Pricing Bermudan options in Lévy process models

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American options in Lévy models

- American options in the **Black-Scholes-Merton** model: Broadie and Detemple (1996)
- **Lévy models**: fit empirical financial data better, explain volatility smiles
- **Bermudan options** (discrete American): can be exercised at any time in a discrete set
- **American options**: increasing monitoring frequency
- **Path dependent** options: Bermudan knock-out barrier, lookback options

Optimal stopping and backward induction

- **Discrete optimal stopping** (for Bermudan puts)

$$V^0(S_0) = \sup_{\tau} \mathbb{E}[e^{-r\tau}(K - S_\tau)^+]$$

where $S_t = S_0 e^{X_t}$, X_t: a Lévy process, τ: stopping time that takes value in $\{0, \Delta, \cdots, N\Delta\}$

- **Change of variable** $X_t = \ln(S_t/K)$, $x = \ln(S/K)$
- **Backward induction**

$$f^N(x) = g(x) = K(1 - e^x)^+$$

$$f^j(x) = \max(g(x), e^{-r\Delta}\mathbb{E}_{j\Delta,x}[f^{j+1}(X_{(j+1)\Delta})]$$

Literature

- Need to compute $\mathbb{E}_{j\Delta,x}[f^{j+1}(X_{(j+1)\Delta})]$
- Longstaff & Schwartz (2001): **least square monte carlo**
- Broadie & Yamamoto (2005): **double exponential fast Gauss transform**
- Fang & Oosterlee (2008): **Fourier cosine series** expansion of the transition density
- Kellezi & Webber (2004): **lattice** approximation of the transition density
- Jackson, Jaimungal & Surkov (2008): conditional expectation is a convolution, its Fourier transform is a product; $f^{j+1} \rightarrow$ **FT of f^{j+1}** $\rightarrow$ multiplied by c.f. $\rightarrow$ FI representation of the conditional expectation $\rightarrow$ take max $\rightarrow f^j \rightarrow$ **FT of f^j**

Hilbert transform method

- **Hilbert transform** of $f \in L^1(\mathbb{R})$

$$\mathcal{H}f(x) = \frac{1}{\pi}\; p.v. \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy$$

- For any $f \in L^1(\mathbb{R})$ with $\hat{f} \in L^1(\mathbb{R})$ ($\hat{f}$: Fourier transform of f)

$$\mathcal{F}(1_{(l,\infty)} \cdot f)(\xi) = \frac{1}{2}\hat{f}(\xi) + \frac{i}{2} e^{i\xi l} \mathcal{H}(e^{-i\eta l}\hat{f}(\eta))(\xi)$$

- Bermudan put: $\exists x_j^* < K$ ($S_0 e^{x_j^*}$ **early exercise boundary**)

$$f^j(x) = g(x) \cdot \mathbf{1}_{(-\infty, x_j^*]}(x) + e^{-r\Delta} \mathbb{E}_{j\Delta,x}[f^{j+1}(X_{(j+1)\Delta})] \cdot \mathbf{1}_{(x_j^*,\infty)}(x)$$

Integrability and Esscher transform

- **Exponential dampening** for integrability: for certain $\alpha > 0$

$$f_\alpha^j(x) = e^{\alpha x} f^j(x) \in L^1(\mathbb{R})$$

- **Esscher transform**: Radon-Nikodým derivative

$$\frac{d\mathbb{P}^\alpha}{d\mathbb{P}}|_{\mathcal{F}_t} = Z_t = e^{-\alpha X_t}/\phi_t(i\alpha)$$

where ϕ_t: characteristic function of X_t

$$e^{\alpha x}\mathbb{E}_{j\Delta,x}[f^{j+1}(X_{(j+1)\Delta})] = \phi_\Delta(i\alpha)\mathbb{E}^\alpha_{j\Delta,x}[f_\alpha^{j+1}(X_{(j+1)\Delta})]$$

- Esscher transformed Lévy process is still a Lévy process with c.f. $\phi_t^\alpha(\xi) = \phi_t(\xi + i\alpha)/\phi_t(i\alpha)$

Backward induction in Fourier space

- From **convolution** theorem, Fourier transform of

$$\mathbb{E}^{\alpha}_{j\Delta,x}[f^{j+1}_{\alpha}(X_{(j+1)\Delta})] = \int_{\mathbb{R}} f^{j+1}_{\alpha}(y) p^{\alpha}_{\Delta}(y-x)dy \quad \text{is} \quad \hat{f}^{j+1}_{\alpha}(\xi)\phi^{\alpha}_{\Delta}(-\xi)$$

- **Backward induction in Fourier space**

$$\hat{f}^{N}_{\alpha}(\xi) = \hat{g}_{\alpha}(\xi)$$

$$\hat{f}^{j}_{\alpha}(\xi) = \mathcal{F}(g_{\alpha} \cdot \mathbf{1}_{(-\infty,x^*_j]})(\xi) + e^{-r\Delta}\left(\frac{1}{2}\hat{f}^{j+1}_{\alpha}(\xi)\phi_{\Delta}(-\xi+i\alpha)\right.$$
$$\left. + \frac{i}{2}e^{i\xi x^*_j}\mathcal{H}(e^{-i\eta x^*_j}\hat{f}^{j+1}_{\alpha}(\eta)\phi_{\Delta}(-\eta+i\alpha))(\xi)\right)$$

$$f^{0}_{\alpha}(x) = \max(g_{\alpha}(x), \frac{1}{2\pi}e^{-r\Delta}\int_{\mathbb{R}} e^{-i\xi x}\hat{f}^{1}_{\alpha}(\xi)\phi_{\Delta}(-\xi+i\alpha)d\xi)$$

Early exercise boundary

- **Early exercise boundary** x_j^* solves

$$g_\alpha(x) = e^{-r\Delta}\phi_\Delta(i\alpha)\mathbb{E}^{\alpha}_{j\Delta,x}[f_\alpha^{j+1}(X_{(j+1)\Delta})]$$

- **Fourier inverse** representation

$$g_\alpha(x) = \frac{1}{2\pi}e^{-r\Delta}\int_{\mathbb{R}} e^{-i\xi x}\hat{f}_\alpha^{j+1}(\xi)\phi_\Delta(-\xi + i\alpha)d\xi$$

- To solve for x_j^*, use **root finding** solver (e.g., Newton-Raphson), with starting point x_{j+1}^* ($x_N^* = K$)

Discrete approximation

- **Discrete Hilbert transform** with step size $h > 0$

$$\mathcal{H}_h f(x) = \sum_{m=-\infty}^{\infty} f(mh) \frac{1 - \cos[\pi(x - mh)/h]}{\pi(x - mh)/h}, \quad x \in \mathbb{R}$$

- For f analytic in a horizontal strip $\{z \in \mathbb{C} : |\Im(z)| < d\}$

$$||\mathcal{H}f - \mathcal{H}_h f||_{L^\infty(\mathbb{R})} \leq \frac{Ce^{-\pi d/h}}{\pi d(1 - e^{-\pi d/h})}$$

- Fourier inverse integrals: **trapezoidal rule**

$$\left| \int_{\mathbb{R}} f(x)dx - \sum_{m=-\infty}^{\infty} f(kh)h \right| \leq \frac{Ce^{-2\pi d/h}}{1 - e^{-2\pi d/h}}$$

Error estimate

- **Discretization error** $\sim O(\exp(-\pi d/h)$
- Truncate infinite sums with truncation level M. With $\phi_t(\xi) \sim \exp(-ct|\xi|^\nu)$, **truncation error** is essentially

$$O(\exp(-\Delta c(Mh)^\nu))$$

- Select $h = h(M)$ according to

$$h(M) = \left(\frac{\pi d}{\Delta c}\right)^{\frac{1}{1+\nu}} M^{-\frac{\nu}{1+\nu}}$$

- Total error: $O(\exp(-CM^{\frac{\nu}{1+\nu}}))$

FFT and computational cost

- Evaluate

$$\mathcal{H}f(\xi) \Leftarrow \sum_{m=-M}^{M} f(mh) \frac{1-\cos[\pi(\xi-mh)/h]}{\pi(\xi-mh)/h}$$

  for $\xi = -Mh, \cdots, Mh$
- Corresponds to Toeplitz matrix vector multiplication
- FFT based method for such multiplications: $O(M\log(M))$
- Fourier inverse integrals: $O(M)$
- Total computational cost of the method: $O(NM\log(M))$

Bermudan put in the NIG model

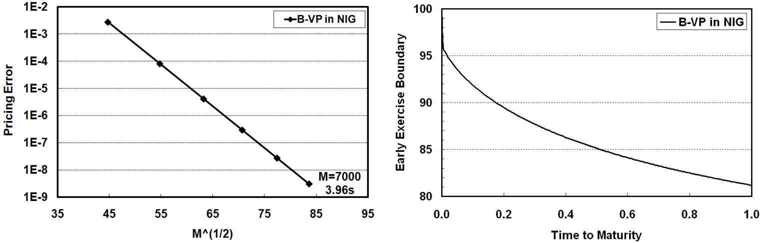


Figure: $T = 1$, **N=252**, $S_0 = 100, K = 100, r = 5\%, q = 2\%, \alpha_{NIG} = 15, \beta_{NIG} = -5, \delta_{NIG} = 0.5$, Matlab R2009a, Lenovo T400 Laptop with 2.53GHz CPU, 2G RAM; average number of NR iterations per time step 4.08

Bermudan barrier/lookback options

- **Bermudan barrier** options

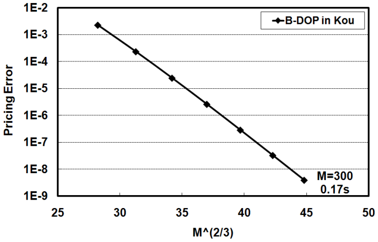
$$f^j(x) = \mathbf{1}_{(l,u)}(x) \cdot \Big(g(x) \cdot \mathbf{1}_{(-\infty, x_j^*]}(x)$$

$$+ e^{-r\Delta} \mathbb{E}_{j\Delta,x}[f^{j+1}(X_{(j+1)\Delta})] \cdot \mathbf{1}_{(x_j^*,\infty)}(x) \Big)$$

- **Bermudan** floating strike **lookback** options: standard backward induction involves two state variables: asset price, maximum asset price
- Can be reduced to one state variable, maximum asset price/asset price

$$f^j(y) = \max(e^y - 1, e^{-q\Delta} \mathbb{E}^*_{j\Delta,y}[f^{j+1}(e^{Y_{(j+1)\Delta}})])$$

Bermudan down-and-out put in Kou's model



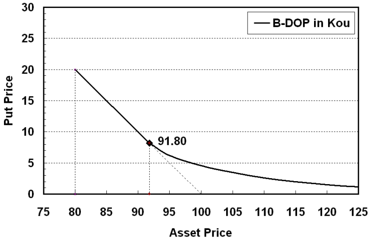


Figure: $T = 1$, **N=252**, $S_0 = 100, K = 100, L = 80, r = 5\%, q = 2\%, \sigma = 0.1, \lambda = 3, p = 0.3, \eta_1 = 40, \eta_2 = 12$, Matlab R2009a, Lenovo T400 Laptop with 2.53GHz CPU, 2G RAM

- $O(1/N)$ convergence of Bermudan options to American options in BSM (Howison (2007))
- **Richardson extrapolation**: from two approximations P_1 with N_1 and P_2 with N_2

$$P_\infty \approx \frac{N_1 P_1 - N_2 P_2}{N_1 - N_2}$$

N	B-VP in NIG	B-VP in BSM	Extrap
5	6.411	6.58462398	
10	6.451	6.62146556	6.65831
20	6.471	6.64073760	6.66001
40	6.481	6.65061811	6.66050
80	6.486	6.65562807	6.66064
160	6.489	6.65815199	6.66068
320	6.490	6.65941858	6.66069
640	6.491	6.66005274	6.66069

Table: American vanilla puts in the NIG model and BSM model.

Summary

- Hilbert transform method for pricing Bermudan style options in Lévy process models
- Accurate with exponentially decaying errors
- Fast with computational cost $O(NM\log(M))$
- Early exercise boundary also obtained
- American options valuation