Utility theory front to back
Inferring preferences from agent's choices

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joint work with

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of Quantitative Finance

(Suppose a market is given in which an agent is investing and/or consuming)

Classical approach: specify agent's preferences (utility) and deduce her optimal behaviour

Inverse approach: given agent's choices infer her preferences

- are the choices compatible with classical utility maximisation?
- do they specify utility uniquely? is it easy to read off agent's characteristics from her actions?
- given agent's consumption, can we infer (the unique) investment strategy?
- etc

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(Some) Important references

- Stream of literature on revealed preferences including
  - Samuelson (1948), Dybvig (1983) (discrete time)
    Houthakker (1950), Richter (1966), Green, Lau and Polemarchakis (1978)
  - Wang (1993), Dybvig and Rogers (1997) (continuous time)
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Outline

Deterministic setup

Consider first a continuous time deterministic setup. Agent has initial wealth x which she consumes over time at a rate $c^*(t, w)$, where $w = w^*(t, x)$ is her remaining wealth at time t. Agent's wealth thus evolves as

$$\frac{\mathrm{d}}{\mathrm{d}t} w^*(t,x) = -c^*(t, w^*(t,x)), \quad w_0^*(x) = x. \tag{1}$$

Inverse approach: when is $c^*(t, w^*(t,x))$ optimal for:

$$v(x) = \sup_{\substack{c_t \geq 0, \\ \int_0^\infty c_t dt \leq x}} \int_0^\infty u(t, c_t)\, dt, \tag{2}$$

and what can we infer about the function u?

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$$\frac{\mathrm{d}}{\mathrm{d}t} w^*(t, x) = -c^*(t, w^*(t, x)), \quad w_0^*(x) = x. \tag{3}$$

Theorem

Suppose $c^*(t, 0) \equiv 0$, $c^*(t, w)$ is continuous and strictly increasing in w, $\int_0^\infty c^*(t, w^*(t, x)) = x$ and $\frac{\partial}{\partial x} c^*(t, w^*(t, x))$ exists and is > 0. Then there exists a function $u(t, c)$ such that $u'(t, c) \geq 0$ and $u''(t, c) \leq 0$, for which the problem:

$$v(x) = \sup_{\substack{c_t \geq 0: \\ \int_0^\infty c_t \, \mathrm{d}t \leq x}} \int_0^\infty u(t, c_t) \, \mathrm{d}t \tag{4}$$

is solved by $c_t = c^*(t, w^*(t, x))$ for each $x \geq 0$.

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Utility specification

Let $D(x) > 0$ be a function satisfying $\int_x^\infty D(z)\mathrm{d}z < \infty$, $x > 0$. Then we can define the utility u by:

$$u_c(t,c) = \int_{y(t,c)}^{\infty} D(z)\mathrm{d}z,$$

where $y = c^*(w^*(\cdot))^{-1}$ i.e. $y\big(t, c^*(t, w^*(t,x))\big) = x$.

In the problem we have no information about agent's comparison of different initial levels of wealth. This is encoded in the function D, which we are free to specify.

Risk attitudes are unspecified and two agents with the same consumption paths could have very different preferences.

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Risk aversion

Define the absolute risk aversion by

$$\gamma(t,c) = -\frac{u_c(t,c)}{u_{cc}(t,c)}.$$

Proposition

An agent has DARA iff $\gamma(t,\cdot)$ is decreasing, if and only if

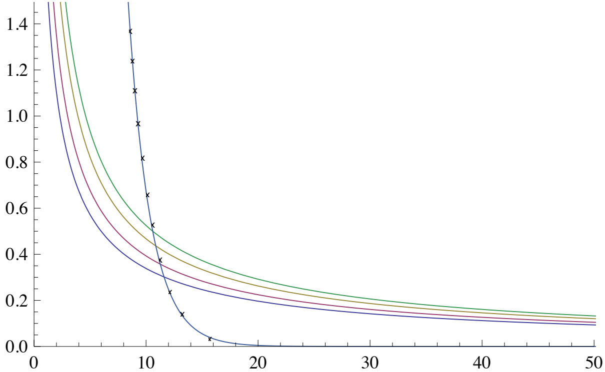
$$\frac{D'(x)}{D(x)} + \frac{D(x)}{\int_x^\infty D(y)\,\mathrm{d}y} \leq \inf_{t\geq 0} \frac{\frac{\partial^2}{\partial x^2} c_t^*(w_t^*(x))}{\frac{\partial}{\partial x} c_t^*(w_t^*(x))}, \quad x > 0.$$

while the agent has IARA if and only if:

$$\frac{D'(x)}{D(x)} + \frac{D(x)}{\int_x^\infty D(y)\,\mathrm{d}y} \geq \sup_{t\geq 0} \frac{\frac{\partial^2}{\partial x^2} c_t^*(w_t^*(x))}{\frac{\partial}{\partial x} c_t^*(w_t^*(x))}, \quad x > 0.$$

Examples

We have explicit time-homogenous and time in-homogenous examples of optimal consumption paths and both DARA and IARA utilities which generate them.



Wealth (solid) $w^*(t,x) = \frac{1}{t}W(tx)$ for $x = 10, 20, 50, 100$, corresponding to $c^*(t,w) = \frac{w^2}{1+tw}$.
Wealth (x—x) $w^*(t,x) = x\exp(-0.5t)$ for $x = 100$, corresponding to $c^*(t,w) = 0.5w$.

Outline

One-period setting

Consider a simple one-period setting. There is a unique investment opportunity which at time 1 yields $Y = \pm 1$, $\mathbb{P}(Y = 1) = p \in (\frac{1}{2}, 1)$. An agent, with initial capital x, decides on

- c – the initial consumption
- π – the investment.

Her total expected utility is given by

$$\mathbb{E}[u_0(c) + u_1(c_1)], \quad \text{where} \quad c_1 = x - c + \pi Y.$$

Classical approach: given u_0, u_1 increasing and concave, $-\infty$ on $\mathbb{R}_-$, we look for c^*, π^* which maximise the expected utility.

Inverse approach: given c, π do there exist u_0, u_1 for which c, π are the optimal ones?

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One-period setting: findings

Q1: Given c, π do there exist u_0, u_1 for which c, π solve

$$\max_{c',\pi'} \mathbb{E}[u_0(c') + u_1(c_1')], \quad \text{with} \quad c_1' = x - c' + \pi' Y \ ?$$

A1 : Given $c(x), \pi(x)$, $x \geq 0$, for one fixed $p = \mathbb{P}(Y = 1)$ there is an infinity of compatible pairs (u_0, u_1).

A2 : Given $c(x), \pi(x)$, $x \geq 0$, for two different values of $\mathbb{P}(Y = 1)$ a compatible pair (u_0, u_1) exists (and is typically unique) only under consistency conditions on agent's actions.

Note that there are many more ways we can twist the question and obtain answers in a similar fashion. E.g.

Q2 : Given $c(x, p)$ can we deduce unique $\pi(x, p)$ which is rational?

Q3 : Consider multi-period model. Given today's choices can we deduce (unique) rational future choice?

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Continuous time stochastic setting

Consider now a Black-Scholes market driven by a geometric Brownian motion

$$\mathrm{d}S_t = \sigma S_t(\mathrm{d}B_t + \theta \mathrm{d}t) + rS_t \mathrm{d}t$$

An agent choses her consumption c_t and investment π_t. Her wealth evolves as

$$\mathrm{d}W_t = rW_t\mathrm{d}t - c_t\mathrm{d}t + \pi_t\sigma(\mathrm{d}B_t + \theta\mathrm{d}t), \quad W_0 = x.$$

Classical approach: given utility function u, find c^*, π^* which solve

$$\sup_{(c_t,\pi_t)\in\mathcal{A}_u} \mathbb{E}\left[\int_0^\infty u(t, c_t)\mathrm{d}t\right],$$

where $\mathcal{A}_u = \{(c_t, \pi_t) : W_t \geq 0, \mathbb{E}\int_0^\infty u(t, c_t)^+\mathrm{d}t < \infty\}$.

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Inverse approach: given agent's choice of actions $c^*(t,w)$, $\pi^*(t,w)$, as function of time and her wealth, decide if they solve

$$\sup_{(c_t,\pi_t)\in\mathcal{A}_u} \mathbb{E}\left[\int_0^\infty u(t,c_t)\mathrm{d}t\right] \qquad (UMP_\infty)$$

and for what u?

Results of Black (1968) and He and Huang (1994)

They consider finite horizon problem

$$\sup_{(c_t,\pi_t)\in\mathcal{A}_u} \mathbb{E}\left[\int_0^T u_1(t,c_t)\mathrm{d}t + u_2(W_T)\right] \qquad (UMP_T).$$

Under fairly strong regularity and growth assumptions on c^* and π^* they show that c^*, π^* solve (UMP_T) if and only if

- they satisfy some consistency and state-independency conditions
- they solve Black's (1968) PDE

$$\pi_t^* + \frac{\sigma^2}{2}\pi^2\pi_{ww} + (rw - c^*)\pi_w^* + \pi^* c_w^* - r\pi^* = 0,$$

where $\pi_t^* = \frac{\partial}{\partial t}\pi^*$ and $\pi_w^* = \frac{\partial}{\partial w}\pi^*$.

Let ξ_t be the state price density, $\mathrm{d}\xi_t = \xi_t(-\theta \mathrm{d}B_t - r\mathrm{d}t)$. Let

$$A(t) = \left(\frac{\theta^2}{2} - r\right) t - \theta \int_0^t G(s)\mathrm{d}s, \;\; G(t) = \int_1^w \frac{\pi_t^*(t,m)}{\pi^*(t,m)^2}\mathrm{d}m + \frac{\sigma^2}{2}\pi_w^*(t,w) + \frac{c^*(t,w)}{\pi^*(t,w)} - r\frac{w}{\pi^*(t,w)}$$

$$F(t,w) = \mathrm{e}^{A(t)} \exp\left(-\theta \int_1^w \frac{\mathrm{d}m}{\pi^*(t,m)}\right) \quad \text{and} \quad y(t, c^*(t,w)) = w.$$

Under mild regularity and integrability assumptions on c^*, π^* we have

Theorem

Fix $x > 0$. Suppose c^*, π^* satisfy Black's (1968) PDE,

$$W_t^{c^*,\pi^*} \geq 0 \quad \textit{and} \quad \mathbb{E}\left[\int_0^\infty \xi_t c^*(t, W_t^{c^*,\pi^*})\mathrm{d}t\right] = x.$$

Then $(c^*, \pi^*) \in \mathcal{A}_u$ are optimal for (UMP_∞) with $u_c(t,c) := F(t, y(t,c))$.

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We deduce that:

- c^*, π^* have to satisfy a PDE and then u is given uniquely (up to a function of time).
- If we assume that $\pi^*(t, w) = \pi^*(w)$ then

$$c^*(t, w) = rw - \frac{\sigma^2}{2}\pi^*(w)\pi^*_w(w) + \eta(t)\pi^*(w),$$

  for some $\eta(t)$.
- Given $\pi^*(w)$ implies a unique (up to a constant η) rational choice of $c^*(t, w) = c^*(w)$ (and vice-versa).
- More generally, given $c^*(t, w)$, $\pi^*(0, w)$ and information about discounting implies a unique rational choice of $\pi^*(t, w)$.

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Time-homogenous actions

Assume that π^* and c^* are function of wealth only.

- If $\pi^*(w) = \phi w$ then $c^*(w) = \psi w$ and we get power utility.
- If $\pi^*(w) = \phi w^\alpha$ for $\alpha > 0$ then in fact $\alpha = 1$.
  More generally, any investment and consumption strategies coming from an (UMP_∞) must be linear in wealth for $w \to 0$ and $w \to \infty$.
- Agent has DARA if and only if

$$\frac{\pi^*_w(w)}{\pi^*(w)} \geq -\frac{c^*_{ww}(w)}{c^*_w(w)}, \quad w > 0.$$

- We can construct non-linear examples:

$$\begin{aligned} \pi^*(w) =& \phi w + \kappa(\sqrt{w+1} - 1) \\ c^*(w) =& (r + \eta\phi - \frac{\sigma^2\phi^2}{2})w + (\sqrt{w+1} - 1)(\kappa\eta - \frac{\sigma^2\phi\kappa}{2}) \\ &- \frac{\sigma^2}{2}\left(\frac{\phi\kappa}{2}\frac{w}{\sqrt{w+1}} + \frac{\kappa^2}{2}\frac{\sqrt{w+1}-1}{\sqrt{w+1}}\right). \end{aligned} \tag{5}$$

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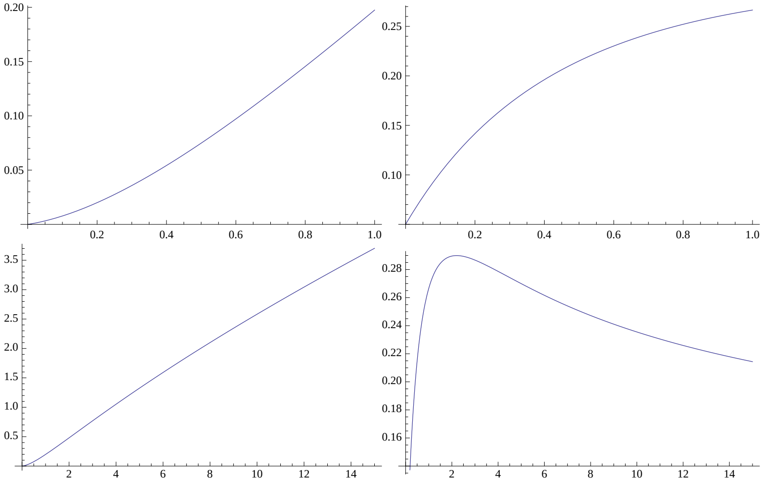
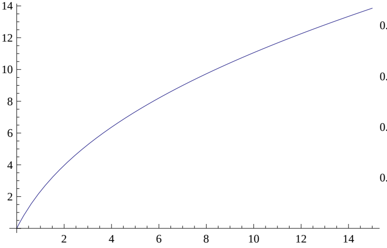
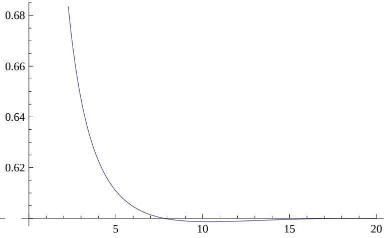


$c^*(w)$ and $c^*_w(w)$ for: $r = 0.05, \eta = \phi = 0.2, \sigma = 0.25$ and $\kappa = 6$.
Here $c^*_w(0) = 0.05$, achieves its maximum of around 2.29 for $w \approx 2.2$ and $c^*_w(w)$ then decreases to the limiting value $c^*_w(\infty) = 0.0875$.

Preferences have DARA.

Consider $\pi^*(w) = \phi w + \kappa \left((w+1)^{1/4} - 1 \right)$.





Optimal consumption   Absolute risk aversion

for: $r = 0.05, \theta = 0.5, \eta = 0.2, \phi = 0.04, \sigma = 0.25$ and $\kappa = 6$.

Here preferences do not have DARA.

Beyond Black's PDE

What about agent's strategies which do not satisfy Black's PDE?

We show they can be seen as solutions to a more general problem of

$$\sup_{\pi_t, c_t} \mathbb{E}\left[\int_0^\infty \left(u(t, c_t) + U(t, W_t)\right) \mathrm{d}t\right].$$

Functions u, U are essentially determined up to a specification of discount factor $A(t)$.

Conclusions and questions

- We propose to take agent's actions as input and deduce her preferences and/or their important properties. We are interested in when this can be done and whether the preferences are given uniquely
- In a deterministic setup agents with very different preferences can have the same consumption paths
- In a one-period setting both situations (under- and over-specification) are possible
- In a BS market strategies have to satisfy a PDE. Time-homogenous strategies solved explicitly. They have to be linear in wealth for $w \longrightarrow 0$ or $w \longrightarrow \infty$.
- More general strategies can be mapped to a more general problem.

- Further analysis of discrete time setup?
- Further examples? Best set of assumptions for BS market?
- Incomplete markets? Case study suggests a picture BS-like.

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THANK YOU!