Pricing index-CDS options in a nonlinear filtering model

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Content

- Short recapitulation of the non-linear filtering model by Frey & Schmidt (2009) and some new additional results.
- Give a very brief recapitulation of the index-CDS
- Present practical formulas for the forward starting index-CDS spreads in the filtering model of Frey & Schmidt (2009).
- Discuss calibration of the model using nonlinear-filter SDE and maximum-likelihood methods with market data on index-CDS spreads
- Present some numerical results in our calibrated model
- Give a short outline of options on index-CDS and how to price them in the model presented here.

The nonlinear filtering model

- We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ where $\mathbb{Q}$ is a risk-neutral measure. Below, all computations are under $\mathbb{Q}$.
- The state of the economy is driven by an unobservable background factor process X modelling the "true" state of the economy.
- X is modelled as finite-state Markov chain on state space $S^X = \{1, 2, \ldots, K\}$ with generator $\mathbf{Q}$ and we define $\mathcal{F}_t^X = \sigma(X_s; s \leq t)$.
- The states in $S^X = \{1, 2, \ldots, K\}$ are ordered so that state 1 represents the best state and K represents the worst state of the economy.
- Market participants only observe the "noisy" history of the state of the economy, i.e. X_t with "noise".

The default times

- Consider m obligors with default times $\tau_1, \tau_2 \ldots, \tau_m$
- Let $\lambda_1, \lambda_2 \ldots, \lambda_m$ be the $\mathcal{F}_t^X$-default intensities for $\tau_1, \tau_2 \ldots, \tau_m$ where $\lambda_i : \{1, 2, \ldots, K\} \mapsto [0, \infty)$ for each obligor
- Hence, each default time τ_i is given by

$$\tau_i = \inf \left\{ t > 0 : \int_0^t \lambda_i(X_s) ds \geq E_i \right\}. \quad (1)$$

  where $E_1, \ldots, E_m$ are iid, $E_i \sim \text{Exp}(1)$, and independent of $\mathcal{F}_\infty^X$.

- The default times $\tau_1, \tau_2 \ldots, \tau_m$ are conditionally independent given the information of the factor process X, that is $\mathcal{F}_\infty^X$.
- By definition of the state space, the mappings $\lambda_i(\cdot)$ are strictly increasing in $k \in \{1, 2, \ldots, K\}$, that is $\lambda_i(k) < \lambda_i(k+1)$

The nonlinear filtering model, cont.

- Let $Y_{t,i} = 1_{\{\tau_i \leq t\}}$ and $Y_t = (Y_{t,1}, \dots, Y_{t,m})$ so that the pure portfolio default history is given by $\mathcal{F}_t^Y = \sigma(Y_s; s \leq t)$
- Market participants do not observe X_t directly, instead they observe Z_t

$$Z_t = \int_0^t \mathbf{a}(X_s)ds + B_t \tag{2}$$

  where B_t is a l-dimensional Brownian motion independent of X_t and Y_t and $\mathbf{a}(\cdot)$ is a function from $\{1, 2, \dots, K\}$ to $\mathbb{R}^l$.
- We define $\mathcal{F}_t^Z = \sigma(Z_s; s \leq t)$ and the information available for market participants denoted by "the market filtration" $\mathcal{F}_t^M$, is given by

$$\mathcal{F}_t^M = \mathcal{F}_t^Y \vee \mathcal{F}_t^Z \tag{3}$$

- So prices of securities are given as conditional expectation with respect to the market filtration $\mathbb{F}^M = (\mathcal{F}_t^M)_{t \geq 0}$

The filtering probabilities

- A central quantity is the filtering probabilities π_t^k defined as
$$\pi_t^k = \mathbb{Q}\left[X_t = k \,|\, \mathcal{F}_t^M\right] \tag{4}$$
and we let $\boldsymbol{\pi}_t \in \mathbb{R}^K$ be the row-vector $\boldsymbol{\pi}_t = \left(\pi_t^1, \ldots, \pi_t^K\right)$.

- The SDE describing the dynamics of π_t^k is well known in nonlinear filtering theory (Kushner-Stratonovic) and connects to the innovation approach. Frey & Schmidt (2009) states the KS-SDE in a heterogeneous credit portfolio.

- We only consider exchangeable credit portfolios, so that $\lambda_i(X_t) = \lambda(X_t)$ for each obligor and we let $\boldsymbol{\lambda} \in \mathbb{R}^K$ be the row-vector $\boldsymbol{\lambda} = (\lambda(1), \ldots, \lambda(K))$.

- Let N_t be the number of defaults up to time t in the portfolio, that is
$$N_t = \sum_{i=1}^{m} Y_{t,i} = \sum_{i=1}^{m} 1_{\{\tau_i \leq t\}}.$$

- The portfolio credit loss at t is given by $L_t = \frac{(1-\phi)}{m} N_t$ where ϕ is the recovery rate for each obligor.

Kushner-Stratonovic equations in exchangeable portfolios

For $j = 1, \cdots, l$, let $\mu_{t,j}$ be a Brownian motion with respect to $\mathcal{F}_t^M$. Then:

The Kushner-Stratonovic SDE in exchangeable credit portfolios

Consider a homogeneous credit portfolio with m obligors. Then, with notation as above, the processes π_t^k satisfies the following K-dimensional system of SDE-s,

$$d\pi_t^k = \gamma^k(\boldsymbol{\pi}_t)dN_t + \boldsymbol{\pi}_t\left(\mathbf{Q}\mathbf{e}_k^\top - \gamma^k(\boldsymbol{\pi}_t)\boldsymbol{\lambda}^\top(m - N_t)\right)dt + \sum_{j=1}^{l}\boldsymbol{\alpha}_j^k(\boldsymbol{\pi}_t)d\mu_{t,j} \quad (5)$$

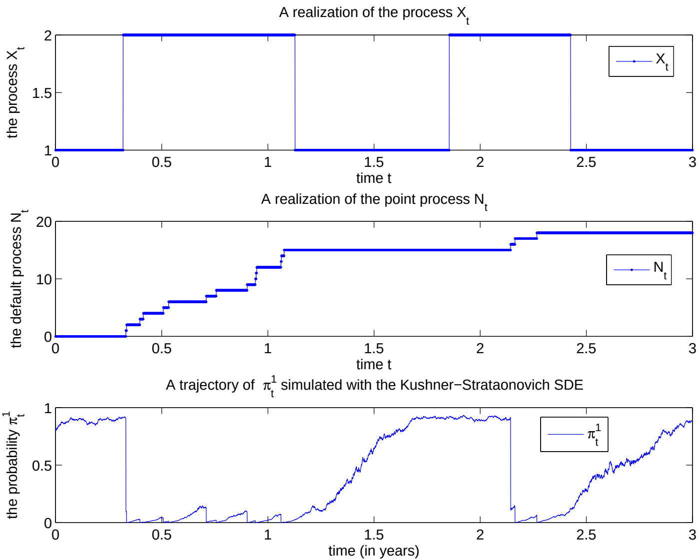
where $\gamma^k(\boldsymbol{\pi}_t)$ and $\boldsymbol{\alpha}^k(\boldsymbol{\pi}_t)$ are given by

$$\gamma^k(\boldsymbol{\pi}_t) = \pi_t^k\left(\frac{\lambda(k)}{\boldsymbol{\pi}_t\boldsymbol{\lambda}^\top} - 1\right) \quad \text{and} \quad \boldsymbol{\alpha}^k(\boldsymbol{\pi}_t) = \pi_t^k\left(\mathbf{a}(k) - \sum_{n=1}^{K}\pi_t^n\mathbf{a}(n)\right). \quad (6)$$

and $\boldsymbol{\alpha}_j^k(\cdot)$ is the j-th component of $\boldsymbol{\alpha}^k(\cdot)$.

Here we set $l = 1$ and denote $\mu_{t,1}$ by μ_t. We also let $a(k) = c \cdot \ln\lambda(k)$.

Example of KS-SDE: $K = 2$, $m = 125$, $l = 1$



A very short recapitulation of the index-CDS

- An index-CDS on a portfolio of m obligors, entered at t with maturity T, gives **A** protection on defaults among the m obligors from **B** up to time T

- **A** pays **B** a fixed fee $S(t, T)$ multiplied what is left in the portfolio at each payment time which are done quarterly in the period $[t, T]$.

- $S(t, T)$ is set so expected discounted cash-flows between **A** and **B** are equal at t and $S(t, T)$ is called the index-CDS spread with maturity $T - t$. Hence,

$$S(t, T) = \frac{\mathbb{E}\left[\int_t^T B(t, s) dL_s \,\middle|\, \mathcal{F}_t^M\right]}{\frac{1}{4}\sum_{n=n_t}^{\lceil 4T \rceil} B(t, t_n)\left(1 - \frac{1}{m}\mathbb{E}\left[N_{t_n} \,\middle|\, \mathcal{F}_t^M\right]\right)} \tag{7}$$

where $B(t, s) = e^{-r(s-t)}$ for constant r and $t_n = \frac{n}{4}$, $n_t = \lceil 4t \rceil + 1$.

- $S(0, T)$ quoted on daily basis on the market for standardized credit portfolios where $T = 3, 5, 7, 10$, see e.g the iTraxx Series.

The index-CDS in the nonlinear filtering model

Given our nonlinear filtering model we can now state the following results

The index-CDS spread in the exchangeable nonlinear filtering model

Consider an index-CDS portfolio in the nonlinear filtering model. Then, with notation as above

$$S(t,T) = \frac{(1-\phi)\left(1-\boldsymbol{\pi}_t\left(e^{\mathbf{Q}_\lambda(T-t)}\left(\mathbf{I}+r\left(\mathbf{Q}_\lambda-r\mathbf{I}\right)^{-1}\right)e^{-r(T-t)}-r\left(\mathbf{Q}_\lambda-r\mathbf{I}\right)^{-1}\right)\mathbf{1}\right)}{\frac{1}{4}\sum_{n=n_t}^{\lceil 4T\rceil}\boldsymbol{\pi}_t e^{\mathbf{Q}_\lambda(t_n-t)}\mathbf{1}e^{-r(t_n-t)}}. \quad (8)$$

where $\mathbf{Q}_\lambda = \mathbf{Q} - \mathbf{I}_\lambda$ and $\mathbf{I}_\lambda$ is a diagonal-matrix such that $(\mathbf{I}_\lambda)_{k,k} = \lambda(k)$ and $\boldsymbol{\pi}_t = \left(\pi_t^1, \ldots, \pi_t^K\right)$.

Note that given $\boldsymbol{\pi}_t$ the formula for $S(t,T)$ is compact and computationally tractable closed-form expressions in terms of $\boldsymbol{\pi}_t$ and $\mathbf{Q}_\lambda$.

Calibrating the models using index-CDS spread data

- Task: estimate $\boldsymbol{\theta} = (\mathbf{Q}, \boldsymbol{\lambda})$
- Let $\{S_M(t,T)\}_{t \in \mathbf{t}^{(s)}}$ be a historical time-series of model spreads observed at $N^{(s)}$ sample time points $\mathbf{t}^{(s)} = \{t_1^{(s)}, \ldots, t_{N^{(s)}}^{(s)}\}$ where $T = t + T_0$ for $t \in \mathbf{t}^{(s)}$.
- For each $t \in \mathbf{t}^{(s)}$ we set $S(t,T)(\omega) = S_M(t,T)$ and rewrite the pricing equation (8) as
$$\boldsymbol{\pi}_t(\omega)\mathbf{C}_t(\boldsymbol{\theta}, S_M(t,T))\mathbf{1} = 1 - \phi \tag{9}$$
where $\mathbf{C}_t(\boldsymbol{\theta}, S_M(t,T))$ is known to us in terms of $\boldsymbol{\theta} = (\mathbf{Q}, \boldsymbol{\lambda})$ and $S_M(t,T)$
- By using $\boldsymbol{\pi}_t(\omega)\mathbf{1} = 1$ with (9) we get linear equation system for $\boldsymbol{\pi}_t(\omega)$, viz.
$$\mathbf{A}_t \boldsymbol{\pi}_t^\top(\omega) = \mathbf{b} \tag{10}$$
So, for fixed $\boldsymbol{\theta} = (\mathbf{Q}, \boldsymbol{\lambda})$ and observed $S_M(t,T)$ and if $\mathbf{A}_t^{-1}$ exists we can find $\boldsymbol{\pi}_t(\omega) = \left(\pi_t^1(\omega), \pi_t^2(\omega), \ldots, \pi_t^K(\omega)\right)$ by solving (10).
- We want to estimate $\boldsymbol{\theta} = (\mathbf{Q}, \boldsymbol{\lambda})$ with maximum likelihood techniques by using time-series data $\{S_M(t,T)\}_{t \in \mathbf{t}^{(s)}}$, Eq. (10) and the KS-SDE (5)

Calibrating the model using index-CDS spread data, cont.

- Let us outline this approach when $K = 2$ (enough to study $\pi_t^1(\omega)$).
- Recall that $\pi_t^1(\omega)$ must satisfy

$$d\pi_t^1 = \gamma^1(\boldsymbol{\pi}_t)dN_t + \boldsymbol{\pi}_t \left(\mathbf{Q}\mathbf{e}_1^\top - \gamma^1(\boldsymbol{\pi}_t)\boldsymbol{\lambda}^\top (m - N_t) \right) dt + \alpha^1(\boldsymbol{\pi}_t)d\mu_t \quad (11)$$

where μ_t is Brownian motion with respect to $\mathcal{F}_t^M$.

- We discretize (11) with $t_{n+1}^{(s)} - t_n^{(s)} = \Delta t$ and assume solution to the discrete SDE is same as solution to (11).
- Let $x_n = S_M(t_n^{(s)}, t_n^{(s)} + T_0)$. The discrete KS-SDE for a fixed $\omega \in \Omega$ is

$$\Delta\pi_{n,1}(\boldsymbol{\theta}, x_n) = g(\boldsymbol{\theta}, x_n)\Delta t + \alpha_1(\boldsymbol{\theta}, x_n)\Delta\mu_{t_n^{(s)}} \quad (12)$$

where $\Delta\pi_{n,1}(\boldsymbol{\theta}, x_n)$, $\alpha_1(\boldsymbol{\theta}, x_n)$ and $g(\boldsymbol{\theta}, x_n)$ are known via x_n, $\boldsymbol{\theta} = (\mathbf{Q}, \boldsymbol{\lambda})$ (our sample contains no defaults, so $N_t(\omega) = 0$ for all $t \in \mathbf{t}^{(s)}$).

The likelihood-function

- Note that R.H.S in (12) is conditionally normally distributed, viz.

$$g(\boldsymbol{\theta}, x_n)\Delta t + \alpha_1(\boldsymbol{\theta}, x_n)\Delta\mu_{t_n^{(s)}} \sim \mathcal{N}\left(g(\boldsymbol{\theta}, x_n)\Delta t, (\alpha_1(\boldsymbol{\theta}, x_n))^2\Delta t\right). \quad (13)$$

  and $\{g(\boldsymbol{\theta}, x_n)\Delta t + \alpha_1(\boldsymbol{\theta}, x_n)\Delta\mu_{t_n^{(s)}}\}_{n=1}^{N^{(s)}}$ are independent.

- Hence, the likelihood function $\mathcal{L}(\boldsymbol{\theta}|x_1, x_2, \ldots, x_{N^{(s)}})$ is

  $\mathcal{L}(\boldsymbol{\theta}|x_1, \ldots, x_{N^{(s)}}) =$

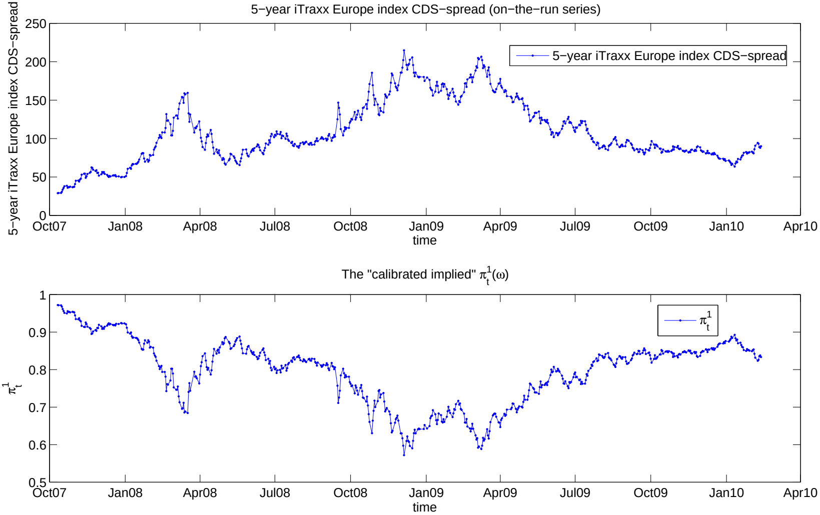
$$\prod_{n=1}^{N^{(s)}} \frac{1}{\sqrt{2\pi(\alpha_1(\boldsymbol{\theta}, x_n))^2\Delta t}} \exp\left(-\frac{(\Delta\pi_{n,1}(\boldsymbol{\theta}, x_n) - g(\boldsymbol{\theta}, x_n)\Delta t)^2}{2(\alpha_1(\boldsymbol{\theta}, x_n))^2\Delta t}\right).$$

- By letting $\ell(\boldsymbol{\theta}|x_1, \ldots, x_{N^{(s)}}) = -\ln\mathcal{L}(\boldsymbol{\theta}|x_1, \ldots, x_{N^{(s)}})$ we retrieve MLE-parameters $\boldsymbol{\theta}_{\text{MLE}}$ as $\boldsymbol{\theta}_{\text{MLE}} = \underset{\boldsymbol{\theta}}{\operatorname{argmin}}\, \ell(\boldsymbol{\theta}|x_1, \ldots, x_{N^{(s)}})$.

- Data: iTraxx Europe on-the-run series ($T - t = 5$ years), Nov 2007-Feb 2010, with 596 observations, $\Delta t = 1/250$, $m = 125$, $r = 3\%$, $\phi = 40\%$.

- Result: $\boldsymbol{\theta}_{\text{MLE}} = (c, \lambda_1, \lambda_2, q_{12}, q_{21}) = (0.2939, 0.001, 0.09, 0.0098, 0.004)$

Time-series $S_M(t, t+5)$ and calibrated implied $\pi_t^1(\omega)$



Options on the index-CDS

- With the calibrated parameters $\boldsymbol{\theta}_{\text{MLE}}$ we can price more complex instruments where the index-CDS is underlying, e.g. options on index CDS-s

- An option on an index-CDS with inception date today, strike K and exercise date t with maturity T gives **A** the right to enter an index-CDS at time t with spread K and maturity $T-t$, sold by **B**.

- Moreover, **B** also pays **A** the accumulated credit loss L_t at time t.

- The payoff $\Pi(t,T;K)$ for this option at time t is

$$\Pi(t,T;K) = \left(PV(t,T)\left(S(t,T)-K\right)+L_t\right)^+ \tag{14}$$

where

$$PV(t,T) = \frac{1}{4}\sum_{n=n_t}^{\lceil 4T\rceil} B(t,t_n)\left(1-\frac{1}{m}\mathbb{E}\left[N_{t_n} \mid F_t^M\right]\right). \tag{15}$$

- It is not correct to use Black-Scholes when finding the price of the option.

Options on the index-CDS in the non-linear filtering model

- Inserting relevant quantities from the filtering model into (14)-(15) yields

$$\Pi(t, T; K) = \left(\boldsymbol{\pi}_t \left[\mathbf{A}(t, T) - K \mathbf{B}(t, T) \right] \mathbf{1} \left(1 - \frac{N_t}{m} \right) + \frac{(1-\phi) N_t}{m} \right)^+ \quad (16)$$

where $\mathbf{A}(t, T)$ and $\mathbf{B}(t, T)$ are defined as

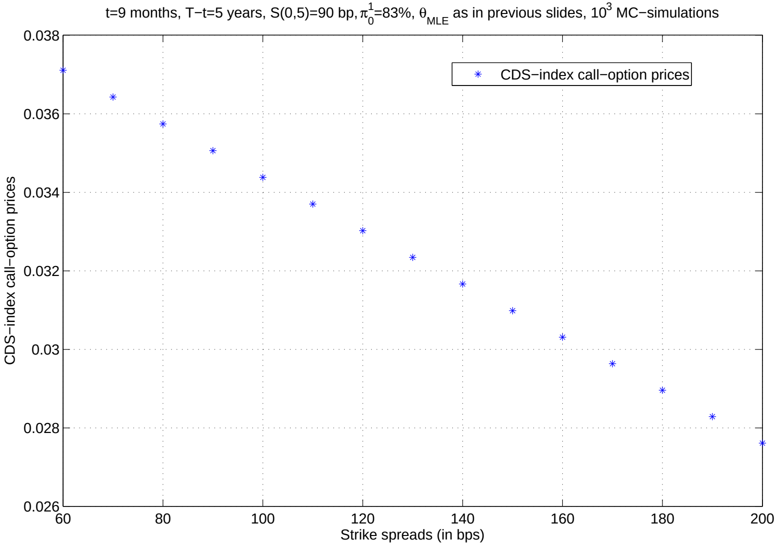
$$\mathbf{A}(t, T) = (1-\phi) \left[\mathbf{I} - e^{\mathbf{Q}_\lambda (T-t)} \left(\mathbf{I} + r \left(\mathbf{Q}_\lambda - r\mathbf{I} \right)^{-1} \right) e^{-r(T-t)} + r \left(\mathbf{Q}_\lambda - r\mathbf{I} \right)^{-1} \right]$$

and

$$\mathbf{B}(t, T) = \frac{1}{4} \sum_{n=n_t}^{\lceil 4T \rceil} e^{\mathbf{Q}_\lambda (t_n - t)} e^{-r(t_n - t)}.$$

- Valuation via MC-simulation: We use e.g. $\boldsymbol{\theta}_{\text{MLE}}$ to simulate $\boldsymbol{\pi}_t$ and then (16) to find $\Pi(t, T; K)$. Note that $\mathbf{A}(t, T)$ and $\mathbf{B}(t, T)$ are deterministic.

Options on the index-CDS: numerical example



Thank you for your attention!