Riding on the Smiles

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Joint work with **J. da Fonseca**

Outline of the presentation:

1. On the calibration of the Heston (1993) model: common pitfalls
2. Calibration of single asset multi-dimensional stochastic volatility models
3. Calibration of multi-asset multi-dimensional stochastic volatility models
4. Price approximations

On the calibration of the Heston (1993) model

$$\begin{aligned} \frac{dS_t}{S_t} &= \sqrt{v_t}dW_t^1 \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^2 \\ dW_t^1 dW_t^2 &= \rho dt \end{aligned}$$

ρ controls the link between vol and asset returns

$\Downarrow$

The Skew or Leverage

Analytic and Financial properties

- Characteristic function of the asset returns

$$\mathbb{E}_t \left[e^{i\omega \log(S_{t+\tau})} \right] = e^{A(\tau)v_t + B(\tau)\log(S_t) + C(\tau)}$$

- $A(\tau)$ solves a Riccati ODE: explicit solution!

- Quasi closed form option prices via Fast Fourier Transform (Carr and Madan 1999)

- Sensitivity analysis, vol of vol asymptotic expansion..

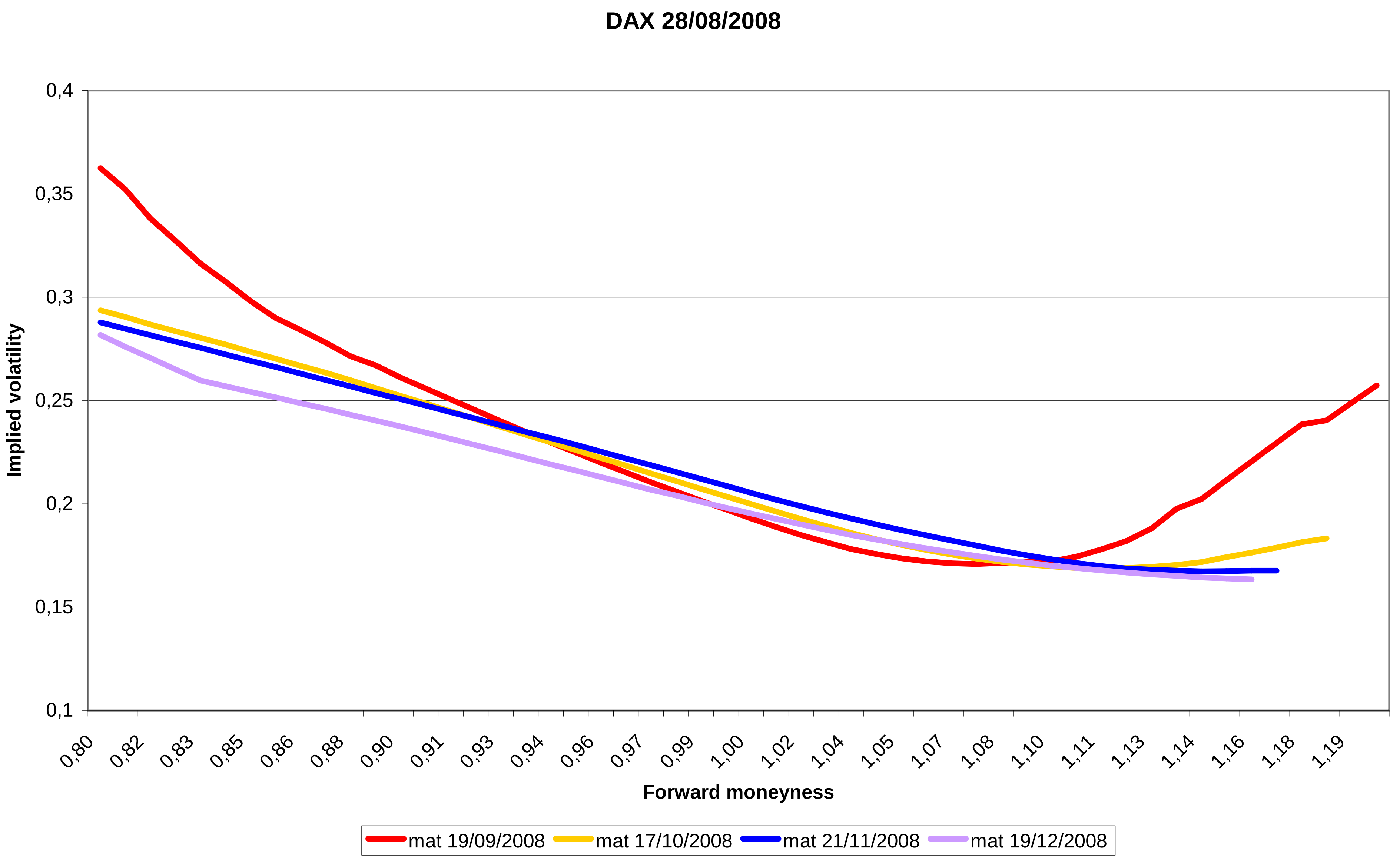
- Each parameter has a clear financial interpretation

Quoting vanilla options

The implied volatility σ_{imp} is the quantity such that

$$\underbrace{C_{mkt}(t,T,S_t,K)}_{\text{market price}} = \underbrace{c_{bs}(t,T,S_t,K,\sigma^2_{imp}(T-t))}_{\text{price in the Black\&Scholes model}} \tag{1}$$

The Smiles


DAX 28/08/2008
Implied volatility
0,4
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Forward moneyness
mat 19/09/2008
mat 17/10/2008
mat 21/11/2008
mat 19/12/2008

Important facts

The skew is controlled by ρ

$\Downarrow$

Term structure of skews

$\Downarrow$

We should have different values for ρ

and

above $T - t > 0.1$ the smiles are similar

The choice of the Criterium: pitfall of the price LSE

Calibration of vanilla options (OTM), maturities available

$$\min \frac{1}{N} \sum_{i=1}^{N} (C_{model}(t, T_i, K_i) - C_{mkt}(t, T_i, K_i))^2 \tag{2}$$

error	ρ	t_{min}
2.25E-07	-0.7095	0.05
2.06E-07	-0.7001	$\underbrace{0.1}_{\text{I don't take the first maturity}}$

- short term options have small (if no) impact on the solution
- the calibration seems to be good
- poor fit of short term options

What is the problem?

short term options have small time value w.r.t long term options

$$\Downarrow$$

small/no impact on the objective

The volatility LSE

$$\min \frac{1}{N} \sum_{i=1}^{N} (\sigma_{imp}^{model}(t, T_i, K_i) - \sigma_{imp}^{mkt}(t, T_i, K_i))^2$$

- more weight on short term options
- adding jumps does not help because jumps impact the very short part of the smile

Calibration tests (Vol norm)

error	ρ	$(T-t)_{min}$
0.00010773	-0.5562	0.05
4.31E-05	-0.6324	0.1

calibration date: 28/08/08

Maturities 0.06= 19/09, 0.13= 17/10 .. 4.31

- to fit the short term skew a low correlation is needed.

Why extending the Heston model?

- The dynamics of the implied volatility surface (vanilla options) and the Variance Swap curve are driven by several factors
- On the FX market the skew is stochastic
- We have a term structure of skew: short term skew $\neq$ long term skew

Double-Heston model

(Christoffensen, Heston, Jacobs 2007)

$$
\begin{aligned}
\frac{dS_t}{S_t} &= \sqrt{v_t^1}\,dZ_t^1 + \sqrt{v_t^2}\,dZ_t^2 \\
dv_t^1 &= \kappa^1(\theta^1 - v_t^1)dt + \sigma^1\sqrt{v_t^1}\,dW_t^1 \\
dv_t^2 &= \kappa^2(\theta^2 - v_t^2)dt + \sigma^2\sqrt{v_t^2}\,dW_t^2 \\
dZ_t^1 dW_t^1 &= \rho^1 dt \\
dZ_t^2 dW_t^2 &= \rho^2 dt
\end{aligned}
$$

but

$$
\underbrace{dZ_t^1 dZ_t^2 = dW_t^1 dW_t^2 = dZ_t^1 dW_t^2 = dZ_t^2 dW_t^1 = 0}_{AFFINITY}
$$

Recall the Duffie-Filipovic-Schachermayer (2003)'s condition

If $X_t = (X_t^1, X_t^2)^\top$ is a vector affine square root process (thus positive):

$$d\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = ...dt + \begin{pmatrix} \times & 0 \\ 0 & \times \end{pmatrix} d\begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix}$$

$$\Downarrow$$

We have strong constraints on the diffusion

$$\Downarrow$$

Strong constraints on the correlation!!

$$\Downarrow$$

We can not correlate v_t^1 and v_t^2 in the Double-Heston

Main question

Is it possible to find an AFFINE model allowing for nontrivial correlation among factors?

$\Downarrow$

Yes, choose a suitable State Space Domain!

Wishart multi-dim Stochastic Vol

- Bru (1991).
- Gourieroux and Sufana (2004).
- Extended by Da Fonseca, Grasselli and Tebaldi (2008)

$$\frac{dS_t}{S_t} = rdt + Tr\left[\sqrt{\Sigma_t}dZ_t\right]$$

- $d\Sigma_t = (\beta Q^\top Q + M\Sigma_t + \Sigma_t M^\top)dt + \sqrt{\Sigma_t}dW_t Q + Q^\top dW_t^\top \sqrt{\Sigma_t}$
- Z_t = Matrix Brownian Motion correlated with W_t (Matrix Brownian Motion)
- $Vol(S_t) = Tr\left[\Sigma_t\right]$ linear combination of the Wishart elements

- $d\Sigma_t = (\beta Q^\top Q + M\Sigma_t + \Sigma_t M^\top)dt + \sqrt{\Sigma_t} dW_t Q + Q^\top dW_t^\top \sqrt{\Sigma_t}$
- $\Omega\Omega^\top = \beta Q^\top Q$ with β large enough (Gindikin's condition)
- M negative definite $\Leftrightarrow$ mean reverting behavior
- Σ_t SYMMETRIC MATRIX SQUARE ROOT PROCESS ($n \times n$)
- Q vol-of-vol.
- $(W_t; t \geq 0)$ is a matrix Brownian motion ($n \times n$)

Correlation in the Wishart model

- $R \in M_n$ (identified up to a rotation) completely describes the correlation structure:

$$\begin{aligned} Z_t &= W_t R^\top + B_t \sqrt{\mathbb{I} - RR^\top} \\ &= \text{Matrix Brownian motion!} \end{aligned}$$

- This choice is compatible with affinity of the model!!

- Other (few) choices are possible but harder to manage.

- The Wishart Affine model is solvable. That is, the conditional characteristic function can be written as:

$$\mathbb{E}_t e^{i\omega \log(S_{t+\tau})} = e^{Tr[A(\tau)\Sigma_t]+B(\tau)\log(S_t)+C(\tau)}$$

- $A(\tau)$ solves a Riccati ODE that can be linearized! (Grasselli and Tebaldi 2008)

Stochastic correlation between stock returns and vol

$$Corr_t\left(dln(S), dVol\left(ln(S)\right)\right) = \quad \rho_t = \quad \frac{2Tr\left[\Sigma_t RQ\right]}{\sqrt{Tr\left[\Sigma_t\right]}\sqrt{Tr\left[\Sigma_t Q^\top Q\right]}}$$

- Stochastic correlation between the stock and its volatility
- Multi-dimensional correlation/volatility SHOULD allow for more complex skew effects

Calibration single-asset stochastic volatility models:

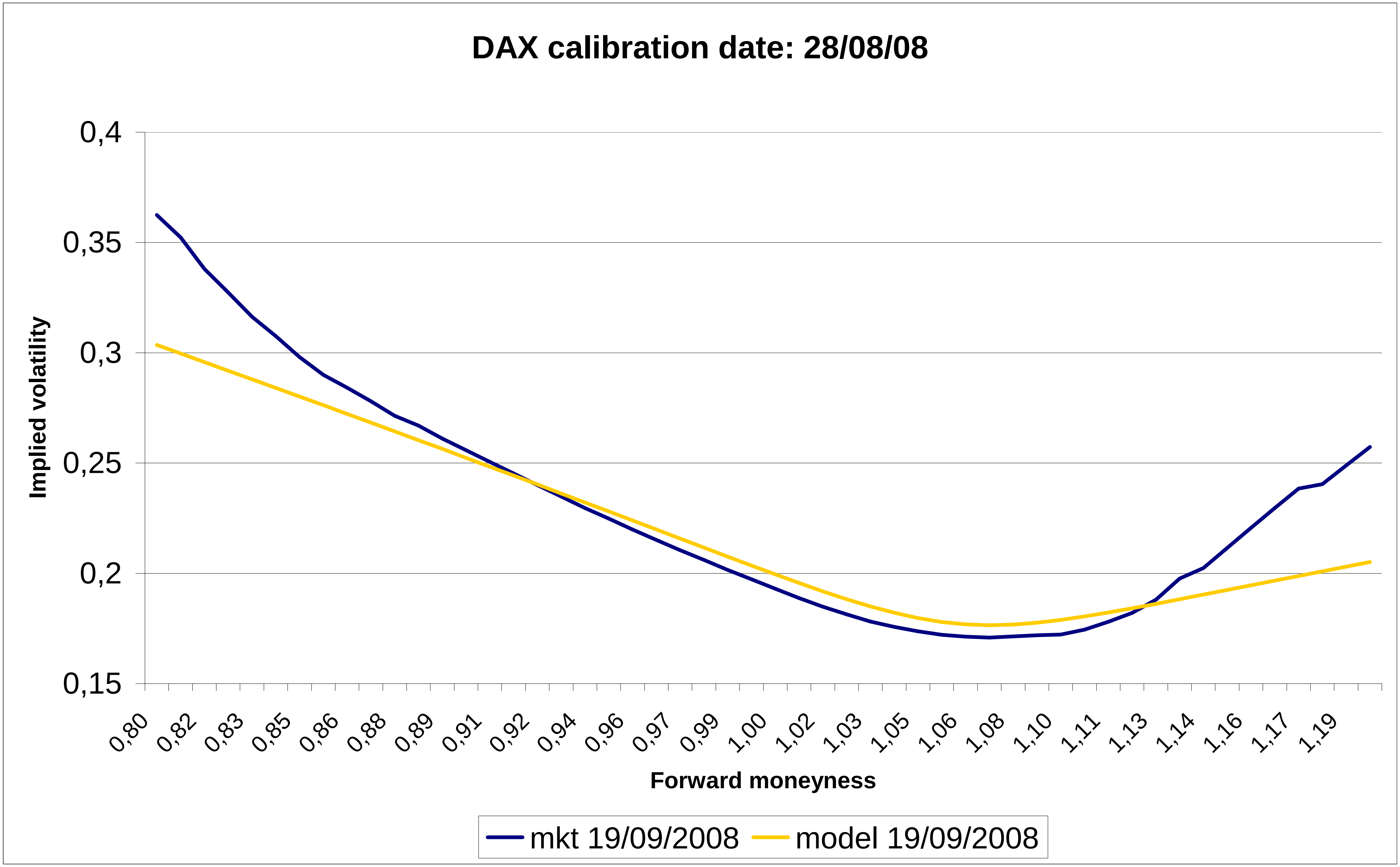
Model	error	$\rho_1(\rho_{11})$	$\rho_2(\rho_{12})$	ρ_{21}	ρ_{22}
Heston	0.00010773	-0.556	xxx		
BiHeston	7.61E-05	-0.393	-0.866		
Wishart	7.19E-05	-0.258	0.017	-0.343	-0.766

- the Wishart/BiHeston perform better than Heston model (not surprising!)
- the Wishart model performs slightly better than BiHeston model but numerical the cost is higher

What about adding jumps?

Model	error	$\rho_1(\rho_{11})$	$\rho_2(\rho_{12})$	ρ_{21}	ρ_{22}
Heston	0.00010773	-0.556	xxx		
BiHeston	7.61E-05	-0.393	-0.866		
Wishart	7.19E-05	-0.258	0.017	-0.343	-0.766
BiBates	2.82E-05	-0.527	0.814		

- Jumps do not change significantly the parameters of the BiHeston
- Improve the very short term fit (less than 3 weeks)
- No conflict with diffuse part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Forward moneyness
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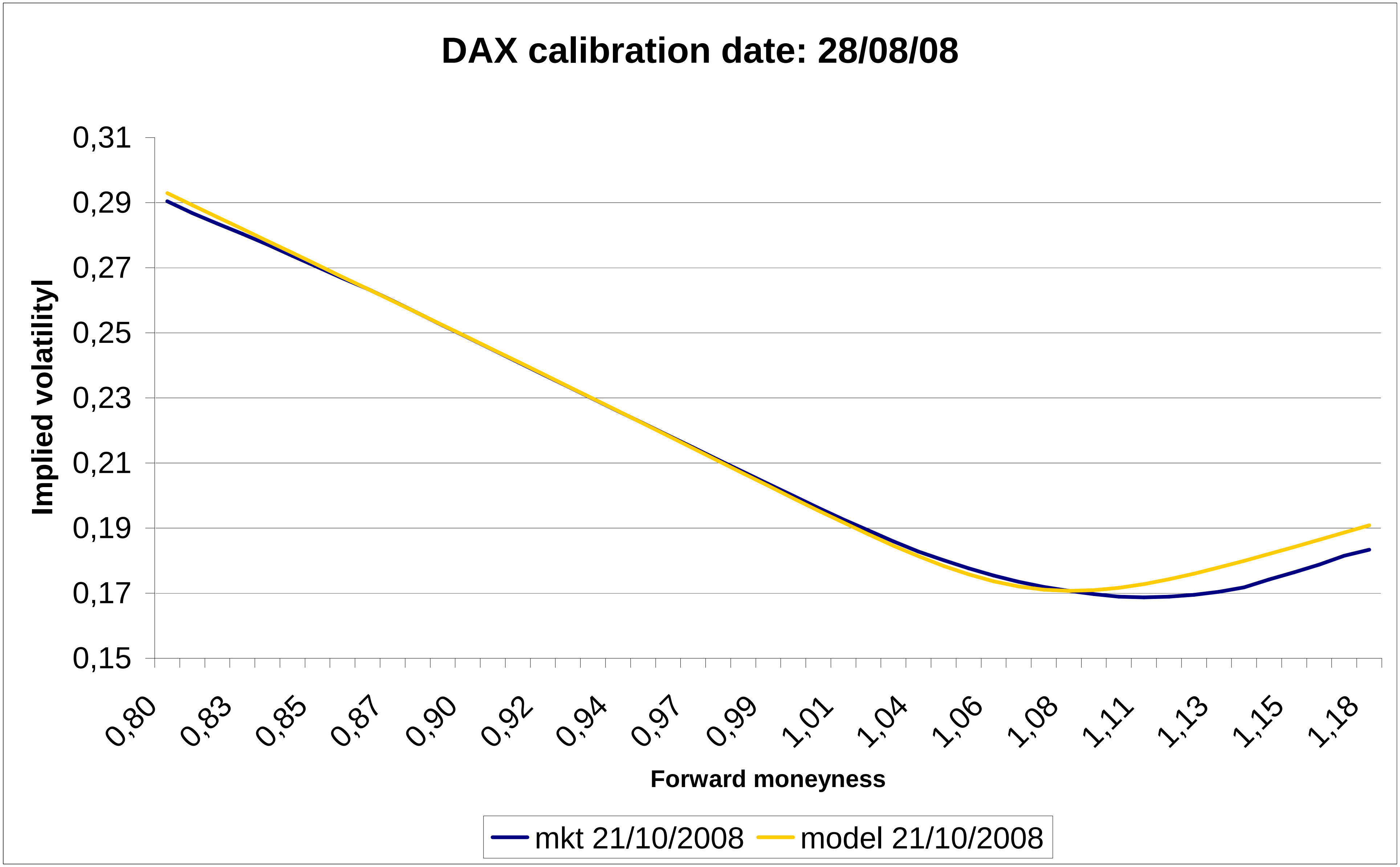
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Forward moneyness
mkt 21/10/2008
model 21/10/2008

A Closer look at the σ_{imp} for short time

Using perturbation method (vol of vol) as in Benabid, Bensusan, El Karoui (2009) we can prove that for ($T - t \sim 0$) as a function of the forward moneyness m_f

$$\sigma_{imp}^2 \sim \mathrm{Tr}[\Sigma_t] + \frac{\mathrm{Tr}[RQ\Sigma_t]}{\mathrm{Tr}[\Sigma_t]} m_f$$

A Double-Heston model would lead to

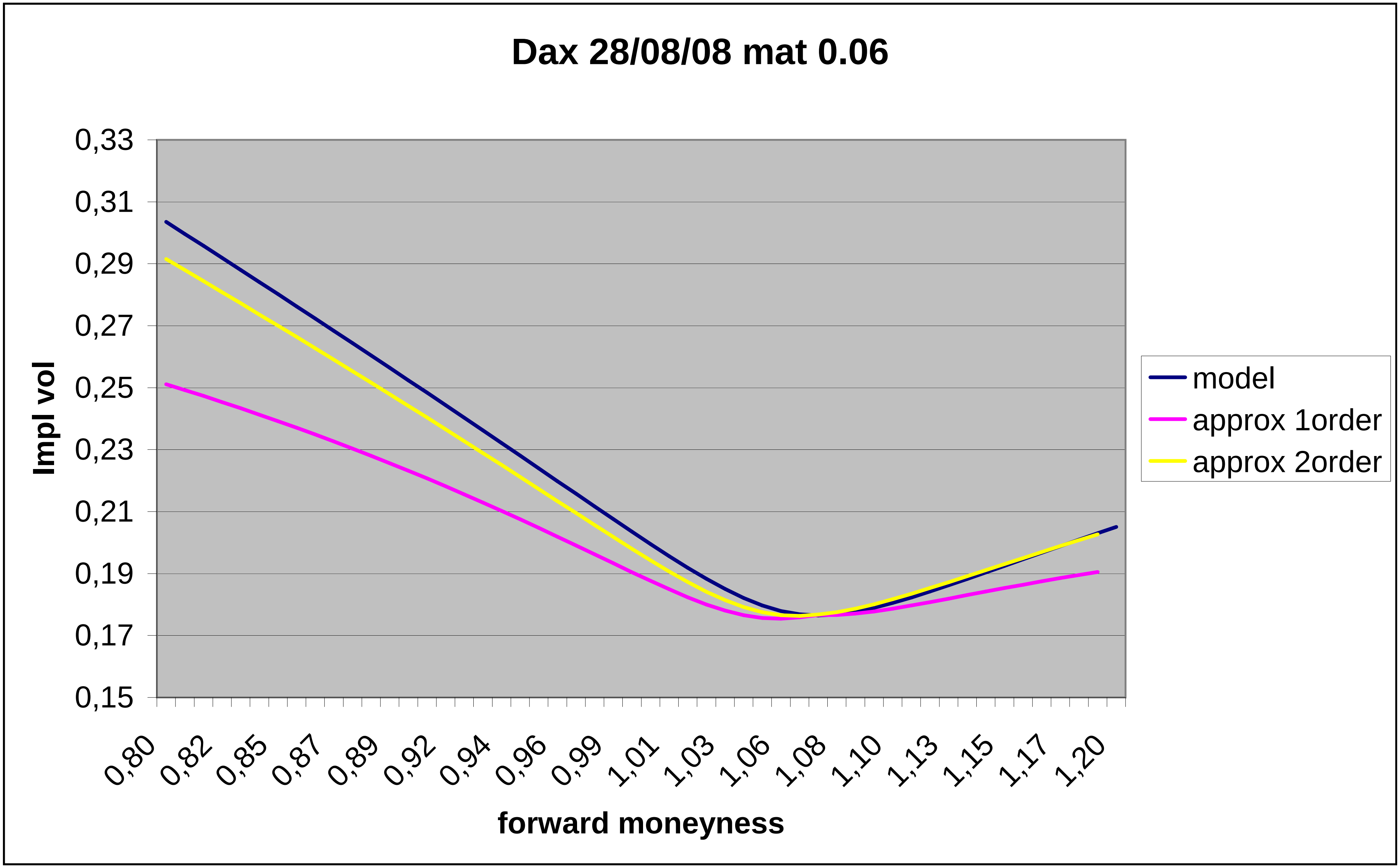
$$\sigma_{imp}^2 \sim v_1 + v_2 + \left(\frac{v_1 \rho_1 \sigma_1 + v_2 \rho_2 \sigma_2}{v_1 + v_2} \right) \frac{m_f}{2}$$

- Σ_{12} controls the slope of the skew and $\Sigma_{11} + \Sigma_{22}$ controls the level of the smile (as far as RQ is non diagonal).
- in the Double-Heston the factors impact both level and skew!

Conclusions

- as far as we are interest with vanilla options the BiHeston and Wishart performs equally
- but the Wishart allows a better management of the implied volatility risks
- the numerical cost of the Wishart model is much more important. How to speed up the pricing process?
- if the calibrated model will be used to price a derivative which is sensitive to the slope of the skew then the Wishart model is of interest
- the selected model depends on
  - the complexity of the smile to be calibrated
  - the sensitivity of the derivative to be priced with the calibrated model
- it raises the problem of how to aggregate the ratios from different models

Numerical results for price approximations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forward moneyness
model
approx 1order
approx 2order

The Multi-asset model

How to build a multi asset framework:

- Consistent with the smile in vanilla options
- With a general correlation structure
- Analytic as much as possible

Using Heston's model

$$
\begin{aligned}
dS_t^1 &= S_t^1 r dt + S_t^1 \sqrt{V_t^1} dZ_t^1 \\
dV_t^1 &= \kappa_1(\theta_1 - V_t^1) dt + \sigma_1 \sqrt{V_t^1} dW_t^1 \\
dS_t^2 &= S_t^2 r dt + S_t^2 \sqrt{V_t^2} dZ_t^2 \\
dV_t^2 &= \kappa_2(\theta_2 - V_t^2) dt + \sigma_2 \sqrt{V_t^2} dW_t^2
\end{aligned}
$$

$dZ^1 dZ^2 = 0 \Leftrightarrow$ Affinity of the model

$$\Downarrow$$

$$\frac{dS^1}{S^1}\frac{dS^2}{S^2} = 0$$

The Wishart Affine Stochastic Correlation model

Da Fonseca, Grasselli and Tebaldi (RDR-2007):

The model: $S_t = (S_t^1, \ldots, S_t^n)^\top$ and $\Sigma_t \in M_{(n,n)}$

$$dS_t = diag[S_t]\left(\mu dt + \sqrt{\Sigma_t} dZ_t\right)$$
$$d\Sigma_t = \left(\Omega\Omega^\top + M\Sigma_t + \Sigma_t M^\top\right)dt + \sqrt{\Sigma_t}dW_t Q + Q^\top (dW_t)^\top \sqrt{\Sigma_t}$$

dZ_t is a vector BM (n,1) and dW_t is a matrix BM (n,n):

$$\frac{dS^i}{S^i}\frac{dS^j}{S^j} = \Sigma^{ij} dt$$

How to correlate dZ and dW?

In Da Fonseca, Grasselli and Tebaldi (RDR-2007):

Affinity of the infinitesimal generator

$$\Updownarrow$$

$$dZ_t = dW_t \rho + \sqrt{1 - \rho^\top \rho} dB_t$$

where ρ is a vector (n,1) and dB is a vector BM(n,1).

- only n parameters to specify the skew
- parsimoniuous model
- Characteristic function has an exponential affine form , it involves the computation of the exponential of a matrix.

Pricing plain vanilla options on single assets

- In the WASC model, the single assets evolve according to a Heston-like dynamics.

- Assets' returns and volatilities are partially correlated:

$$Corr_t\left(Noise(Y^1), Noise(Vol(S^1))\right) = \frac{Q_{11}\rho_1 + Q_{21}\rho_2}{\sqrt{Q_{11}^2 + Q_{21}^2}}$$

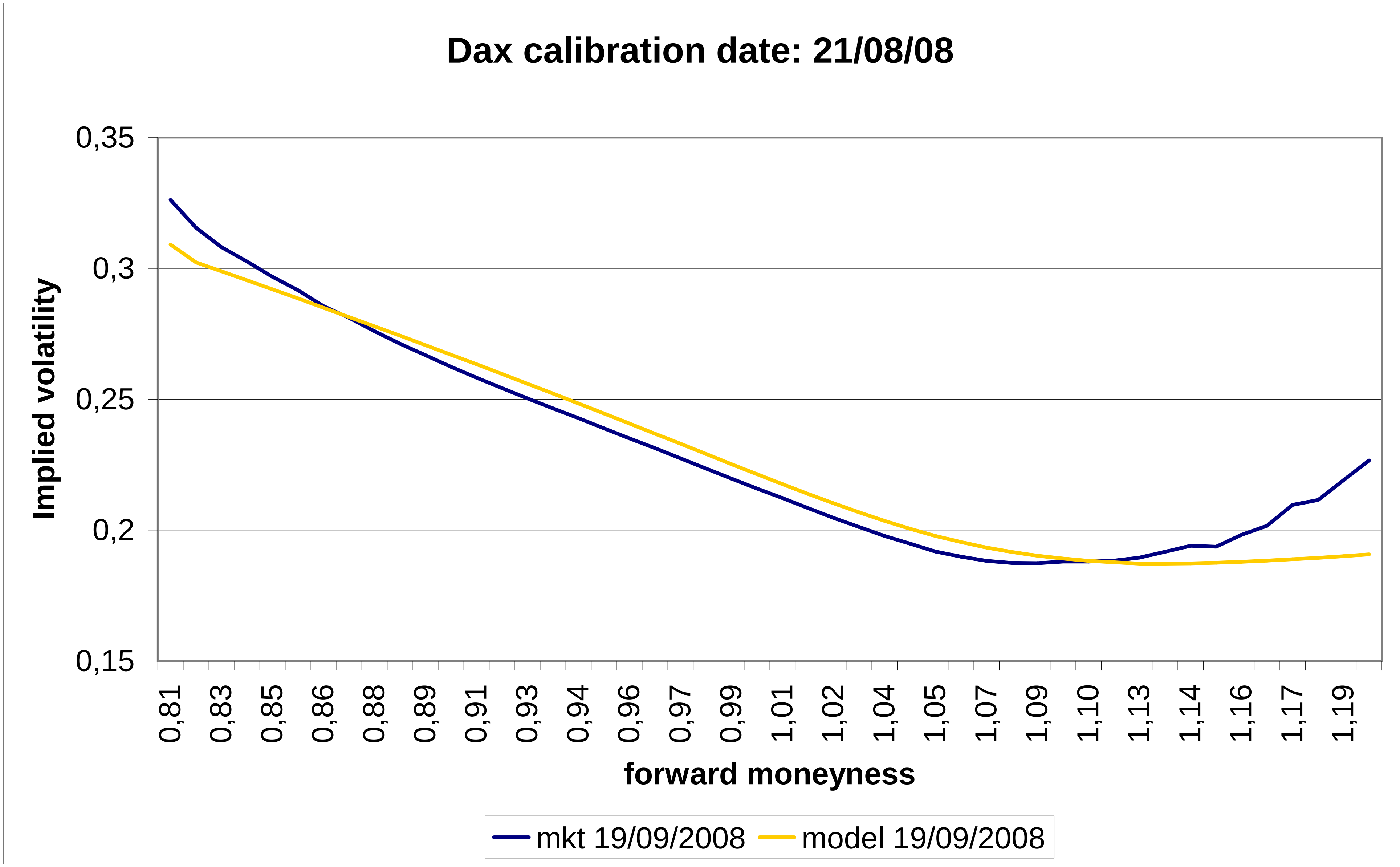
- $\text{Vol-Of-Vol}(S_1) = 2\sqrt{Q_{11}^2 + Q_{21}^2}$

- Skew in the implied volatility is related with the correlation, cross-asset effects appear(systematic vs specific dependence)

Calibration results in the multi-asset model

Stock	error(WASC)	error(Heston)
Dax	2.52E-05	1.105E-04
SP	1.39E-04	1.59E-04

- we calibrate a stochastic correlation model using only vanilla options!
- vanilla options are basket products


Dax calibration date: 21/08/08
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forward moneyness
mkt 19/09/2008
model 19/09/2008

A closer look at σ_{imp} for short time

We can prove

$$\sigma_{imp}^{Dax} \;=\; \Sigma_t^{11} + (\rho_1 Q_{11} + \rho_2 Q_{21}) m_f + m_f^2 \left[\frac{4(Q_{11}^2 + Q_{21}^2) - 7(\rho_1 Q_{11} + \rho_2 Q_{21})^2}{6 \Sigma_t^{11}} \right]$$

Recall for Heston we have

$$\sigma_{imp}^2 = v + \sigma \frac{\rho}{2} m_f + \frac{\sigma^2}{2} \frac{m_f^2 (4 - 7\rho^2)}{24 v}$$

- the expansions for the smile are similar
- the same problem as for Heston: we have a concave relation! those asymptotics can not be used to build a starting point for the calibration!
- at first order ρ and σ appear as a product $\rightarrow$ identification problem (same for Wasc)
- this aggregation of parameters allows to understand the parameter values

A competitor

$$
\begin{aligned}
ds_1(t) &= s_1(t)(\sqrt{v_1(t)}dw_1(t) + \sqrt{v_0(t)}dw_0(t)) \\
ds_2(t) &= s_2(t)(\sqrt{v_2(t)}dw_2(t) + \sqrt{v_0(t)}dw_0(t)) \\
dv_1(t) &= \kappa_1(\theta_1 - v_1(t))dt + \sigma_1\sqrt{v_1(t)}(\rho_1 dw_1(t) + \sqrt{1-\rho_1^2}d\tilde{w}_1(t)) \\
dv_2(t) &= \kappa_2(\theta_2 - v_2(t))dt + \sigma_2\sqrt{v_2(t)}(\rho_2 dw_2(t) + \sqrt{1-\rho_2^2}d\tilde{w}_2(t)) \\
dv_0(t) &= \kappa_0(\theta_0 - v_0(t))dt + \sigma_0\sqrt{v_0(t)}(\rho_0 dw_0(t) + \sqrt{1-\rho_0^2}d\tilde{w}_0(t))
\end{aligned}
$$

- this model allows stochastic correlation and is more tractable (the CF is computationally less complicated than the Wasc).
- in this model we have a factor model for the covariance matrix whereas for the Wasc model the covariance matrix is the factor, might be of interest when dealing with estimation

Conclusions

- we build a model which is tractable
- this model allows for stochastic volatilities and stochastic correlation
- we provide some results on calibration using single underlying options with the consequence that vanilla options are basket products.

some open problems

- building estimation strategy, for the Wasc see Da Fonseca, Grasselli, Ielpo (2009).
- how to increase the dimension of the model and still be able to estimate it
- how to aggregate the risks of different models: Heston, BiHeston, Wishart, Wasc and others...

Thanks for your attention!