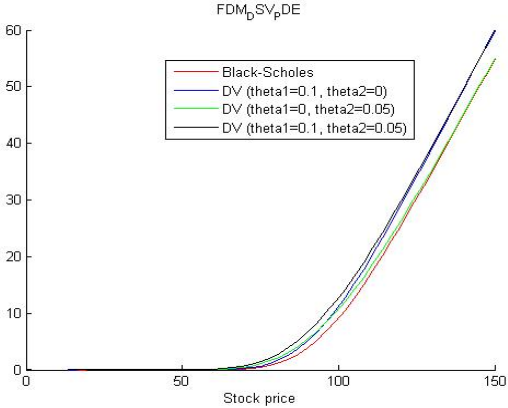


strike price=100, $\xi(x) = x^{\theta_1}, \ \eta(x) = x^{0.1}$



strike price=100, $\xi(x) = x^{0.2}, \ \eta(x) = x^{\theta_2}$

Comparison of European call option price for DSV model and Black-Scholes model



strike price=100, $\xi(x) = x^{\theta_1}, \ \eta(x) = x^{\theta_2}$

The "blue" line is a case where the delay term is in only drift term, the "green" line is a case where the delay term is in only volatility term and the "black" line is a case where the delay term is in both terms

Conclusion

- Introduced a new Non-Markovian Stochastic Volatility model.
- The price by DSV model is more flexible to market than BS model.
- Performed asymptotic analysis.
- Still on-going research - Mathematical rigor, Data fitting, and etc.