Conditional Density Models for Asset Pricing

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The conditional density process

We shall consider the problem of how best to model the dynamics of an asset price when we are given option prices for a range of strikes and maturities as initial data.

This problem has a long history.

When option pricing theory was being developed in the 1970s, it was originally thought that the option price could be modelled as a function of two variables—the value of the underlying asset, and time.

It was eventually recognised however that option prices have the potential to carry more information than that simply entailed in the current level of the underlying asset.

To put it another way, option prices to some extent have a life of their own, and need to be modelled along side the dynamics of the underlying asset.

This point was implicitly recognised in the work of Breeden & Litzenberger (1978) who showed that the (risk-neutral) probability density for the value of an asset at some future time T could be obtained from the system of call option prices with maturity T by differentiating the price twice with respect to the strike price.

This observation was one of a series of developments that eventually led to the idea that option prices should be used as "inputs" to asset pricing models (rather than only as "outputs").

Since then a great deal of work has been carried out on how to use option price data as inputs in asset pricing models—but even now there is no generally accepted or universal scheme for carrying this out.

A major step forward was taken by Dupire (1994) and others, who considered the situation where the asset price could be modelled by a diffusion process.

In particular, it was noted that the Fokker-Planck equation (or Kolmogorov forward equation) for the risk-neutral probability density can be used to show that the local price volatility (when regarded as a function of the value of the asset and of time) is completely determined by the system of option prices for all strikes and a band of maturities.

On the other hand, it appears to be too restrictive to assume from the outset that asset prices can be modelled by simple diffusion processes.

There does not seem to be any economic justification for assuming that volatility should be given by a function of the current level of the asset.

On the other hand, the category of general stochastic volatility models is very large, and once one leaves the special case of simple diffusion models it is not so clear how to organise the general theory, and in particular how to treat the "input" problem (calibration) from a broad perspective.

Keeping all this in mind, our approach will be to model the conditional probability density process, with respect to a suitable choice of measure, for the asset price at some fixed future time.

Modelling the conditional density process

The ingredients will consist of a filtered probability space $(\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}, \mathbb{Q})$ together with an $\{\mathcal{F}_t\}$-adapted Brownian motion $\{W_t\}$.

Here $\mathbb{Q}$ denotes the martingale measure associated with the choice of some non-dividend-paying asset as numeraire.

We let $\{S_t\}$ denote the $\{\mathcal{F}_t\}$-adapted price process of a non-dividend-paying asset, and we assume that prices are given in units of the numeraire asset.

It follows then that $\{S_t\}$ is a martingale, and that the $\mathbb{Q}$-dynamics of $\{S_t\}$ are of the form

$$\mathrm{d}S_t = \Sigma_t \mathrm{d}W_t, \tag{1}$$

for some stochastic volatility process $\{\Sigma_t\}$.

We shall not attempt to model the dynamics of the volatility of the asset directly.

There are of course a number of models of this type (Heston, SABR, etc), but we shall take a different approach.

Instead, we shall model the dynamics of the conditional probability density of the asset price at some fixed time T.

More specifically, we fix a time $T > 0$, and assume for $t < T$ the existence of an $\mathcal{F}_t$-conditional probability density $f_{tT}(x) > 0$ for S_T.

In particular:

(a) For all bounded, measurable functions $g(x)$ we have

$$\int_{-\infty}^{\infty} g(x) f_{tT}(x) \mathrm{d}x = \mathbb{E}\left[g(S_T) \,|\, \mathcal{F}_t \right]. \tag{2}$$

(b) We have the martingale property

$$\mathbb{E}\left[f_{tT}(x) \,|\, \mathcal{F}_s \right] = f_{sT}(x), \qquad 0 \leq s \leq t < T. \tag{3}$$

(c) The asset price can be expressed in terms of the density by writing

$$S_t = \int_{-\infty}^{\infty} x f_{tT}(x) \mathrm{d}x. \tag{4}$$

It follows that if $C_{tT}(K)$ denotes the price at time t of a T-maturity, K-strike call option, then

$$C_{tT}(K) = \int_{-\infty}^{\infty} (x-K)^{+} f_{tT}(x)\mathrm{d}x. \tag{5}$$

On the other hand, by differentiating $C_{tT}(K)$ twice with respect to the strike K we obtain

$$\frac{\partial^2 C_{tT}(x)}{\partial x^2} = f_{tT}(x). \tag{6}$$

Clearly knowledge of the T-maturity call option prices at time t for all K allows us to reconstruct the density $f_{tT}(x)$.

The goal now is to model the dynamics of the density process $\{f_{tT}(x)\}$.

Modelling the volatility structure

We assume that the dynamical equation for $\{f_{tT}(x)\}$ takes the form

$$\mathrm{d}f_{tT}(x) = f_{tT}(x)V_{tT}(x)\,\mathrm{d}W_t. \tag{7}$$

The normalisation condition

$$\int_{-\infty}^{\infty} f_{tT}(x)\mathrm{d}x = 1 \tag{8}$$

then implies that $V_{tT}(x)$ is of the form

$$V_{tT}(x) = \sigma_{tT}(x) - \int_{-\infty}^{\infty} \sigma_{tT}(y)f_{tT}(y)\mathrm{d}y, \tag{9}$$

for some $\{\sigma_{tT}(x)\}$.

Thus, once we specify $\{\sigma_{tT}(x)\}$ and the initial density $f_{0T}(x)$, the dynamics of the density are given by

$$\mathrm{d}f_{tT}(x) = f_{tT}(x)\left[\sigma_{tT}(x) - \int_{-\infty}^{\infty} \sigma_{tT}(y)f_{tT}(y)\mathrm{d}y\right]\mathrm{d}W_t. \tag{10}$$

Now we are in a position to say what we mean by a "model" for $\{f_{tT}(x)\}$.

In this specification we are motivated by advances in the studies of infinite-dimensional SDEs and SPDEs in the context of interest-rate theory.

By a "model" for the density process $\{f_{tT}\}$ we understand the following:

(1) A specification of an *initial density* $f_{0T}(x)$.

(2) A specification of the *volatility structure* $\{\sigma_{tT}(x)\}$ in the form of a functional

$$\sigma_{tT}(x) = \Phi[f_{tT}(\cdot), t, x]. \tag{11}$$

The initial density $f_{0T}(x)$ is determined by the specification of initial option price data for maturity T and all strikes K.

$$C_0(T, K) = \mathbb{E}[(S_T - K)^+] = \int (x - K)^+ f_{0T}(x)\mathrm{d}x. \tag{12}$$

Assuming that C_0 is twice differentiable with respect to the strike, we have:

$$f_{0T}(x) = \frac{\partial^2 C_0(T, x)}{\partial x^2}. \tag{13}$$

Ideally, we would then like to be able to specify the functional $\Phi[f_{tT}(\cdot), t, x]$ modulo just enough freedom to let us input initial option price data for all maturities τ in the range $0 < \tau < T$, and all strikes K.

That is one scenario that would lead to an interesting class of models.

But more generally we might like to be in a position to specify more data—or less data—to fix the model, depending on the context in which the model will be used.

The idea is to develop a methodology that is sufficiently flexible to accommodate various different types of option markets.

In what follows we present some specific examples of models showing how these goals can be realised, at least in part.

Example I: Bachelier model

The so-called Bachelier model, assuming $S_0 = 0$, is given by

$$S_t = \nu W_t, \tag{14}$$

where ν is a constant. This is a very simple model, too simple for practical application, but it is not without interest.

The conditional density is

$$f_{tT}(x) = \frac{1}{\nu\sqrt{2\pi(T-t)}} \exp\left(-\frac{1}{2}\frac{(x-S_t)^2}{\nu^2(T-t)}\right) \qquad (0 \le t < T). \tag{15}$$

Applying Ito's Lemma to (15), we obtain the following SDE for $\{f_{tT}(x)\}$:

$$\mathrm{d}f_{tT}(x) = f_{tT}(x)\left[\frac{x}{\nu(T-t)} - \int_{-\infty}^{\infty} \frac{y}{\nu(T-t)} f_{tT}(y)\mathrm{d}y\right]\mathrm{d}W_t. \tag{16}$$

Hence we see that the "master" equation

$$\mathrm{d}f_{tT}(x) = f_{tT}(x)\left[\sigma_{tT}(x) - \int_{-\infty}^{\infty} \sigma_{tT}(y) f_{tT}(y)\mathrm{d}y\right]\mathrm{d}W_t \tag{17}$$

is indeed satisfied if we set:

$$\begin{aligned} f_{0T}(x) &= \frac{1}{\nu\sqrt{2\pi T}}\exp\left(-\frac{1}{2}\frac{x^2}{\nu^2 T}\right), \\ \sigma_{tT}(x) &= \frac{x}{\nu(T-t)}. \end{aligned} \tag{18}$$

The pair $f_{0T}(x)$, $\sigma_{tT}(x)$ indicated above thus gives the Bachelier model.

Note: The initial density $f_{0T}(x)$ is already fully determined in this model, apart from the specification of the parameter ν, and so is the volatility structure $\sigma_{tT}(x)$.

We may therefore regard the Bachelier model as almost completely "rigid", since there is rather little scope for the input of initial option data.

We observe in particular that the Bachelier volatility structure (18) is deterministic and "semi-linear".

Example II: Bachelier volatility structure, with an arbitrary initial density

Again we consider the dynamics

$$\mathrm{d}f_{tT}(x) = f_{tT}(x)\left[\sigma_{tT}(x) - \int_{-\infty}^{\infty}\sigma_{tT}(y)f_{tT}(y)\mathrm{d}y\right]\mathrm{d}W_t. \tag{19}$$

In this example we take an arbitrary initial density and combine this with a semilinear ("Bachelier-type") volatility structure:

$$\begin{aligned} f_{0T}(x) &= \bar{f}_{0T}(x) \\ \sigma_{tT}(x) &= \frac{\sigma T}{T-t}x. \end{aligned} \tag{20}$$

Here $\bar{f}_{0T}(x)$ is the initial density implied by T-maturity option data at time 0.

This model can also be solved explicitly, though the form the solution takes is more subtle.

In fact we are able to show that $f_{tT}(x)$ takes the following form:

$$f_{tT}(x) = \frac{f_{0T}(x)\exp\left[-\frac{1}{2}\frac{T}{t(T-t)}\left(\xi_{tT}-\sigma t x\right)^2\right]}{\int_{-\infty}^{\infty} f_{0T}(x)\exp\left[-\frac{1}{2}\frac{T}{t(T-t)}\left(\xi_{tT}-\sigma t x\right)^2\right]\mathrm{d}x}. \tag{21}$$

Here $\{\xi_{tT}\}_{0\le t\le T}$ is a so-called information process, defined by

$$\xi_{tT} = \sigma t S_T + \beta_{tT}. \tag{22}$$

The process $\{\beta_{tT}\}_{0\le t\le T}$ is a *Brownian bridge* that is taken to be independent of the random variable S_T describing the terminal asset-price value.

In this model the filtration $\{\mathcal{F}_t\}$ is generated by $\{\xi_{tT}\}$. We are then able to show that the density process satisfies the master equation,

$$\mathrm{d}f_{tT}(x) = f_{tT}(x)\left[\sigma_{tT}(x) - \int_{-\infty}^{\infty}\sigma_{tT}(y)f_{tT}(y)\mathrm{d}y\right]\mathrm{d}W_t. \tag{23}$$

Here the process $\{W_t\}$ is defined by:

$$W_t = \xi_{tT} + \int_0^t \frac{1}{T-s}\xi_{sT}\mathrm{d}s - \sigma T\int_0^t \frac{1}{T-s}\int_{-\infty}^{\infty} x f_{sT}(x)\,\mathrm{d}x\,\mathrm{d}s. \tag{24}$$

It is an exercise in martingale theory (not trivial!) to show that $\{W_t\}$ is an $\{\mathcal{F}_t\}$-Brownian motion.

That shows that $\{f_{tT}(x)\}$ satisfies the required dynamical equation.

This model shows much more flexibility than the elementary Bachelier model.

In fact it allows a full calibration to the implied density $\bar{f}_{0T}(x)$ derived from the initial T-maturity option data $C_{0T}(K)$ for all K.

Example III: deterministic volatility structure

Now we look at a model for the density process that has the scope to be calibrated with initial option data for all strikes and for a family of maturities τ such that $0 < \tau \leq T$.

Again we begin with the dynamical equation $f_{tT}(x)$.

We consider the following model:

$$f_{0T}(x) = \bar{f}_{0T}(x),$$

$$\sigma_{tT}(x) = v(t,x). \tag{25}$$

Here $v(t,x)$ is a deterministic function of two variables. The solution takes the following form:

Let the filtration $\{\mathcal{F}_t\}$ for this model be generated by the process $\{Z_t\}$ defined by:

$$Z_t = B_t + \int_0^t v(s, S_T)\,\mathrm{d}s. \tag{26}$$

Here $\{B_t\}$ is a Brownian motion which is assumed to be independent of S_T.

The solution of the master equation for $\{f_{tT}(x)\}$ is then given by

$$f_{tT}(x) = \frac{f_{0T}(x)\exp\left[\int_0^t v(s,x)\mathrm{d}Z_s - \frac{1}{2}\int_0^t v^2(s,x)\mathrm{d}s\right]}{\int_{-\infty}^{\infty} f_{0T}(x)\exp\left[\int_0^t v(s,x)\mathrm{d}Z_s - \frac{1}{2}\int_0^t v^2(s,x)\mathrm{d}s\right]\mathrm{d}x}. \tag{27}$$

In particular, we find that the density process satisfies

$$\mathrm{d}f_{tT}(x) = f_{tT}(x)\left[v(t,x) - \int_{-\infty}^{\infty} v(t,y) f_{tT}(y)\mathrm{d}y\right]\mathrm{d}W_t, \tag{28}$$

where

$$W_t = Z_t - \int_{s=0}^{t}\int_{x=-\infty}^{\infty} f_{sT}(x)\, v(s,x)\mathrm{d}x\mathrm{d}s. \tag{29}$$

A calculation then verifies that $\{W_t\}$ is indeed an $\{\mathcal{F}_t\}$-Brownian motion.

In this model we can calibrate the initial density $f_{0T}(x)$ to the implied density $\bar{f}_{0T}(x)$ coming from options with maturity T.

But we can also, at least in principle, calibrate the model to data coming from options with maturities τ less than T.

This is because:

$$
\begin{aligned}
C_{0\tau}(K) &= \mathbb{E}\left[(S_\tau - K)^+\right] \qquad (\tau < T) \\
&= \mathbb{E}\left[\left(\int_{-\infty}^{\infty} x f_{\tau T}(x)\mathrm{d}x - K\right)^+\right] \\
&= \Theta\left[\{v(t,\cdot)\}_{0\le t\le\tau}, \tau, K\right], \qquad (30)
\end{aligned}
$$

where the functional Θ depends only on $\{v(t,\cdot)\}_{0\le t\le\tau}$, τ, and K.

For each input density $\bar{f}_{0T}(x)$ and volatility structure function $v(t,x)$ we then obtain a corresponding price surface $C_{0\tau}(K)$ for $0 \le \tau \le T$ and $-\infty < K < \infty$.

In conclusion, we see the implied density method offers the scope for a fresh approach to the dynamics of the volatility surface, allowing it to be treated in a way that is similar in some respects the theory of interest rate models.

It is interesting that though the approach lies within the scope of infinite-dimensional SDE theory, it leads in some instances to surprisingly tractable and simple results, including exact solutions in some cases.

The connection with nonlinear filtering theory and information-based asset pricing is also surprising.

In some ways the theory is most natural if we regard the assets under consideration as being characterised entirely by the cash flows that they eventually deliver.

Then the option pricing problem ceases to be a theory of the volatility term structure as such.

Instead, it becomes a theory of the dynamics of the conditional densities of the random variables that determine the cash flows.

References

D. C. Brody & L. P. Hughston (2001) Interest rates and information geometry. *Proc. R. Soc. Lond.* A **457**, 1343-1363.

D.C. Brody & L.P. Hughston (2001) Applications of Information Geometry to Interest Rate Theory, 281-288 in *Disordered and Complex Systems*, eds. P. Sollich, T. Coolen, L.P. Hughston & R.F. Streater (American Institute of Physics).

D.C. Brody & L.P. Hughston (2002) Entropy and Information in the Interest Rate Term Structure, *Quantitative Finance* **2**, 70-80.

D.C. Brody & L.P. Hughston (2004) Chaos and Coherence: a New Framework for Interest Rate Modelling, *Proceedings of the Royal Society* A **460**, 85-110.

D.C. Brody, L.P. Hughston & A. Macrina (2007) Beyond Hazard Rates: a New Approach to Credit Risk Modelling. In: *Advances in Mathematical Finance, Festschrift Volume in Honour of Dilip Madan*, eds. R. Elliott, M. Fu, R. Jarrow, and Ju-Yi Yen (Birkhauser/Springer).

D. C. Brody, L. P. Hughston & A. Macrina (2008) Information-based asset pricing. *Int. J. Theo. App. Fin.* **11**, No. 1, 107-142.

D. T. Breeden & R.H. Litzenberger; Prices of state-contingent claims implicit in option prices, *J. Business*, 51, 621-651, 1978.

M. Davis (2004) Complete-market models of stochastic volatility. *Proc. R. Soc. Lond.* A **460**, 11-26.

B. Dupire (1994) Pricing with a smile. *Risk* **7**, 18-20.

D. Filipovic (2001) Consistency Problems for Heath-Jarrow-Morton Interest Rate Models, *Lecture Notes in Mathematics* **1760**, Springer-Verlag, Berlin.

D. Filipovic, L. P. Hughston & A. Macrina (2010) Conditional density models for asset pricing. Working paper.

D. Filipovic & J. Teichmann (2004) On the Geometry of the Term Structure of Interest Rates *Proc. Roy. Soc. Lond.* A. **460**, 129-167,

J. Gatheral (2006) The implied volatility surface: a practitioner's guide. Wiley Finance.

L.P. Hughston (2003) The Past, Present, and Future of Term Structure Modelling, chapter 7 in *Modern Risk Management: A History, introduced by Peter Field*, Risk Publications.

L.P. Hughston & A. Macrina (2008) Information, Interest, and Inflation. To appear in *Advances in Mathematics of Finance*, ed. L. Stettner. Banach Center Publications, Warsaw. ArXiv: math-pr/0710.2876.

L.P. Hughston & A. Rafailidis (2005) A Chaotic Approach to Interest Rate Modelling, *Finance and Stochastics* **9**, 43-65.

J. Jacod and P. Protter (2006) Risk Neutral Compatibility with Option Prices.

P. J. Schönbucher (1999) A market model for stochastic implied volatility. *Phil. Trans. R. Soc. Lond.* A **357**, 2071-2092.

M. Schweizer & J. Wissel (2008) Term structure of implied volatilities: Absence of arbitrage and existence results. *Mathematical Finance* **18**, 77-114.

M. Schweizer & J.Wissel (2008) Arbitrage-free market models for option prices: The multi-strike case. *Finance and Stochastics*.