Pricing Options on Variance in Affine Stochastic Volatility Models

Johannes Muhle-Karbe

Joint work with Jan Kallsen and Moritz Voß

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Outline

Introduction

Realized variance

Realized variance of a stock $S = S_0 \exp(X)$ for fixings $0 = t_0 < \ldots < t_N = T$:

$$\sum_{n=1}^{N} \log(S_{t_n}/S_{t_{n-1}})^2 = \sum_{n=1}^{N} (X_{t_n} - X_{t_{n-1}})^2$$

Options on variance:

- Variance swap
- Volatility swap
- Puts on variance, variance calls, etc.

Tractable pricing formulas in realistic models?

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Introduction
Quadratic variation

For $\sup_{n=1,\dots,N} |t_n - t_{n-1}| \to 0$:

$$\sum_{n=1}^{N} (X_{t_n} - X_{t_{n-1}})^2 \to [X, X]_T \quad \text{in probability}$$

- Sepp (2008), Broadie & Jain (2008): Typically good approximation via **quadratic variation** $[X, X]$ for daily fixings
- Exception: Short-dated call options
- Pointed out by Bühler (2006), analyzed in Keller-Ressel & M-K (2010) $\Rightarrow$ Next talk!
- Here: Use approximation via quadratic variation $[X, X]$
- What type of structure of X makes this tractable?

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Introduction

Literature

For **continuous** stock prices **without leverage**:

- Benth et al. (2007): BNS model
- Carr & Lee (2007, 2009), Gatheral & Friz (2005): Model-free formulas

Models with **jumps**:

- Carr et al.(2005): Lévy processes
- Sepp (2008), Broadie & Jain (2008): Heston models with specific compound Poisson jumps
- Carr & Itkin (2009): Options on predictable quadratic variation $\langle X, X \rangle$ in time-changed Lévy models

Unifying framework including jumps, stochastic volatility and the leverage effect?

Introduction

Fourier-Laplace methods

Carr & Madan (1999), Raible (2000): Consider European-style option (e.g. put, call) with payoff

$$f(X_T) = \int_{R-i\infty}^{R+i\infty} l(z) e^{zX_T} dz, \quad R \in \mathbb{R}$$

- Price unter risk-neutral measure Q given by

$$E_Q[f(X_T)] = \int_{R-i\infty}^{R+i\infty} l(z) E_Q[e^{zX_T}] dz$$

- Tractable via numerical quadrature, if Fourier-Laplace transform $E_Q[e^{zX_T}]$ is known, likewise for $[X, X]$
- Flexible model class where this is the case: **Affine processes** characterized by Duffie et al. (2003)

Affine Stochastic Volatility Models

Definition

- Affine local characteristics of X and volatility v:

$$b^{(v,X)} = \beta_0 + \beta_1 v_-, \quad c^{(v,X)} = \gamma_0 + \gamma_1 v_-,$$
$$K^{(v,X)}(dx) = \kappa_0(dx) + \kappa_1(dx) v_-$$

- Affine conditional Fourier-Laplace transform:

$$E[e^{zX_T} | \mathcal{F}_t] = \exp(\Psi_0(T-t,z) + \Psi_1(T-t,z)v_t + zX_t),$$

where $\Psi_0(t,z) = \int_t^T \psi_0(\Psi_1(t,z),z)dt$ and

$$\partial_t \Psi_1(t,z) = \psi_1(\Psi_1(t,z),z), \quad \Psi_1(0,z) = 0$$

- Generalized Riccati PIDE with

$$\psi_i(z) = \beta_i^\top z + \frac{1}{2} z^\top \gamma_i z + \int (e^{zx} - 1 - zx)\kappa_i(dx)$$

Affine Stochastic Volatility Models

Examples

Includes most models from the option pricing literature:

- **Lévy models**
- **CIR-time-change models** (generalized Heston models):

$$X_t = L_{\int_0^t v_s ds} + \varrho(v_t - v_0) + \text{Drift}$$
$$dv_t = (\eta - \lambda v_t)dt + \sigma\sqrt{v_t}dZ_t$$

  for Lévy process L, Wiener process Z

- **OU-time-change models** (generalized BNS models):

$$X_t = L_{\int_0^t v_s ds} + \varrho Z_t + \text{Drift}$$
$$dv_t = -\lambda v_{t-}dt + dZ_t$$

  for Lévy process L, subordinator Z

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Affine Stochastic Volatility Models

- Definition:

$$[X, X]_t = \langle X^c, X^c \rangle_t + \sum_{s \leq t} \Delta X_s^2$$

- Local characteristics:

$$b^{[X,X]} = c^X + \int x^2 K^X(dx), \quad c^{[X,X]} = 0,$$
$$K^{[X,X]}(G) = \int 1_G(x^2) K^X(dx) \quad \forall G \in \mathcal{B}^2$$

- **Key observation**: $(v, X, [X, X])$ is affine in v!
- Still analytically tractable, characteristic function via generalized Riccati equations
- Compare $(r, \int_0^\cdot r_t dt)$ in affine short-rate models

Affine Stochastic Volatility Models

Quadratic variation: Characteristic function

Fourier-Laplace transform of $[X, X]_T$:

- Need to solve generalized Riccati PIDE
- No quadratic term, since $[X, X]$ is of finite variation
- But need to evaluate terms of the form

$$\int (e^{zx^2} - 1 - zx^2) K^X(dx),$$

  since $\Delta[X, X]_t = \Delta X_t^2$
- In many models of interest, this can be done using special functions
- Only difference compared to evaluation of stock options
- Then: Swaps via differentiation, options via integration

Affine Stochastic Volatility Models

Quadratic variation: Characteristic function ct'd

Example 1: Generalized Heston model of Carr et al. (2003):

$$X_t = L_{\int_0^t v_s ds} + \rho(v_t - v_0) + \text{Drift}, \quad dv_t = (\eta - \lambda v_t)dt + \sigma\sqrt{v_t}dZ_t$$

for Lévy process L with triplet (b^L, c^L, K^L), Wiener process Z.
Then:

$$E[e^{z[X,X]_T}|\mathcal{F}_t) = e^{\Psi_0(T-t,z)+\Psi_1(T-t,z)v_t+z[X,X]_t}$$

- $\Psi_1(t,z) = \frac{2g(z)(e^{f(z)t}-1)}{f(z)-\lambda+e^{f(z)t}(f(z)+\lambda)}$
- $\Psi_0(t,z) = \frac{2\eta}{\sigma^2}\log\left(\frac{2f(z)e^{t(f(z)+\lambda)/2}}{f(z)-\lambda+e^{f(z)t}(f(z)+\lambda)}\right)$
- $f(z) = \sqrt{\lambda^2 - 2\sigma^2 g(z)}$, $g(z) = (\sigma^2\rho^2 + c^L)z + \int(e^{zx^2} - 1)K^L(dx)$
  typically known in terms of special functions

Affine Stochastic Volatility Models

Quadratic variation: Characteristic function ct'd

Example 2: Model of Barndorff-Nielsen & Shephard (2001):

$$dX_t = (\text{Drift})dt + \sqrt{v_{t-}}dW_t + \rho dZ_t, \quad dv_t = -\lambda v_{t-}dt + dZ_t$$

for compound poisson process Z with rate a and $\exp(b)$-jumps. Then:

$$E[e^{z[X,X]_T}|\mathcal{F}_t) = e^{\Psi_0(T-t,z)+\Psi_1(T-t,z)v_t+z[X,X]_t}$$

- $\Psi_1(t,z) = \frac{1-e^{-\lambda t}}{\lambda}z$
- $\Psi_0(t,z) = \frac{ab}{2\sqrt{-\rho^2 z}}\int_0^t U\left(\frac{1}{2},\frac{1}{2},\frac{(b-\Psi_1(s,z))^2}{-4\rho^2 z}\right)ds - at$ for hypergeometric U-function
- One extra dt-integral compared to generalized Heston

Pricing Options on Variance

Variance swaps

- Choose swap rate K_{var} such that

$$E_Q([X,X]_T - K_{var}) = 0$$

- Differentiation of the characteristic function:

$$E_Q([X,X]_T|\mathscr{F}_t) = [X,X]_t + \partial_u \Psi_0(T-t,0) + \partial_u \Psi_1(T-t,0) v_t$$

- Variance swap dynamics are (inhomogeneously) affine!
- Opens the door to mean-variance hedging etc.
- Moreover: Explicit formulas for K_{var} in concrete models, e.g.,

$$K_{var} = \left(\frac{e^{-\lambda T} - 1 + \lambda T}{\lambda^2}\right)\frac{a}{b} + \frac{2a\varrho^2}{b^2}T + \frac{1-e^{-\lambda T}}{\lambda}v_0$$

for BNS model from above

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Pricing Options on Variance

European payoffs $f([X,X]_T)$

- Volatility swap: $f(x) = \sqrt{x} - K_{vol}$, hence

$$K_{vol} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - E_Q[e^{-z[X,X]_T}]}{z^{3/2}} dz$$

- Put on variance: $f(x) = (K - x)^+$, hence

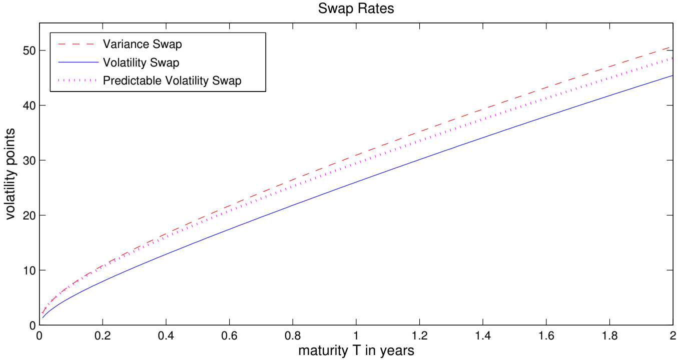
$$E_Q[(K-x)^+] = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} \frac{e^{-Kz}}{z^2} E_Q[e^{z[X,X]_T}] dz, \quad R < 0$$

- Evaluation via numerical quadrature
- Similar but simpler formulas for $\langle X, X \rangle$. No special functions, just one dz-integration
  $\Rightarrow$ Good approximation?

Numerical Illustration

Variance and volatility swaps

Above BNS model with calibrated parameters of Schoutens (2003):



- Considerable difference between quadratic variation $[X, X]$ and its predictable counterpart $\langle X, X \rangle$

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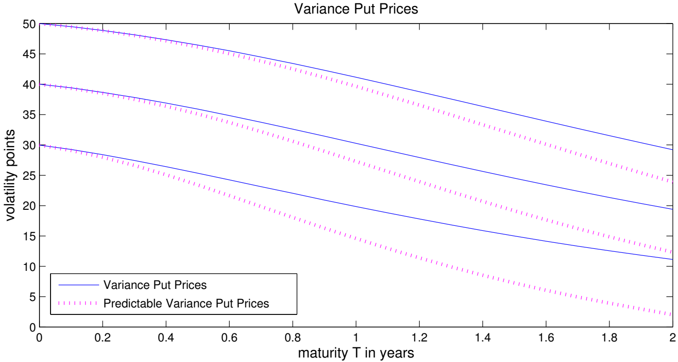
Numerical Illustration

Puts on variance

Above BNS model with calibrated parameters of Schoutens (2003):



- Again systematic error for approximation of $[X, X]$ with $\langle X, X \rangle$

Summary

Pricing options on variance in affine stochastic volatility models

- Approximation of realized variance by $[X, X]$
- Affine structure of (v, X) passed on to $(v, X, [X, X])$
- Characteristic function via generalized Riccati equations
- Variance swap prices via differentiation, volatility swaps, puts, calls etc. via numerical quadrature
- Integrands somewhat more involved than for stock options (special functions!), but still tractable
- Price processes of variance swaps are inhomogeneously affine

For more details:

- Kallsen, J., Muhle-Karbe, J., and M. Voß (2010). Pricing options on variance in affine stochastic volatility models. Forthcoming in *Mathematical Finance*. Available at `www.mat.univie.ac.at/~muhlekarbe`

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