

# Carr-Wiener-Hopf method and refined fast Fourier transforms for pricing barrier options

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# Overview

We price knock-out options with one or two barriers in a regime switching Lévy model modulated by a continuous-time finite-state Markov chain.

The class of Lévy processes we allow is rather general, and includes hyper-exponential jump-diffusions (of which Kou's model is a special case), the V.G. model, NIG processes, KoBoL/CGMY, and the  $\beta$ -class.

We allow an arbitrary number of states in the modulating Markov chain, so we can approximate models with stochastic volatility and interest rates.

We use a combination of techniques, which we collectively refer to as the “BBL method” (the term “BoyarLeven methodology” appeared in some other works, but we prefer the former one since it is shorter).

# Ingredients in the BBL method

The dates refer to the first time each method was applied in option pricing:

1. Analytic method of lines (P. Carr & D. Faguet 1996), or, in a probabilistic interpretation, Carr's randomization (P. Carr 1998)
2. Operator form of the Wiener-Hopf factorization (S. Boyarchenko and S. Levendorskiĭ, 1998) and its interpretation in terms of expected present value operators (S. Boyarchenko and S. Levendorskiĭ, 2002)
3. Finite element method (A. Eydeland, 1994)
4. Refined fast Fourier transforms techniques (M. Boyarchenko and S. Levendorskiĭ, 2008) – required for processes whose characteristic exponent is not rational, for example, V.G., NIG, KoBoL/CGMY

# Advantages from the viewpoint of speed

- Option prices are calculated for a (rather fine) uniformly spaced grid of initial log-spot prices (as opposed to one initial spot price).
- This allows us to calculate the deltas and gammas of the option at the points of the same grid using numerical differentiation.
- The prices and sensitivities corresponding to log-spot prices that do not lie on the grid are found using interpolation (the additional computational cost of interpolation is negligible).
- Even for a single spot price, the BBL method performs faster (sometimes, 5–10 times faster) than the competing algorithms.

# Example

**Table:** Prices of double barrier foreign exchange options in a regime-switching HEJD model with two states and 4 double exponential summands in each state

Initial state	1	2	1	2
Transition rates	1.85/1.1	1.85/1.1	10/10	10/10
Vanilla call	4.40036	4.00855	4.40383	4.33705
Vanilla put	2.84040	2.52717	2.81875	2.76598
Digital call	0.45971	0.43212	0.45313	0.44872
Digital put	0.38852	0.36614	0.38332	0.37970

Option parameters (same for both states):  $S_0 = 220$ ,  $r_{dom} = 0.046$ ,  $r_{for} = 0.051$ ,  $L = 195$ ,  $U = 250$ ,  $T = 0.9$ ,  $K = 218$ . Rebate:

$R_L = R_U = 0.25$ , paid at maturity if either barrier is crossed.

The parameters are taken from Ambrose, Carr and Crosby (2009).

The calculation of each option price took  $\approx 1$  second.

Table: Prices obtained by Ambrose, Carr and Crosby in the same setup

Initial state	1	2	1	2
Transition rates	1.85/1.1	1.85/1.1	10/10	10/10
Vanilla call	4.39899	4.00910	4.40469	4.33799
Vanilla put	2.83956	2.52770	2.81620	2.76183
Digital call	0.46099	0.43320	0.45554	0.45108
Digital put	0.38718	0.36507	0.38284	0.37919

Table: Relative differences between our prices and A-C-C prices

Initial state	1	2	1	2
Transition rates	1.85/1.1	1.85/1.1	10/10	10/10
Vanilla call	0.00031	-0.00014	-0.00020	-0.00022
Vanilla put	0.00030	-0.00021	0.00091	0.00150
Digital call	-0.00278	-0.00249	-0.00529	-0.00523
Digital put	0.00346	0.00293	0.00125	0.00135

# Advantages from the viewpoint of accuracy

- The BBL method is numerically stable (no high precision arithmetic).
- When the underlying process has infinite jump activity, the competing methods replace it with a finite jump activity process. This leads to significant errors in the approximation near the barrier(s).
- By contrast, Carr's randomization approximation (where the underlying process remains unchanged) exhibits the correct asymptotic behavior near the boundary (Levendorskii, 2009).

# Outline of the remainder of the talk

- 1 Specification of the model and of the options
- 2 Carr's randomization: theory and implementation
- 3 Wiener-Hopf factorization and expected present value operators
- 4 Calculation of the action of EPV operators
- 5 Calculation of the Wiener-Hopf factors
- 6 Application of the finite element method
- 7 Application of refined FFT techniques
- 8 References
- 9 Appendix



# Double barrier options in regime-switching models

- For simplicity we consider a constant riskless rate  $r > 0$ .
- $\mathbb{X} = (Y_t, X_t)_{t \geq 0}$  temporally homogeneous Markov process with state space  $\{1, 2, \dots, m\} \times \mathbb{R}$ , constructed using pairwise independent Lévy processes  $X^{(j)} = (X_t^{(j)})_{t \geq 0}$  ( $1 \leq j \leq m$ ) and a continuous time Markov chain  $Y = (Y_t)_{t \geq 0}$  defined by transition rates  $(\lambda_{jk})_{1 \leq j \neq k \leq m}$ .
- Price of the underlying:  $S_t = \exp(c_{Y_t} X_t + d_{Y_t})$  ( $c_j > 0$ ,  $d_j \in \mathbb{R}$ ).
- In each state  $1 \leq j \leq m$ , have log-barriers  $-\infty < h_-^j < h_+^j < \infty$  and terminal payoff function  $g^j(x)$  defined for  $h_-^j < x < h_+^j$ .
- The option expires worthless if for some  $t \leq T = \text{maturity}$  we have  $X_t \notin (h_-^j, h_+^j)$ , where  $j = Y_t$ . Otherwise at  $t = T$  the owner of the option receives payoff  $g^j(e^{X_t})$ , where  $j = Y_T$ .

# Carr's randomization

- Carr's randomization (a.k.a. "Canadization") was originally discovered (P. Carr, 1998) as a probabilistic interpretation of the "analytic method of lines" (used by P. Carr and D. Faguet, 1996).
- Carr proposed to approximate a finite-lived option pricing problem by replacing a deterministic maturity date  $T$  with a suitably chosen random maturity date whose mean is equal to  $T$ .
- When this random maturity is a sum of independent exponentially distributed maturity dates, the new pricing problem often reduces to a sequence of perpetual pricing problems, which are easier to solve.
- We believe that this idea has a very wide scope of applications. For the time being, the efficiency of Carr's randomization for American and barrier options has been well documented in the literature.

# Carr's randomization for barrier options: setup

- We will write  $\vec{h}_{\pm} = (h_{\pm}^j)_{j=1}^m$  and  $\vec{g} = (g^j)_{j=1}^m$ .
- $V_j(x, T; \vec{h}_{\pm}; \vec{g})$  = value function of the option in state  $j$  (the no-arbitrage price of the option above assuming  $\mathbb{X}$  starts at  $(j, x)$ ).
- $v_j(x; q; \vec{h}_{\pm}; \vec{g})$  = value function of the knock-out continuous cash flow  $\{e^{-qt} g^{Y_t}(X_t)\}_{t \geq 0}$  in state  $j$ , where  $q > 0$  is the killing rate.
- Important remark: suppose  $q = r + \Delta^{-1}$  for some  $\Delta > 0$ . Then  $\Delta^{-1} \cdot v_j(x; q; \vec{h}_{\pm}; \vec{g})$  can be interpreted as the value function of a *finite lived* option with *random* maturity date  $T \sim \text{Exp } \Delta^{-1}$ .
- We will write  $\vec{V} = (V_j)_{j=1}^m$ ,  $\vec{v} = (v_j)_{j=1}^m$  (vector-valued functions).

# Carr's randomization via backward induction

**Step I.** Choose a partition,  $\mathcal{P}$ , of the interval  $[0, T]$ . Thus  $\mathcal{P}$  is a finite collection of points  $0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T$ , where  $N$  is a positive integer. (Typically,  $t_s = sT/N$  for all  $s$ .)

**Step II.** For every  $0 \leq s \leq N - 1$ , set  $\Delta_s = t_{s+1} - t_s$  and  $q_s = r + \Delta_s^{-1}$ .

**Step III.** Put  $V_j^N(x) = g^j(x)$  for all  $1 \leq j \leq m$ .

**Step IV.** In a cycle with respect to  $s = N - 1, N - 2, \dots, 1, 0$ , calculate

$$\vec{V}^s(x) = \Delta_s^{-1} \cdot \vec{v}(x; q_s; \vec{h}_{\pm}; \vec{V}^{s+1}).$$

**Step V.** Put  $\vec{V}_{\mathcal{P}}(x, T; \vec{h}_{\pm}; \vec{g}) = \vec{V}^0(x)$ , where  $\vec{V}^0(x)$  is obtained at the end of the cycle in Step IV. Then  $\vec{V}_{\mathcal{P}}$  is Carr's randomization approximation to  $\vec{V}(x, T; \vec{h}_{\pm}; \vec{g})$ , defined by the partition  $\mathcal{P}$ .

# Perpetual pricing problem: iterative procedure

- Problem: calculate  $v_j(x) := v_j(x; q; \vec{h}_\pm; \vec{g})$  for  $1 \leq j \leq m$ .
- The functions  $v_j(x)$  solve the following system of PIDE:

$$\begin{cases} (q - L_j)v_j(x) - \sum_{k=1}^m \lambda_{jk} v_k(x) = g^j(x), & h_-^j < x < h_+^j; \\ v_j(x) = 0, & x \leq h_-^j \quad \text{or} \quad x \geq h_+^j, \end{cases}$$

where  $L_j$  is the infinitesimal generator of the Lévy process  $X^{(j)}$ .

- We construct a sequence of approximations  $\vec{v}^0, \vec{v}^1, \vec{v}^2, \dots$  to  $\vec{v}$ .
- Put  $\Lambda_j = -\sum_{k \neq j} \lambda_{jk}$ . Set  $\vec{v}^0 = 0$  and for  $n = 1, 2, \dots$  solve

$$\begin{cases} (q + \Lambda_j - L_j)v_j^n(x) = g^j(x) + \sum_{k \neq j} \lambda_{jk} v_k^{n-1}(x), & h_-^j < x < h_+^j; \\ v_j^n(x) = 0, & \text{otherwise.} \end{cases}$$

- This is a perpetual pricing problem in a Lévy model with one state.

# Normalized EPV operators of a Lévy process

- Next goal: explain how to price knock-out continuous cash flows with one or two barriers in a 1-dimensional Lévy model.
- The *supremum* and *infimum* processes of  $X = \{X_t\}_{t \geq 0}$  are

$$\bar{X}_t = \sup_{0 \leq s \leq t} X_s \quad \text{and} \quad \underline{X}_t = \inf_{0 \leq s \leq t} X_s.$$

- The normalized expected present value (EPV) operators are

$$(\mathcal{E}_q f)(x) = \mathbb{E} \left[ \int_0^\infty q e^{-qt} f(x + X_t) dt \right],$$

$$(\mathcal{E}_q^+ f)(x) = \mathbb{E} \left[ \int_0^\infty q e^{-qt} f(x + \bar{X}_t) dt \right],$$

$$(\mathcal{E}_q^- f)(x) = \mathbb{E} \left[ \int_0^\infty q e^{-qt} f(x + \underline{X}_t) dt \right].$$

# Wiener-Hopf factorization (WHF)

- Let  $T_q \sim \text{Exp } q$  be an exponentially distributed random variable with mean  $q^{-1}$ , which is independent of the Lévy process  $X = \{X_t\}_{t \geq 0}$ .
- Probability form of the WHF formula:

$$\mathbb{E}[e^{i\xi X_{T_q}}] = \mathbb{E}[e^{i\xi \bar{X}_{T_q}}] \cdot \mathbb{E}[e^{i\xi \underline{X}_{T_q}}] \quad \forall \xi \in \mathbb{R}.$$

- The last identity follows from the following facts:
  - (1) we have  $X_{T_q} = \bar{X}_{T_q} + (X_{T_q} - \bar{X}_{T_q})$ ;
  - (2) the random variables  $\bar{X}_{T_q}$  and  $X_{T_q} - \bar{X}_{T_q}$  are independent (deep!);
  - (3) the random variables  $\underline{X}_{T_q}$  and  $X_{T_q} - \bar{X}_{T_q}$  are identical in law;
  - (4) the characteristic function of the sum of two independent random variables is equal to the product of their characteristic functions.

## Two other WHF formulas

- Define the *Wiener-Hopf factors*  $\phi_q^\pm(\xi)$  (for  $\xi \in \mathbb{R}$ ) by the formulas

$$\phi_q^+(\xi) = \mathbb{E}[e^{i\xi \bar{X}_{T_q}}], \quad \phi_q^-(\xi) = \mathbb{E}[e^{i\xi X_{T_q}}].$$

- $\phi_q^\pm(\xi)$  admit analytic continuation without zeroes into the upper/lower half plane.
- They are related to the normalized EPV operators  $\mathcal{E}_q^\pm$  via

$$\mathcal{E}_q^\pm(e^{i\xi x}) = \phi_q^\pm(\xi) \cdot e^{i\xi x} \quad \forall \xi \in \mathbb{R}.$$

- Writing  $\psi(\xi)$  for the characteristic exponent of  $X$ , one has

$$\mathcal{E}_q(e^{i\xi x}) = q \cdot (q + \psi(\xi))^{-1} \cdot e^{i\xi x} \quad \forall \xi \in \mathbb{R}.$$

- Analytic form of the WHF formula:  $q \cdot (q + \psi(\xi))^{-1} = \phi_q^+(\xi)\phi_q^-(\xi)$ .
- Operator form of the WHF formula:  $\mathcal{E}_q = \mathcal{E}_q^+ \mathcal{E}_q^- = \mathcal{E}_q^- \mathcal{E}_q^+$ .



# The Wiener-Hopf method for one barrier

- Now we return to pricing a continuous cash flow  $\{e^{-qt}g(\ln S_t)\}_{t \geq 0}$ .
- Consider the down-and-out case; let  $0 < L < \infty$  be the barrier.
- Write  $x = \ln S_0$  and  $h_- = \ln L$ . The flow is terminated as soon as the price of the underlying  $S_t = e^{x+X_t}$  reaches or falls below  $L = e^{h_-}$ .
- The price of this down-and-out continuous cash flow equals

$$V_{down-and-out}(x; q; h_-; g) = q^{-1} \cdot \mathcal{E}_q^- (\mathbb{1}_{(h_-, +\infty)}(x) \cdot (\mathcal{E}_q^+ g)(x)).$$

- Similar formula in the up-and-out case ( $U = e^{h_+}$  is the upper barrier):

$$V_{up-and-out}(x; q; h_+; g) = q^{-1} \cdot \mathcal{E}_q^+ (\mathbb{1}_{(-\infty, h_+)}(x) \cdot (\mathcal{E}_q^- g)(x)).$$

# The Wiener-Hopf method for two barriers

- Consider two barriers,  $0 < L < U < +\infty$ . Put  $h_- = \ln L$ ,  $h_+ = \ln U$ .
- Value of a knock-out cash flow  $\{e^{-qt}g(\ln S_t)\}_{t \geq 0}$  with barriers  $(L, U)$ :

$$\begin{aligned}v(x; q; h_{\pm}; g) = & G^0(x) - G_+^1(x) - G_-^1(x) + G_+^2(x) + G_-^2(x) \\ & - G_+^3(x) - G_-^3(x) + G_+^4(x) + G_-^4(x) - \dots\end{aligned}$$

- To find the terms on the RHS, first calculate  $G^0(x) = q^{-1} \cdot (\mathcal{E}_q g)(x)$ .
- Next, use the formulas

$$G_+^0(x) = G^0(x)|_{[h_+, +\infty)}, \quad G_-^0(x) = G^0(x)|_{(-\infty, h_-]},$$

$$G_+^n(x) = \mathcal{E}_q^- \left( \mathbb{1}_{(-\infty, h_-]}(x) \cdot ((\mathcal{E}_q^-)^{-1} G_-^{n-1})(x) \right) \quad \forall n \geq 1,$$

$$G_-^n(x) = \mathcal{E}_q^+ \left( \mathbb{1}_{[h_+, +\infty)}(x) \cdot ((\mathcal{E}_q^+)^{-1} G_+^{n-1})(x) \right) \quad \forall n \geq 1.$$

# Using the BBL method in practice

Suppose we can efficiently calculate the action of the normalized EPV operators  $\mathcal{E}_q^\pm$  for a 1-dimensional Lévy process (no regime switching).

Using that as a building block, we obtain a fast and accurate pricing algorithm for barrier options based on the Carr-Wiener-Hopf method.

The numerical realization of  $\mathcal{E}_q^\pm$  uses the finite element method.

For processes with rational characteristic exponent (e.g., HEJD),  $\exists$  very explicit formulas for  $\mathcal{E}_q^\pm$ , and no FFT is required.

For other processes, one uses *refined FFT* (discussed below).

# Normalized EPV operators in the Black-Scholes model

- Introduce two types of integral operators:

$$(I_{\beta}^{+} f)(x) = \int_0^{\infty} \beta e^{-\beta y} f(x + y) dy \quad (\beta > 0),$$

$$(I_{\beta}^{-} f)(x) = \int_{-\infty}^0 (-\beta) e^{-\beta y} f(x + y) dy \quad (\beta < 0).$$

- Assume that  $X = \{X_t\}_{t \geq 0}$  is a BM with volatility  $\sigma$  and drift  $\mu$ .
- Denote by  $\beta^{-} < 0 < \beta^{+}$  the roots of the *characteristic equation*

$$\frac{\sigma^2}{2} \beta^2 + \mu \beta - q = 0.$$

- Then  $\phi_q^{\pm}(\xi) = \beta^{\pm} \cdot (\beta^{\pm} - i\xi)^{-1}$  and  $\mathcal{E}_q^{\pm} = I_{\beta^{\pm}}^{\pm}$ .
- $\exists$  similar formulas for HEJD involving linear combinations of finitely many integral operators of the form  $I_{\beta}^{\pm}$ .

# Normalized EPV Operators and Fourier Transforms

- $X = \{X_t\}_{t \geq 0}$  is a Lévy process with characteristic exponent  $\psi(\xi)$
- fix  $q > 0$  and let  $\phi_q^\pm(\xi)$  be the Wiener-Hopf factors of  $q \cdot (q + \psi(\xi))^{-1}$
- PDO realization of the normalized EPV operators of  $X$ :

$$(\mathcal{E}_q^\pm f)(x) = \mathcal{F}_{\xi \rightarrow x}^{-1}(\phi_q^\pm(\xi) \cdot \widehat{f}(\xi))$$

- convolution realization of the normalized EPV operators:

$$(\mathcal{E}_q^+ f)(x) = \int_0^{+\infty} f(x+y) p_q^+(dy), \quad (\mathcal{E}_q^- f)(x) = \int_{-\infty}^0 f(x+y) p_q^-(dy),$$

where  $p_q^\pm(dy)$  are Borel probability measures on  $\mathbb{R}$  supported on the positive and the negative half axis, respectively

- the Fourier transforms of  $p_q^\pm$  are given by  $\widehat{p}_q^\pm(\xi) = \phi_q^\pm(\xi)$

# Integral Formulas for the Wiener-Hopf Factors

Under certain regularity conditions on the characteristic exponent  $\psi(\xi)$ ,

$$\phi_q^\pm(\xi) = \exp \left[ \pm \frac{1}{2\pi i} \int_{\text{Im } \eta = \omega_\mp} \frac{\xi \cdot \ln(1 + q^{-1}\psi(\eta))}{\eta(\xi - \eta)} d\eta \right],$$

where  $\omega_- < 0 < \omega_+$  are suitably chosen. Main requirements:

- $\psi(\xi)$  admits analytic continuation into an open horizontal strip in  $\mathbb{C}$  that contains the closed strip  $\{\xi \in \mathbb{C} \mid \text{Im } \xi \in [\omega_-, \omega_+]\}$ , and
- $\text{Re}(q + \psi(\xi)) > 0$  for all  $\xi$  in this closed strip.

## Practical applications of the above formula

One calculates the values of  $\phi_q^\pm(\xi)$  on a suitable grid  $\vec{\xi} = (\xi_k)_{k=1}^M$  by applying the trapezoid rule to discretize the integral above and using FFT.

# Fourier Transforms and FFT

## Fourier transforms on the real line

$$\widehat{f}(\xi) = (\mathcal{F}f)(\xi) = (\mathcal{F}_{x \rightarrow \xi} f)(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x) dx$$

$$(\mathcal{F}^{-1}g)(\xi) = (\mathcal{F}_{\xi \rightarrow x}^{-1}g)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} g(\xi) d\xi$$

## Fast Fourier transforms

Consider uniformly spaced grids  $\vec{x} = (x_j)_{j=1}^M$  and  $\vec{\xi} = (\xi_k)_{k=1}^M$  with mesh  $\Delta$  and  $\zeta$ , respectively. Replace  $(\mathcal{F}_{x \rightarrow \xi} f)(\xi)$  and  $(\mathcal{F}_{\xi \rightarrow x}^{-1}g)(x)$  with

$$(\mathcal{F}_{fast} f)(\xi) = \Delta \cdot \sum_{j=1}^M f(x_j) e^{-i\xi x_j}, \quad (\mathcal{F}_{fast}^{-1} g)(x) = \frac{\zeta}{2\pi} \cdot \sum_{k=1}^M g(\xi_k) e^{i\xi_k x}.$$

# Standard FFT techniques

- Let  $\vec{f} = (f_j)_{j=1}^M$  be an array of complex numbers. Set

$$\text{dft}(\vec{f})_k = \sum_{j=1}^M f_j \cdot e^{-2\pi i(j-1)(k-1)/M}, \quad 1 \leq k \leq M.$$

Standard FFT algorithms are designed for fast calculation of the vector  $\text{dft}(\vec{f})$  (“fast” means  $O(M \cdot \ln M)$  arithmetic operations).

- $\mathcal{F}_{fast}^{\pm 1}$  can be expressed in terms of  $\text{dft}$  provided the identity  $M \cdot \Delta \cdot \zeta = 2\pi$  holds (“Nyquist relation” or “uncertainty principle”).
- $\text{dft}$  can also be used for very fast calculation of sums of the form  $h_k = \sum_{j=1}^M f_j g_{k-j}$  ( $1 \leq k \leq M$ ), where  $\vec{f} = (f_j)_{j=1}^M$  and  $\vec{g} = (g_\ell)_{\ell=1-M}^{M-1}$  are arrays of complex numbers.



# Finite element method via piecewise linear interpolation

Consider a grid  $\vec{x} = (x_j)_{j=1}^M$ , where  $x_j = x_1 + (j-1)\Delta$  for all  $1 \leq j \leq M$ , and  $\Delta > 0$  is fixed. Approximating  $f$  with a piecewise linear function yields

$$(\mathcal{E}_q^+ f)(x_k) \approx -d_k^+ \cdot f_M + \sum_{j=k}^M c_{k-j}^+ \cdot f_j \quad (1 \leq k \leq M),$$

where  $f_j = f(x_j)$  for  $1 \leq j \leq M$ ,

$$d_k^+ = \frac{\Delta}{2\pi} \int_{-\infty}^{\infty} e^{i(k-M)\Delta\xi} \cdot (\phi_q^+(\xi) - \phi_q^+(\infty)) \cdot \frac{e^{-i\xi\Delta} + i\xi\Delta - 1}{(i\xi\Delta)^2} d\xi$$

for  $1 \leq k \leq M$ ,

$$c_\ell^+ = \frac{\Delta}{2\pi} \int_{-\infty}^{\infty} e^{i\ell\Delta\xi} \cdot (\phi_q^+(\xi) - \phi_q^+(\infty)) \cdot \frac{e^{i\xi\Delta} + e^{-i\xi\Delta} - 2}{(i\xi\Delta)^2} d\xi$$

for  $1-M \leq \ell \leq -1$ , and  $c_0^+ = 1 - \sum_{1-M \leq \ell \leq -1} c_\ell^+$ .

There are also similar formulas for calculating the action of  $\mathcal{E}_q^-$ .

# Comments on the practical implementation

## General scheme of numerical realization of $\mathcal{E}_q^\pm$

1. Numerically calculate  $\phi_q^\pm(\xi)$  on a suitable  $\xi$ -grid using fast convolution.
2. Compute the coefficients  $d_k^\pm, c_\ell^\pm$  using inverse FFT.
3. The action of  $\mathcal{E}_q^\pm$  on any function is calculated using fast convolution.

## Important issue (resolved via the use of refined FFT)

- For the calculation of  $d_k^\pm, c_\ell^\pm$  to be accurate enough, the  $\xi$ -grid must be sufficiently long and sufficiently fine.
- Not only does this violate the Nyquist relation  $M \cdot \Delta \cdot \zeta = 2\pi$ , but it also forces us to use *more* points in the  $\xi$ -grid than in the  $x$ -grid.

# Explanation of refined FFT via a simple example

- Suppose we are given uniformly spaced grids  $\vec{x} = (x_j)_{j=1}^M$  and  $\vec{\xi} = (\xi_k)_{k=1}^M$  of mesh  $\Delta$  and  $\zeta$ ; and the relation  $M\Delta\zeta = 2\pi$  holds.
- Given functions  $f(x)$  and  $g(\xi)$ , we can (quickly) calculate  $(\mathcal{F}_{fast} f)(\xi_k)$  and  $(\mathcal{F}_{fast}^{-1} g)(x_j)$  using standard FFT techniques.
- Now suppose we wish to halve the mesh of the  $\xi$ -grid and double the number of points in it, while leaving the  $x$ -grid intact.
- Call the new grid  $\vec{\xi}' = (\xi'_k)_{k=1}^{2M}$ . It has mesh equal to  $\zeta/2$ .
- The points  $\{\xi'_1, \xi'_3, \xi'_5, \dots, \xi'_{2M-1}\}$  and  $\{\xi'_2, \xi'_4, \xi'_6, \dots, \xi'_{2M}\}$  form two uniformly spaced grids with mesh  $\zeta$ .
- Apply the standard FFT technique twice, and we are in good shape.
- Similarly we can stretch and refine the  $\xi$ -grid by arbitrary factors. (The details are provided in the appendix to this set of slides.)

# References

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# An Improved Setup for FFT and Inverse FFT

We assume that a uniformly spaced grid  $\vec{x} = (x_j)_{j=1}^M$  of points in  $\mathbb{R}$  is given, where  $x_j = x_1 + (j - 1)\Delta$ , and both  $M$  and  $\Delta > 0$  are fixed. One should choose two positive integers,  $M_2$  and  $M_3$ , that will be responsible, respectively, for refining and stretching the  $\xi$ -grid. One should also choose  $\xi_1 \in \mathbb{C}$ , the desired initial point of the  $\xi$ -grid.

The total number of points in the  $\xi$ -grid equals  $M_1 = MM_2M_3$ . Let us define  $\zeta = 2\pi/(M\Delta)$ . The mesh of the  $\xi$ -grid equals  $\zeta_1 = \zeta/M_2$ . Hence the  $\xi$ -grid has length  $M_3 \cdot (2\pi/\Delta)$ . Explicitly, the  $\xi$ -grid is given by

$$\vec{\xi} = (\xi_k)_{k=1}^{M_1}, \quad \xi_k = \xi_1 + (k - 1)\zeta_1 = \xi_1 + (k - 1) \cdot \frac{\zeta}{M_2}.$$

# Implementing FFT in the New Setup

We would like to calculate the values of  $\mathcal{F}_{fast} f$  at all the points of the grid  $\vec{\xi}$ . The best one can hope for is to reduce the calculation to  $M_2 \cdot M_3$  applications of FFT for arrays of length  $M$  (since the input array has length  $M$  and the output array has length  $M \cdot M_2 \cdot M_3$ ). To this end, we represent the grid  $\vec{\xi} = (\xi_k)_{k=1}^{M_1}$  as a disjoint union of  $M_2 \cdot M_3$  grids, each of which has  $M$  points and mesh  $\zeta$ , and apply ordinary FFT to each of them:

$$\begin{aligned} & (\xi_{M_2 \cdot (k-1)+1})_{k=1}^M, \quad (\xi_{M_2 \cdot (k-1)+2})_{k=1}^M, \quad \dots, \quad (\xi_{M_2 \cdot k})_{k=1}^M, \\ & (\xi_{M_2 \cdot (k-1+M)+1})_{k=1}^M, \quad (\xi_{M_2 \cdot (k-1+M)+2})_{k=1}^M, \quad \dots, \quad (\xi_{M_2 \cdot (k+M)})_{k=1}^M, \\ & \dots, \\ & (\xi_{M_2 \cdot (k-1+(M_3-1)M)+1})_{k=1}^M, \quad \dots, \quad (\xi_{M_2 \cdot (k+(M_3-1)M)})_{k=1}^M. \end{aligned}$$

# Implementing Inverse FFT in the New Setup

Let  $g(\xi)$  be a function whose domain contains the grid  $\vec{\xi}$ . We would like to calculate the values of the function  $\mathcal{F}_{fast}^{-1}g$  on the grid  $\vec{x}$ . To this end, for each  $1 \leq j \leq M_3$  and each  $1 \leq \ell \leq M_2$ , let  $g_{j,\ell}$  be the restriction of  $g$  to the sub-grid  $\vec{\xi}(j, \ell) = (\xi_{M_2 \cdot (k-1 + (j-1)M) + \ell})_{k=1}^M$ . The values of  $\mathcal{F}_{fast}^{-1}g_{j,\ell}$  on the grid  $\vec{x}$  can be calculated using the standard FFT techniques. Note that for each pair  $(j, \ell)$ , we only need to calculate a single FFT for a vector of length  $M$ . Finally, it follows immediately from the definitions that

$$\mathcal{F}_{fast}^{-1}g = \frac{1}{M_2} \sum_{j=1}^{M_3} \sum_{\ell=1}^{M_2} \mathcal{F}_{fast}^{-1}(g_{j,\ell}).$$

This method of calculating  $\mathcal{F}_{fast}^{-1}g$  requires only  $O(M_1 \cdot \ln M)$  arithmetic operations, which, again, is the best one can hope for.

# Fast Discrete Convolution via FFT

- Goal: compute  $h_k = \sum_{j=1}^M f_j g_{k-j}$ , where  $\vec{f} = (f_j)_{j=1}^M$  and  $\vec{g} = (g_\ell)_{\ell=1-M}^{M-1}$  are complex arrays of lengths  $M$  and  $2M - 1$ .
- Let  $\tilde{f}$  be the array of length  $2M$  with entries

$$\tilde{f}_j = \begin{cases} f_j, & 1 \leq j \leq M; \\ 0, & M + 1 \leq j \leq 2M. \end{cases}$$

- Let  $\tilde{g}$  be the array of length  $2M$  with entries

$$g_0, g_1, \dots, g_{M-1}, 0, g_{1-M}, g_{2-M}, \dots, g_{-1}$$

- Calculate the array  $\tilde{h} = (\tilde{h}_\ell)_{\ell=1}^{2M}$  with entries

$$\tilde{h}_\ell = \text{dft}(\tilde{f})_\ell \cdot \text{dft}(\tilde{g})_\ell.$$

- For all  $1 \leq k \leq M$ , we have  $h_k = \text{idft}(\tilde{h})_k$  (where  $\text{idft} = \text{dft}^{-1}$ ).



# Convergence of Carr's randomization

**Theorem.** Suppose that each Lévy process  $X^{(j)}$  has either nonzero diffusion component or infinite jump activity (or both). Assume also that each  $g^j(x)$  is a bounded continuous function on  $(h_-^j, h_+^j)$ . Then for fixed  $j$  and  $x$ , we have  $V_{\mathcal{P},j}(x, T; \vec{h}_{\pm}; \vec{g}) \rightarrow V_j(x, T; \vec{h}_{\pm}; \vec{g})$  as  $\text{mesh}(\mathcal{P}) \rightarrow 0$ , where  $\text{mesh}(\mathcal{P}) = \max_{0 \leq s \leq N-1} \Delta_s$ .

## Important ingredient in the proof

**Theorem.** Under the same assumptions, for fixed  $j$  and  $x$ ,  $V_j(x, T; \vec{h}_{\pm}; \vec{g})$  is continuous as a function of  $T$ .

The assumption on  $X^{(j)}$  is used in the following way: it is equivalent to requiring that  $\mathbb{P}[X_t^{(j)} = a] = 0$  for any  $t > 0$ ,  $1 \leq j \leq m$  and  $a \in \mathbb{R}$ .