Prices and sensitivities of barrier and first touch digital options in Lévy-driven models, near barrier

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Objectives

We study the asymptotics of the price of barrier and first-touch digital options near the barrier, under several classes of Lévy processes. Reasons:

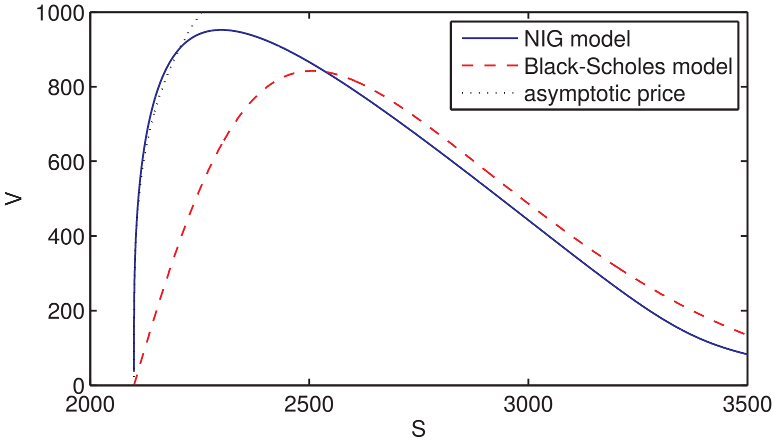
I. Interesting in its own right: the region near the boundary is where it is most difficult to obtain a good estimate of the price with any numerical method. In particular, the results show that, for some processes, the price can have a large discontinuity at the barrier.

II. By comparing the (exact) asymptotic prices with with those produced by various numerical methods, it is possible to check the accuracy of these methods.

Results

1. Classification of the possible shapes of the barrier price near the boundary for several classes of Lévy processes.
2. Comparison with numerical prices calculated with the FFT-based technique of S.Boyarchenko and Levendorskiĭ (2002) and Boyarchenko and Levendorskiĭ (2008) (henceforth the BBL methodology), shows that the latter is extremely accurate near the barrier, where it produces very different results from HEJD approximations.
3. We use a modification of the BBL algorithm to calculate directly the option's delta. Comparison with the delta obtained by numerical differentiation of the BBL price shows that the latter is very accurate, even near the barrier.

Barrier option price comparison



The price of a down-and-out put option in the NIG and Black-Scholes models (parameters from Jeannin and Pistorius, 2007). The strike is $K = 3500$, the barrier is $H = 2100$, the time to maturity is $T = 0.25$ years, the riskless rate is 3%, and the underlying stock pays no dividends.

Outline

Outline

Lévy processes: general definitions

Every Lévy process $X = (X_t)$ has a characteristic exponent ψ, defined by

$$E\left[e^{i\xi X_t}\right] = e^{-t\psi(\xi)},$$

where $\xi \in \mathbb{R}$. In 1D, the Lévy-Khintchine formula gives an expression for ψ

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{\mathbb{R}\setminus\{0\}} (1 - e^{ix\xi} + ix\xi \mathbb{1}_{|x|<1}(x))F(dx),$$

where $F(dx)$, the Lévy measure, satisfies $\int_{\mathbb{R}\setminus\{0\}} \min\{|x|^2, 1\}F(dx) < \infty$.

Lévy processes: general definitions

E.g. for a KoBoL (a.k.a. CGMY) process, $F(dx)$ takes the form

$$F(dx) = c_+ x^{-\nu-1} e^{\lambda_- x} \mathbb{1}_{\{x>0\}}\, dx + c_- |x|^{-\nu-1} e^{\lambda_+ |x|} \mathbb{1}_{\{x<0\}}\, dx, \quad \nu \in [0,2). \quad (1)$$

Other model classes: Normal Tempered Stable, with $\nu \in (0,2)$ (the $\nu = 1$ case being Normal Inverse Gaussian), Generalized Hyperbolic, β-family, etc.

The Lévy density has asymptotic form $O(|x|^{-\nu-1})$ for $x \to 0$, where ν is the *order* of the process. The process is of finite (infinite) variation if $\nu < 1$ ($\nu \geq 1$).

Outline

The asymptotics near the boundary

1. Down-and-out barrier option: as $x = \ln S \downarrow h = \ln H$,

$$V(T,x) = \kappa(T)(x-h)^{\nu_-} + O((x-h)^{\nu_- + s}),$$

where $s > 0$.

2. Down-and-in first-touch digital option (pays 1 as S crosses H from above)

$$V(T,x) = 1 - \kappa_{\mathrm{f.t.}}(T)(x-h)^{\nu_-} + O((x-h)^{\nu_- + s}),$$

where $s > 0$.

The value of ν_- depends on the type, order and drift of the process.

The main qualitatively different shapes near the boundary

1. Processes with non-trivial BM component, or infinite-activity processes with finite variation (except VG) and drift pointing towards the boundary have $\nu_- = 1$, and the price near the barrier looks similar to the BM case;

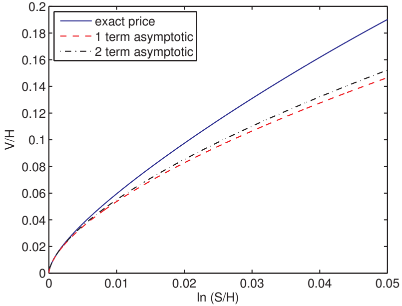
2. for infinite activity and infinite variation processes, $\nu_- \in (0, \nu)$. In particular, if the Lévy measure has asymptotic form $c|x|^{-\nu-1}$ as $x \to 0$, then $\nu_- = \nu/2$;

3. NIG: depending on the drift, ν_- can take any value in $(0, 1)$;

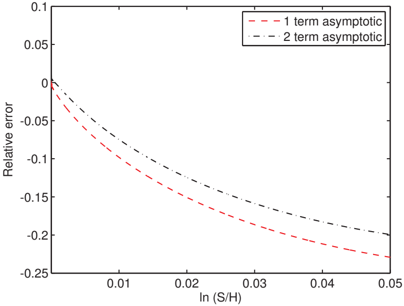
4. KoBoL processes of finite variation (incl. VG) with the drift pointing away from the barrier: $\nu_- = 0$, so that the prices of the down-and-out barrier options and first-touch digitals are discontinuous at the boundary.

Outline

Numerical example: DO put price (KoBoL, $1 \leq \nu < 2$)



(a) Price

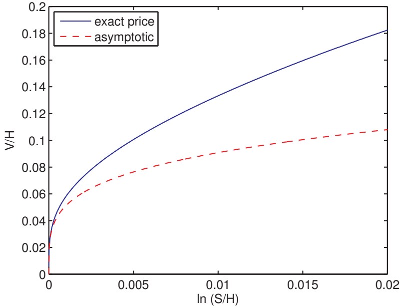


(b) Relative Error

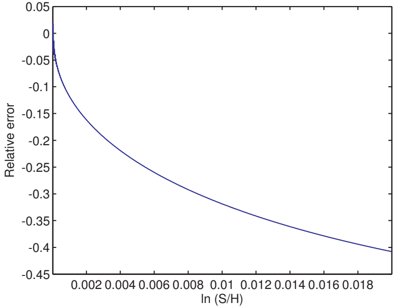
Infinite variation KoBoL with $\nu = 1.25$, $\lambda_- = -9$, $\lambda_+ = 8$, $c = 0.15$, $\mu = 0.03$, riskless rate $r = 0.03$, $T = 0.25$. The option parameters are the same as in the previous graph.

Here and in the following examples, the "exact" price is the one obtained by the BBL method.

Numerical example: DO put price (KoBoL, $\nu < 1$, $\mu = 0$)



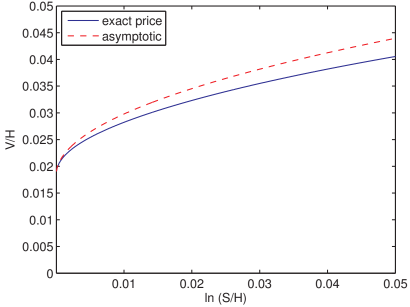
(a) Price



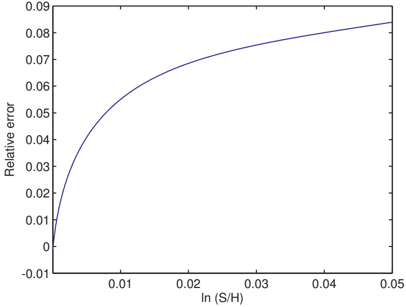
(b) Relative Error

Finite variation KoBoL with $\nu = 0.5$, $\lambda_- = -8$, $\lambda_+ = 9$, $c = 1$, $\mu = 0$, riskless rate $r \approx 0.072309571491738$, $T = 0.25$.

Numerical example: DO put price (KoBoL, $\nu < 1$, $\mu > 0$)



(a) Price

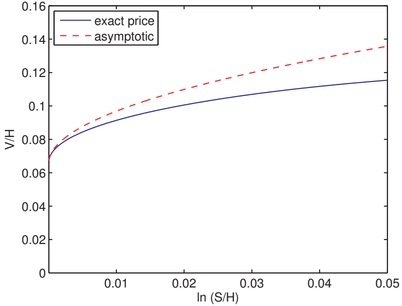


(b) Relative Error

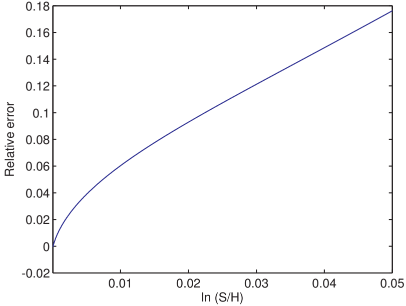
Finite variation KoBoL with positive drift. Example from Asmussen, Madan and Pistorius (2008), calibrated to vanilla options on Ford: $\nu = 0.5$, $\lambda_- = -11.0187$, $\lambda_+ = 1.9458$, $c = 0.6506$, $\mu \approx 0.4356$, riskless rate $r = 0.03$, $T = 1.5$.

This situation often arises from calibration to market data. Note the large discontinuity at the boundary. This would not be present if we approximated the process by a jump-diffusion.

Numerical example: DO put price (KoBoL, $\nu < 1$, $\mu > 0$)



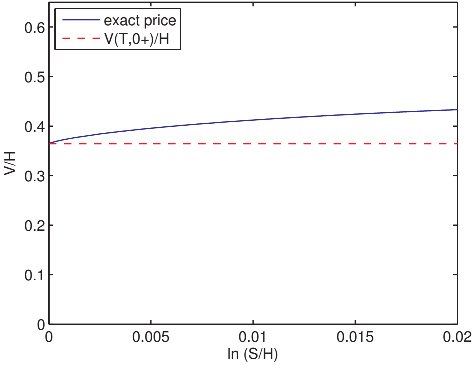
(a) Price



(b) Relative Error

Finite variation KoBoL with positive drift. Example from Asmussen, Madan and Pistorius (2008), calibrated to vanilla options on General Motors: $\nu = 0.5$, $\lambda_- = -5.8031$, $\lambda_+ = 1.0084$, $c = 0.2171$, $\mu \approx 0.2006$, riskless rate $r = 0.03$, $T = 1.5$. Again, we have a discontinuity at the barrier.

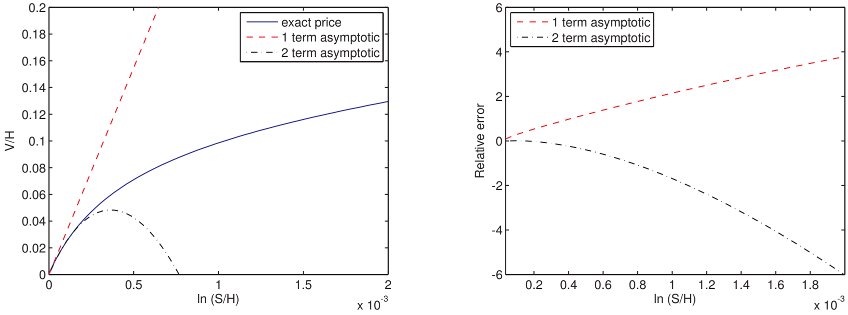
Numerical example: DO put price (VG, $\mu > 0$)



Finite variation KoBoL of order 0 (VG) with positive drift. Example from Jeannin and Pistorius (2007), calibrated to market prices of vanilla options on Stoxx50E: $\lambda_- = 11.876$, $\lambda_+ = 4.667$, $c = 0.925$, $\mu \approx 0.1282$, riskless rate $r = 0.03$, $T = 0.25$.

For VG we only calculated the leading term, which corresponds to the size of the gap at the barrier.

Numerical example: DO put price (KoBoL, $\nu < 1$, $\mu < 0$)



(a) Price (b) Relative Error

Finite variation KoBoL with negative drift. Example from Crosby, Le Saux and Mijatović (2009): $\nu = 0.25$, $\lambda_- = -8$, $\lambda_+ = 9$, $c = 1$, $\mu \approx -0.0140$, riskless rate $r = 0.03$, $T = 0.25$. For $\nu < 1$ and negative drift, the behaviour near the barrier is similar to the BM case.

Outline

Normalized EPV operators

Recall that the supremum and infimum processes are defined by

$$\overline{X}_t = \sup_{0 \le s \le t} X_s, \quad \underline{X}_t = \inf_{0 \le s \le t} X_s.$$

We define the normalized expected present value (EPV) operators by

$$(\mathcal{E}_q f)(x) := q\mathbb{E}\left[\int_0^{+\infty} e^{-qt} f(x + X_t) dt\right]$$

$$(\mathcal{E}_q^+ f)(x) := q\mathbb{E}\left[\int_0^{+\infty} e^{-qt} f(x + \overline{X}_t) dt\right],$$

$$(\mathcal{E}_q^- f)(x) := q\mathbb{E}\left[\int_0^{+\infty} e^{-qt} f(x + \underline{X}_t) dt\right].$$

$\mathcal{E}_q f$, $\mathcal{E}_q^+ f$ and $\mathcal{E}_q^- f$ have a natural financial interpretation as q times the expected present value of the payment streams $f(x + X_t)$, $f(x + \overline{X}_t)$ and $f(x + \underline{X}_t)$ with constant discount rate q.

Wiener-Hopf factorization formula

Three versions:

1. Let $T_q \sim \mathrm{Exp}(q)$ be the exponential random variable of mean q^{-1}, independent of the process X. For $\xi \in \mathbb{R}$,

$$\mathbb{E}[e^{i\xi X_{T_q}}] = \mathbb{E}[e^{i\xi \overline{X}_{T_q}}]\mathbb{E}[e^{i\xi \underline{X}_{T_q}}];$$

2. For $\xi \in \mathbb{R}$,

$$\frac{q}{q+\psi(\xi)} = \phi_q^+(\xi)\phi_q^-(\xi),$$

   where $\phi_q^\pm(\xi)$ admits the analytic continuation into the corresponding half-plane and does not vanish there

3. $\mathcal{E}_q = \mathcal{E}_q^- \mathcal{E}_q^+ = \mathcal{E}_q^+ \mathcal{E}_q^-$.

3 is valid in appropriate function spaces, and can be either proved as 1 or deduced from 2 because $\mathcal{E}_q = q(q+\psi(D))^{-1}$, $\mathcal{E}_q^\pm = \phi_q^\pm(D)$.

The down-and-out European option price

The following result was proven in S.Boyarchenko and Levendorskiĭ, (2002), Boyarchenko and Levendorskiĭ (2008).

Theorem

Let G be a function satisfying certain regularity conditions. Then the Laplace transform w.r.t. T of the price of a DO European option with maturity T, payoff G and log-barrier h is

$$\hat{V}(q,x) = (q+r)^{-1}\mathcal{E}^{-}_{q+r}\mathbb{1}_{(h,+\infty)}\mathcal{E}^{+}_{q+r}G(x), \tag{2}$$

where r is the risk free rate.

A similar result holds for first-touch digital options.

The down-and-out European option price

The time 0 price of a DO option with maturity T and payoff G is given by the following integral

$$V(x, T; G) = \frac{e^{-rT}}{2\pi i(-T)^k} \int_{\mathrm{Re}\, q=\sigma} e^{qT} \partial_q^k \left(q^{-1} \mathcal{E}_q^- \mathbb{1}_{(h,+\infty)} \mathcal{E}_q^+ G\right)(x) dq,$$

for some $k \in \mathbb{Z}_+$, $\sigma > 0$.

Outline

The form of the asymptotic coefficient

The $\phi_q^{\pm}(\xi)$ have the following asymptotic form as $\xi \to \infty$ in the corresponding half-plane:

$$\phi_q^{\pm}(\xi) = \phi_{q,\infty}^{\pm}(1 \mp i\xi)^{-\nu_{\pm}}(1 + O(|\xi|^{-\rho})),$$

where $\rho > 0$, $\nu_{\pm} \in [0, \nu]$. The $\nu_{\pm}$ are determined by ν and the drift μ.

In the same way as ν_-, ν_+ determines the asymptotics of the price near the boundary for up-and-out barrier options and up-and-in first-touch digitals.

The form of the asymptotic coefficient

We prove that there exists a $\sigma > 0$ such that $\kappa(T)$ for a down-and-out barrier option with payoff function G is given by

$$\kappa(T) = \frac{e^{-rT}}{2\pi i \Gamma(1+\nu_-)(-T)} \int_{\mathrm{Re}\, q=\sigma} e^{qT} \partial_q (q^{-1} \phi^-_{q,\infty} \tilde{G}(q, h+0)) dq,$$

where the integral is absolutely convergent for $\mathrm{Re}\, q \geq \sigma$.

The functions $q \mapsto \tilde{G}(q, h+0) = \mathcal{E}^+_q G(h+0)$ and $q \mapsto \phi^{\pm}_{q,\infty}$ are analytic for $\mathrm{Re}\, q \geq \sigma$.

Explicit analytical formulas for $\phi^{\pm}_{q,\infty}$ and $\tilde{G}(q, h+0)$ were derived, for several model processes.

For some processes, expressions for the second term of asymptotics were obtained as well.

Outline

Calculation of Sensitivities

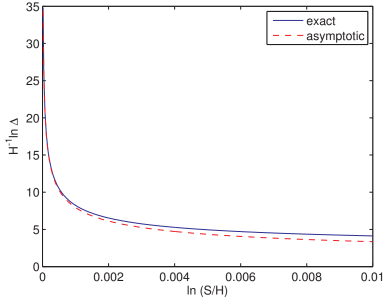
de Innocentis and Levendorskiĭ (work in progress).

Objective: investigate the accuracy of numerical differentiation for the results of the BBL pricing method.

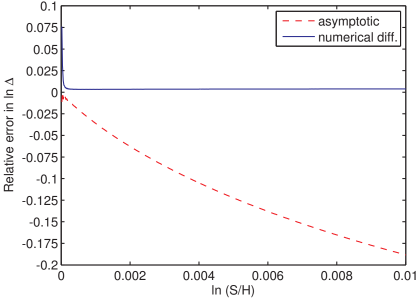
We only look at the case of delta.

If the payoff G satisfies some regularity conditions, one can differentiate step by step in the algorithm for the price, and obtain the corresponding value of the log-delta without using numerical differentiation (by using the regularity of G and certain properties of the EPV operators as PDOs).

Numerical example: DO put delta (KoBoL, $1 \leq \nu < 2$)



(a) Log-delta



(b) Relative Error

Infinite variation KoBoL with $\nu = 1.25$, $\lambda_{-} = -9$, $\lambda_{+} = 8$, $c = 0.15$, $\mu = 0.03$, riskless rate $r = 0.03$, $T = 0.25$. The relative error of numerical vs. "exact" log-delta is less than 0.3%, outside of a tiny region near the barrier. Even where the price seems fairly regular, the delta is not (gamma is even worse).

(Convergence of the asymptotic coefficient for the sensitivities, in the $\nu > 1$ case, was proven in Levendorskiĭ, 2009).

Conclusions

1. The study of the asymptotic price for barrier options and first-touch digitals, and the comparison with the BBL price, shows that the latter is extremely accurate, even near the barrier, where other methods - e.g. approximation by a jump-diffusion, Monte Carlo simulation - tend to be less reliable.
2. It also shows that, for some process parameters, we can have a large discontinuity at the barrier (including examples obtained by fitting to market data).
3. The delta of the option can be calculated with a modification of the algorithm for the price.
4. Comparison with the delta obtained by numerical differentiation of the BBL price shows that the latter is very accurate, even near the barrier.