Rating Based Lévy Libor Model

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Joint work with Ernst Eberlein

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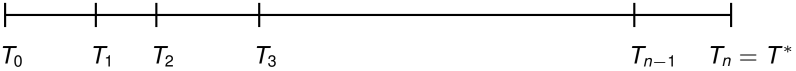
Libor market models – introduction

Discrete tenor structure: $0 = T_0 < T_1 < \ldots < T_n = T^*$, with $\delta_k = T_{k+1} - T_k$



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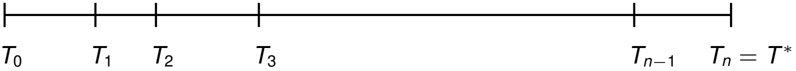

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Defaultable zero coupon bonds with credit ratings: $B_C(\cdot, T_1), \ldots, B_C(\cdot, T_n)$

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$$L_C(t, T_k) = \frac{1}{\delta_k}\left(\frac{B_C(t, T_k)}{B_C(t, T_{k+1})} - 1\right)$$

Libor modeling

- modeling under forward martingale measures, i.e. risk-neutral measures that use zero-coupon bonds as numeraires

- on a given stochastic basis, construct a family of Libor rates $L(\cdot, T_k)$ and a collection of mutually equivalent probability measures $\mathbb{P}_{T_k}$ such that

$$\left(\frac{B(t, T_j)}{B(t, T_k)}\right)_{0 \leq t \leq T_k \wedge T_j}$$

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- in addition model defaultable Libor rates $L_C(\cdot, T_k)$ such that

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Defaultable bonds with ratings

- Credit ratings identified with elements of a finite set $\mathcal{K} = \{1, 2, \ldots, K\}$, where 1 is the best possible rating and K is the default event
- Credit migration is modeled by a conditional Markov chain C with state space $\mathcal{K}$
- Default time τ: the first time when C reaches the absorbing state K, i.e.

$$\tau = \inf\{t > 0 : C_t = K\}$$

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- We consider defaultable bonds $B_C(\cdot, T_k)$ with credit migration process C and fractional recovery of Treasury value $q = (q_1, \ldots, q_{K-1})$ upon default:

$$B_C(t, T_k) = \sum_{i=1}^{K-1} B_i(t, T_k) \mathbf{1}_{\{C_t = i\}} + q_{C_{\tau-}} B(t, T_k) \mathbf{1}_{\{C_t = K\}},$$

We have $B_i(T_k, T_k) = 1$, for all i.

Canonical construction of C

Let $(\Omega, \mathcal{F}_{T^*}, \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T^*}, \mathbb{P}_{T^*})$ be a given complete stochastic basis.

- Let $\Lambda = (\Lambda_t)_{0 \le t \le T^*}$ be a matrix-valued $\mathbb{F}$-adapted stochastic process

$$\Lambda(t) = \begin{bmatrix} \lambda_{11}(t) & \lambda_{12}(t) & \dots & \lambda_{1K}(t) \\ \lambda_{21}(t) & \lambda_{22}(t) & \dots & \lambda_{2K}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

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- Enlarge probability space

$$(\Omega, \mathcal{F}_{T^*}, \mathbb{P}_{T^*}) \rightarrow (\tilde{\Omega}, \mathcal{G}_{T^*}, \mathbb{Q}_{T^*})$$

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The process C is a *conditional Markov chain* relative to $\mathbb{F}$, i.e. for every $0 \le t \le s$ and any function $h : \mathcal{K} \to \mathbb{R}$

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s)|\mathcal{F}_t \vee \mathcal{F}_t^C] = \mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s)|\mathcal{F}_t \vee \sigma(C_t)],$$

where $\mathbb{F}^C = (\mathcal{F}_t^C)$ denotes the filtration generated by C.

The progressive enlargement of filtration

$$\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{F}_t^C, \quad t \in [0, T^*],$$

satisfies the $(\mathcal{H})$-hypothesis:

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$(\mathcal{H}1)$ $\mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_{T^*}] = \mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_t]$,

for any bounded, $\mathcal{F}_t^C$-measurable random variable Y (Brémaud and Yor (1978) or Elliot, Jeanblanc and Yor (2000))

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But this follows easily from property

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_B|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_B|\mathcal{F}_t], \qquad t \le s, B \in \mathcal{F}_t^C,$$

which is proved as a consequence of the canonical construction.

Risk-free Lévy Libor model

(Eberlein and Özkan, 2005)

Let $(\Omega, \mathcal{F}_{T^*}, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T^*}, \mathbb{P}_{T^*})$ be a complete stochastic basis.

- as driving process take a time-inhomogeneous Lévy process $X = (X^1, \ldots, X^d)$ whose Lévy measures satisfy certain integrability conditions
- X is a special semimartingale with canonical decomposition

$$X_t = \int_0^t b_s \mathrm{d}s + \int_0^t \sqrt{c_s} \mathrm{d}W_s^{T^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu^{T^*})(\mathrm{d}s, \mathrm{d}x),$$

where W^{T^*} denotes a $\mathbb{P}_{T^*}$-standard Brownian motion and
μ is the random measure of jumps of X with $\mathbb{P}_{T^*}$-compensator ν^{T^*}.
We assume that $b = 0$.

Construction of Libor rates (backward induction):

Starting from $k = n - 1$, we have for each T_k:

(i) define the forward measure $\mathbb{P}_{T_{k+1}}$ via

$$\left.\frac{d\mathbb{P}_{T_{k+1}}}{d\mathbb{P}_{T^*}}\right|_{\mathcal{F}_t} = \prod_{l=k+1}^{n-1} \frac{1 + \delta_l L(t, T_l)}{1 + \delta_l L(0, T_l)} = \frac{B(0, T^*)}{B(0, T_{k+1})} \frac{B(t, T_{k+1})}{B(t, T^*)}.$$

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(ii) the dynamics of the Libor rate $L(\cdot, T_k)$ under this measure

$$L(t, T_k) = L(0, T_k) \exp\left(\int_0^t b^L(s, T_k)\mathrm{d}s + \int_0^t \sigma(s, T_k)\mathrm{d}X_s^{T_{k+1}}\right), \tag{1}$$

where

$$X_t^{T_{k+1}} = \int_0^t \sqrt{c_s}\mathrm{d}W_s^{T_{k+1}} + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu^{T_{k+1}})(\mathrm{d}s, \mathrm{d}x)$$

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with $\mathbb{P}_{T_{k+1}}$-Brownian motion $W^{T_{k+1}}$ and

$$\nu^{T_{k+1}}(\mathrm{d}s,\mathrm{d}x) = \prod_{l=k+1}^{n-1}\left(\frac{\delta_l L(s-,T_l)}{1+\delta_l L(s-,T_l)}(e^{\langle\sigma(s,T_l),x\rangle}-1)+1\right)\nu^{T^*}(\mathrm{d}s,\mathrm{d}x).$$

The drift term $b^L(s,T_k)$ is chosen such that $L(\cdot,T_k)$ becomes a $\mathbb{P}_{T_{k+1}}$-martingale.

More precisely,

$$\begin{aligned} b^L(s, T_k) &= -\frac{1}{2}\langle \sigma(s, T_k), c_s \sigma(s, T_k)\rangle \\ &\quad - \int_{\mathbb{R}^d} \left(e^{\langle \sigma(s, T_k), x\rangle} - 1 - \langle \sigma(s, T_k), x\rangle \right) F_s^{T_{k+1}}(\mathrm{d}x). \end{aligned}$$

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- This construction guarantees that the forward bond price processes

$$\left(\frac{B(t,T_j)}{B(t,T_k)}\right)_{0\le t\le T_j\wedge T_k}$$

are martingales for all $j = 1,\dots,n$ under the forward measure $\mathbb{P}_{T_k}$ associated with the date T_k ($k = 1,\dots,n$).

- The arbitrage-free price at time t of a contingent claim with payoff X at maturity T_k is given by

$$\pi_t^X = B(t,T_k)\mathbb{E}_{\mathbb{P}_{T_k}}[X|\mathcal{F}_t].$$

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(6) Define on this space the forward measures $\mathbb{Q}_{T_k}$ by:

for each tenor date T_k $\mathbb{Q}_{T_k}$ is obtained from $\mathbb{Q}_{T^*}$ in the same way as $\mathbb{P}_{T_k}$ from $\mathbb{P}_{T^*}$ $(k = 1, \ldots, n-1)$

Note that

$$\frac{\mathrm{d}\mathbb{Q}_{T_k}}{\mathrm{d}\mathbb{Q}_{T^*}} = \psi^k,$$

where ψ^k is an $\mathcal{F}_{T_k}$-measurable random variable with expectation 1.

Conditional Markov chain C *under forward measures*

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Theorem

Let C be a canonically constructed conditional Markov chain with respect to $\mathbb{Q}_{T^*}$. Then C is a conditional Markov chain with respect to every forward measure $\mathbb{Q}_{T_k}$ and

$$p_{ij}^{\mathbb{Q}_{T_k}}(t,s) = p_{ij}^{\mathbb{Q}_{T^*}}(t,s)$$

i.e. the matrices of transition probabilities under $\mathbb{Q}_{T^*}$ and $\mathbb{Q}_{T_k}$ are the same.

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Theorem

The $(\mathcal{H})$-hypothesis holds under all $\mathbb{Q}_{T_k}$, i.e. every $(\mathbb{F}, \mathbb{Q}_{T_k})$-local martingale is a $(\mathbb{G}, \mathbb{Q}_{T_k})$-local martingale.

Rating-dependent Libor rates

- The forward Libor rate for credit rating class i

$$L_i(t, T_k) := \frac{1}{\delta_k}\left(\frac{B_i(t, T_k)}{B_i(t, T_{k+1})} - 1\right), \quad i = 1, 2, \ldots, K-1$$

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- The corresponding discrete-tenor forward inter-rating spreads

$$H_i(t, T_k) := \frac{L_i(t, T_k) - L_{i-1}(t, T_k)}{1 + \delta_k L_{i-1}(t, T_k)}$$

Observe that the Libor rate for the rating i can be expressed as

$$\begin{aligned} 1 + \delta_k L_i(t, T_k) &= (1 + \delta_k L_{i-1}(t, T_k))(1 + \delta_k H_i(t, T_k)) \\ &= \underbrace{(1 + \delta_k L(t, T_k))}_{\text{default-free Libor}} \prod_{j=1}^{i} \underbrace{(1 + \delta_k H_j(t, T_k))}_{\text{spread } j-1 \to j} \end{aligned}$$

Observe that the Libor rate for the rating *i* can be expressed as

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Idea: model $H_j(\cdot, T_k)$ as exponential semimartingales and thus ensure automatically the *monotonicity* of Libor rates w.r.t. the credit rating:

$$L(t, T_k) \leq L_1(t, T_k) \leq \cdots \leq L_{K-1}(t, T_k)$$

$\Longrightarrow$ worse credit rating, higher interest rate

Pre-default term structure of rating-dependent Libor rates

For each rating i and tenor date T_k we model $H_i(\cdot, T_k)$ as

$$H_i(t, T_k) = H_i(0, T_k)\exp\left(\int_0^t b^{H_i}(s, T_k)\mathrm{d}s + \int_0^t \gamma_i(s, T_k)\mathrm{d}X_s^{T_{k+1}}\right) \tag{2}$$

with initial condition

$$H_i(0, T_k) = \frac{1}{\delta_k}\left(\frac{B_i(0, T_k)B_{i-1}(0, T_{k+1})}{B_{i-1}(0, T_k)B_i(0, T_{k+1})} - 1\right).$$

$X^{T_{k+1}}$ is defined as earlier and $b^{H_i}(s, T_k)$ is the drift term (we assume $b^{H_i}(s, T_k) = 0$, for $s > T_k \Rightarrow H_i(t, T_k) = H_i(T_k, T_k)$, for $t \geq T_k$).

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$\Rightarrow$ the forward Libor rate $L_i(\cdot, T_k)$ is obtained from relation

$$1 + \delta_k L_i(t, T_k) = (1 + \delta_k L(t, T_k)) \prod_{j=1}^{i} (1 + \delta_k H_j(t, T_k)).$$

Theorem

Assume that $L(\cdot, T_k)$ and $H_i(\cdot, T_k)$ are given by (1) and (2). Then:

(a) *The rating-dependent forward Libor rates satisfy for every T_k and $t \leq T_k$*

$$L(t, T_k) \leq L_1(t, T_k) \leq \cdots \leq L_{K-1}(t, T_k),$$

i.e. Libor rates are monotone with respect to credit ratings.

(b) *The dynamics of the Libor rate $L_i(\cdot, T_k)$ under $\mathbb{P}_{T_{k+1}}$ is given by*

$$L_i(t, T_k) = L_i(0, T_k) \exp\left(\int_0^t b^{L_i}(s, T_k)\mathrm{d}s + \int_0^t \sqrt{c_s}\sigma_i(s, T_k)\mathrm{d}W_s^{T_{k+1}} + \int_0^t \int_{\mathbb{R}^d} S_i(s, x, T_k)(\mu - \nu^{T_{k+1}})(\mathrm{d}s, \mathrm{d}x) \right),$$

where

$$\sigma_i(s, T_k) := \ell_i(s-, T_k)^{-1}\Big(\ell_{i-1}(s-, T_k)\sigma_{i-1}(s, T_k) + h_i(s-, T_k)\gamma_i(s, T_k)\Big)$$
$$= \ell_i(s-, T_k)^{-1}\Big[\ell(s-, T_k)\sigma(s, T_k) + \sum_{j=1}^{i} h_j(s-, T_k)\gamma_j(s, T_k)\Big]$$

represents the volatility of the Brownian part and

$$S_i(s, x, T_k) := \ln\Big(1 + \ell_i(s-, T_k)^{-1}(\beta_i(s, x, T_k) - 1)\Big)$$

controls the jump size. Here we set

$$h_i(s, T_k) := \frac{\delta_k H_i(s, T_k)}{1 + \delta_k H_i(s, T_k)},$$
$$\ell_i(s, T_k) := \frac{\delta_k L_i(s, T_k)}{1 + \delta_k L_i(s, T_k)},$$

and

$$\beta_i(s, x, T_k) := \beta_{i-1}(s, x, T_k)\Big(1 + h_i(s-, T_k)(e^{\langle \gamma_i(s, T_k), x\rangle} - 1)\Big)$$
$$= \Big(1 + \ell(s-, T_k)(e^{\langle \sigma(s, T_k), x\rangle} - 1)\Big)$$
$$\times \prod_{j=1}^{i}\Big(1 + h_j(s-, T_k)(e^{\langle \gamma_j(s, T_k), x\rangle} - 1)\Big).$$

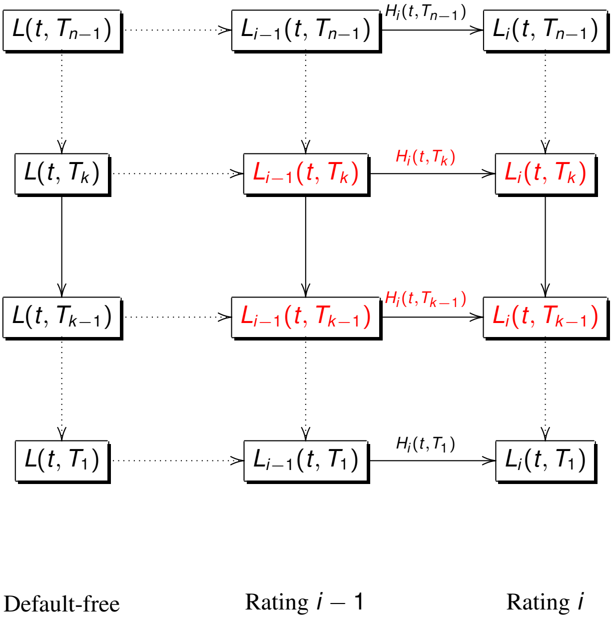

Figure: Connection between subsequent Libor rates

No-arbitrage condition for the rating based model

Recall the defaultable bond price process with fractional recovery of Treasury value q

$$B_C(t, T_k) = \sum_{i=1}^{K-1} B_i(t, T_k)\mathbf{1}_{\{C_t=i\}} + q_{C_{\tau\_}} B(t, T_k)\mathbf{1}_{\{C_t=K\}}.$$

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Note: the forward bond price process

$$\frac{B_C(\cdot, T_k)}{B(\cdot, T_j)}$$

is a $\mathbb{Q}_{T_j}$-local martingale for every $k, j = 1, \ldots, n-1$

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is a $\mathbb{Q}_{T_k}$-local martingale for every $k = 1, \dots, n-1$.

We postulate that the forward bond price process is given by

$$\frac{B_C(t,T_k)}{B(t,T_k)} := \sum_{i=1}^{K-1} \underbrace{\prod_{j=1}^{i}\prod_{l=0}^{k-1}\frac{1}{1+\delta_l H_j(t,T_l)}}_{:=\mathbb{H}(t,T_k,i)} e^{\int_0^t \lambda_i(s)\mathrm{d}s}\mathbf{1}_{\{C_t=i\}} + q_{C_{\tau\_}}\mathbf{1}_{\{C_t=K\}}$$

$$= \sum_{i=1}^{K-1} \mathbb{H}(t,T_k,i)e^{\int_0^t \lambda_i(s)\mathrm{d}s}\mathbf{1}_{\{C_t=i\}} + q_{C_{\tau\_}}\mathbf{1}_{\{C_t=K\}}, \quad (3)$$

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Note that this specification is consistent with the definition of H_i which implies the following connection of bond prices and inter-rating spreads:

$$\frac{B_j(t,T_k)}{B_{j-1}(t,T_k)} = \frac{B_j(t,T_{k-1})}{B_{j-1}(t,T_{k-1})}\frac{1}{1+\delta_{k-1}H_j(t,T_{k-1})}$$

and relation

$$\frac{B_i(t,T_k)}{B(t,T_k)} = \frac{B_1(t,T_k)}{B(t,T_k)}\prod_{j=2}^{i}\frac{B_j(t,T_k)}{B_{j-1}(t,T_k)}.$$

Lemma

Let T_k be a tenor date and assume that $H_j(\cdot, T_k)$ are given by (2)*. The process $\mathbb{H}(\cdot, T_k, i)$ has the following dynamics under $\mathbb{P}_{T_k}$*

$$\mathbb{H}(t, T_k, i) = \mathbb{H}(0, T_k, i) \times \mathcal{E}_t \Bigg(\int_0^{\cdot} b^{\mathbb{H}}(s, T_k, i)\mathrm{d}s - \int_0^{\cdot} \sqrt{c_s} \sum_{j=1}^{i} \sum_{l=1}^{k-1} h_j(s-, T_l)\gamma_j(s, T_l)\mathrm{d}W_s^{T_k} + \int_0^{\cdot} \int_{\mathbb{R}^d} \left(\prod_{j=1}^{i} \prod_{l=1}^{k-1} \left(1 + h_j(s-, T_l)(e^{\langle \gamma_j(s,T_l), x\rangle} - 1)\right)^{-1} - 1 \right) \times (\mu - \nu^{T_k})(\mathrm{d}s, \mathrm{d}x) \Bigg),$$

where $b^{\mathbb{H}}(s, T_k, i)$ is the drift term.

No-arbitrage condition

Theorem

Let T_k be a tenor date. Assume that the processes $H_j(\cdot, T_k)$, $j = 1, \dots, K-1$, are given by (2). Then the process $\frac{B_C(\cdot, T_k)}{B(\cdot, T_k)}$ defined in (3) is a local martingale with respect to the forward measure $\mathbb{Q}_{T_k}$ and filtration $\mathbb{G}$ iff:
for almost all $t \leq T_k$ on the set $\{C_t \neq K\}$

$$b^{\mathbb{H}}(t, T_k, C_t) + \lambda_{C_t}(t) = \left(1 - q_{C_t} \frac{e^{-\int_0^t \lambda_{C_t}(s)\mathrm{d}s}}{\mathbb{H}(t-, T_k, C_t)}\right) \lambda_{C_t K}(t) + \sum_{j=1, j \neq C_t}^{K-1} \left(1 - \frac{\mathbb{H}(t-, T_k, j) e^{\int_0^t \lambda_j(s)\mathrm{d}s}}{\mathbb{H}(t-, T_k, C_t) e^{\int_0^t \lambda_{C_t}(s)\mathrm{d}s}}\right) \lambda_{C_t j}(t). \quad (4)$$

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Sketch of the proof: Use the fact that the jump times of the conditional Markov chain C do not coincide with the jumps of any $\mathbb{F}$-adapted semimartingale, use martingales related to the indicator processes $\mathbf{1}_{\{C_t = i\}}$, $i \in \mathcal{K}$, and stochastic calculus for semimartingales.

Defaultable forward measures

Assume that $\frac{B_C(\cdot,T_k)}{B(\cdot,T_k)}$ is a *true martingale* w.r.t. forward measure $\mathbb{Q}_{T_k}$. (back to DFP)

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$$\left.\frac{\mathrm{d}\mathbb{Q}_{C,T_k}}{\mathrm{d}\mathbb{Q}_{T_k}}\right|_{\mathcal{G}_t} := \frac{B(0,T_k)}{B_C(0,T_k)} \frac{B_C(t,T_k)}{B(t,T_k)}.$$

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Proposition

The defaultable Libor rate $L_C(\cdot, T_k)$ is a martingale with respect to $\mathbb{Q}_{C,T_{k+1}}$ and

$$\left.\frac{\mathrm{d}\mathbb{Q}_{C,T_k}}{\mathrm{d}\mathbb{Q}_{C,T_{k+1}}}\right|_{\mathcal{G}_t} = \frac{B_C(0,T_{k+1})}{B_C(0,T_k)}(1+\delta_k L_C(t,T_k)).$$

Pricing problems I: Defaultable bond

Proposition

The price of a defaultable bond with maturity T_k and fractional recovery of Treasury value q at time $t \leq T_k$ is given by

$$B_C(t,T_k)\mathbf{1}_{\{C_t \neq K\}} = B(t,T_k)\sum_{i=1}^{K-1}\mathbf{1}_{\{C_t=i\}}\Bigg[\mathbb{E}_{\mathbb{Q}_{T_k}}[1-p_{iK}(t,T_k)|\mathcal{F}_t] + \sum_{j=1}^{K-1}\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{t<\tau\leq T_k\}}\mathbf{1}_{\{C_t=i\}}\mathbf{1}_{\{C_{\tau-}=j\}}q_j|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]}\Bigg].$$

Pricing problems II: Credit default swap

- consider a maturity date T_m and a defaultable bond with fractional recovery of Treasury value q as the underlying asset
- protection buyer pays a fixed amount S periodically at tenor dates $T_1, \ldots, T_{m-1}$ until default
- protection seller promises to make a payment that covers the loss if default happens:

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has to paid at T_{k+1} if default occurs in $(T_k, T_{k+1}]$

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Proposition

The swap rate S at time 0 is equal to

$$S = \frac{\sum_{k=2}^{m} B(0,T_k) \sum_{j=1}^{K-1} \mathbb{E}_{\mathbb{Q}_{T_k}}[(1-q_j)\mathbf{1}_{\{T_{k-1}<\tau\le T_k, C_{\tau-}=j\}}]}{\sum_{k=1}^{m-1} B(0,T_k)\mathbb{E}_{\mathbb{Q}_{T_k}}[1-p_{iK}(0,T_k)]},$$

if the observed class at time zero is i.

Pricing problems III: use of defaultable measures

Proposition

Let Y be a promised $\mathcal{G}_{T_k}$-measurable payoff at maturity T_k of a defaultable contingent claim with fractional recovery q upon default and assume that Y is integrable with respect to $\mathbb{Q}_{T_k}$.
The time-t value of such a claim is given by

$$\pi^t(Y) = B_C(t, T_k)\mathbb{E}_{\mathbb{Q}_{C,T_k}}[Y|\mathcal{G}_t].$$

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Example: a cap on the defaultable forward Libor rate

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Example: a cap on the defaultable forward Libor rate

The time-t price of a caplet with strike K and maturity T_k on the defaultable Libor rate is given by

$$C_t(T_k, K) = \delta_k B_C(t, T_{k+1})\mathbb{E}_{\mathbb{Q}_{C,T_{k+1}}}[(L_C(T_k, T_k) - K)^+|\mathcal{G}_t]$$

and the price of the defaultable forward Libor rate cap at time $t \leq T_1$ is given as a sum

$$\mathbb{C}_t(K) = \sum_{k=1}^{n} \delta_{k-1} B_C(t, T_k)\mathbb{E}_{\mathbb{Q}_{C,T_k}}[(L_C(T_{k-1}, T_{k-1}) - K)^+|\mathcal{G}_t].$$