On the Convergence of Higher Order Hedging Schemes

Magnus Wiktorsson Mats Brodén

Centre for Mathematical Sciences
Mathematical Statistics
Lund University
magnusw@maths.lth.se

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Outline

Hedging Error

Arbitrage Theory in Continuous Time: In a complete market setting every contingent claim can be replicated by continuously trade in the underlying.

In practice: Continuous trading is impossible.

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Hedging error $\mathcal{R}$, i.e. the value of the hedge portfolio differ by some amount $\mathcal{R}$ from the value of the derivative.

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Setting

Risky asset under $\mathbb{Q}$: $dX(t) = rX(t)dt + \sigma(X(t))X(t)dW(t)$.
Bank account: $dB(t) = rB(t)dt$.
Derivative prices: $F_i(t, X(t)) = e^{-r(T_i - t)}\mathbb{E}[\Phi_i(X(T_i))|\mathcal{F}_t]\,,\ i \in \{1, 2\}$.

Assumptions:
Let $\tilde{\sigma}(y) = \sigma(e^y)$.

A1. (i) There is a positive constant σ_0 such that $\tilde{\sigma}(y) \geq \sigma_0$ for all $y \in \mathbb{R}$.
(ii) The function $\tilde{\sigma}$ is bounded, uniformly Lipschitz continuous in compact subsets of $\mathbb{R}$ and uniformly Hölder continuous.

A2. The functions $(\partial^k/\partial y^k)\tilde{\sigma}(y)$, $i \in \{1, 2, 3, 4\}$, are bounded.

A3. $\Phi_1(x) = (x - K_1)^+$, $\Phi_2(x) = (x - K_2)^+$ and $T_2 > T_1$.

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Δ-hedging

Find a self-financing portfolio $\{h^X, h^B\}$ such that

$$h^X(t)X(t) + h^B B(t) = F_1(t, X(t))$$

for all $t \in [0, T_1]$.

Solution: let $h^X(t) = \frac{\partial F_1}{\partial x}(t, X(t)) = F_{1,x}(t, X(t))$.

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Γ-Hedging

Introduce one more derivative: F_2 with Φ_2 and $T_2 > T_1$.
Form a hedge-portfolio $\{h^X, h^{F_2}, h^B\}$ and match the first and second derivatives w.r.t. X:

$$\begin{aligned} F_1(t,X(t)) &= h^X(t)X(t) + h^{F_2}(t)F_2(t,X(t)) + h^B(t)B(t)\,, \\ \Delta^{F_1}(t,X(t)) &= h^X(t) + h^{F_2}(t)\Delta^{F_2}(t,X(t))\,, \\ \Gamma^{F_1}(t,X(t))) &= h^{F_2}(t)\Gamma^{F_2}(t,X(t))\,. \end{aligned}$$

This yields the portfolio

$$\begin{aligned} h^X(t) &= \Delta^{F_1}(t,X(t)) - \frac{\Gamma^{F_1}(t,X(t))}{\Gamma^{F_2}(t,X(t))}\Delta^{F_2}(t,X(t))\,, \\ h^{F_2}(t) &= \frac{\Gamma^{F_1}(t,X(t))}{\Gamma^{F_2}(t,X(t))}\,, \\ h^B(t) &= \frac{F_1(t,X(t)) - h^X(t)X(t) - h^{F_2}(t)F_2(t,X(t))}{B(t)}\,. \end{aligned}$$

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Discrete Time Hedging

Since the portfolio processes in both the Δ-hedging and the Γ-hedging case are continuous processes the hedge portfolio must be rebalanced at every time instant in order for the hedging error to equal zero.

- In practice this is not possible.
- Re-balance at an equidistant time grid, i.e. $t_i = i/n$.
- Let $\mathcal{R}(n)$ denote the hedging error using an equidistant time grid with n re-balancing points. What properties of $\mathcal{R}(n)$ do we get?

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Numerical experiment: Δ-hedging

Model: Black and Scholes. Parameters: $s_0 = 100$, $K_1 = 100$, $K_2 = 120$, $T_1 = 0.5$, $T_2 = 1.5$, $r = 0.03$ and $\sigma = 0.2$.

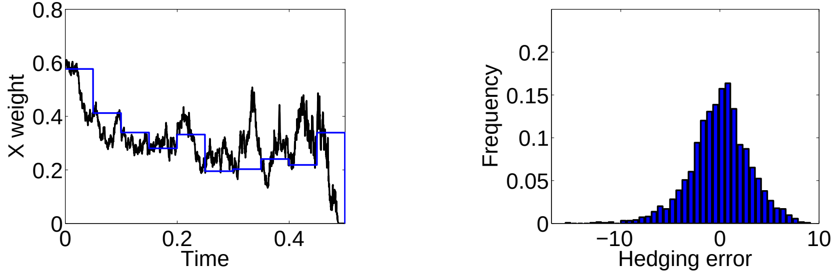

Figure: Δ-hedging. Blue line: $n = 10$,

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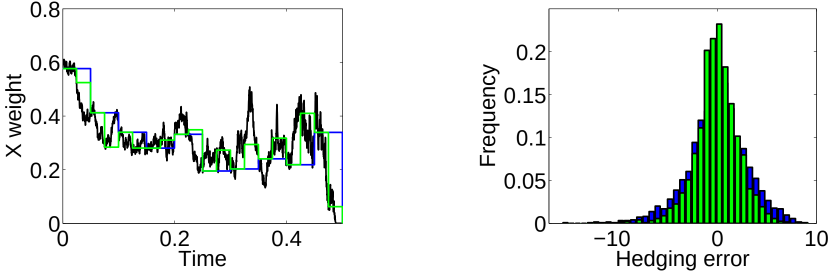


Figure: Δ-hedging. Blue line: $n = 10$, green line: $n = 20$.

Numerical experiment: Γ-hedging

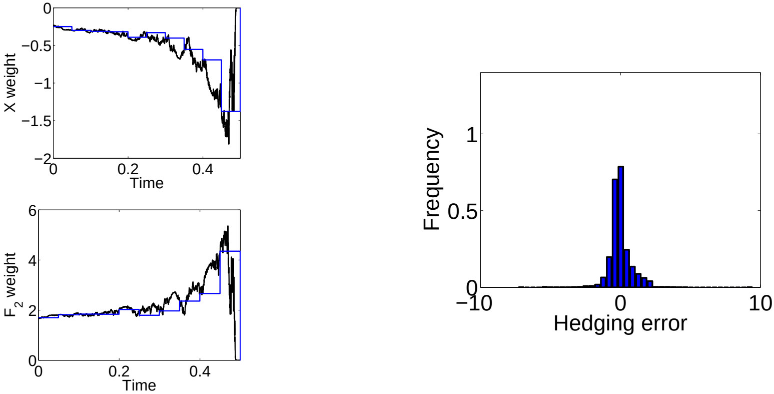
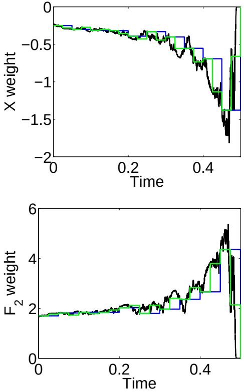


Figure: Γ-hedging. Blue line: $n = 10$,

Numerical experiment: Γ-hedging



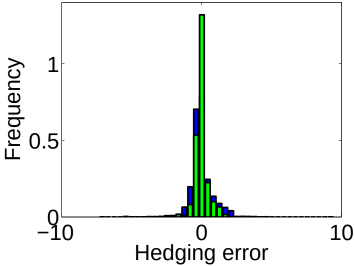


Figure: Γ-hedging. Blue line: $n = 10$, green line: $n = 20$.

Numerical experiment: order of convergence

Assume that: $\mathbb{E}[\mathcal{R}^2(n)] = Cn^{\alpha}$ then
$\log_{10}(\mathbb{E}[\mathcal{R}^2(n)]) = \log_{10}(C) + \alpha \log_{10}(n)$.

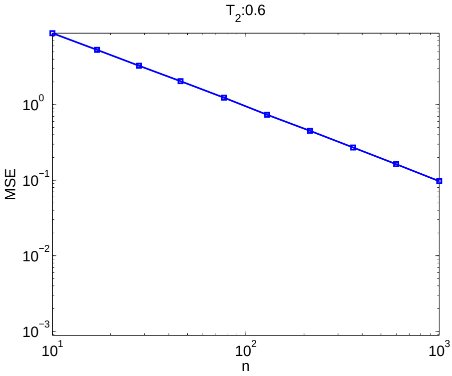


Figure: Squares (□): Δ-hedging,

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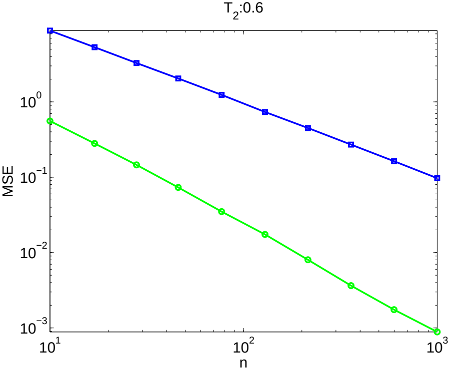

Figure: Squares (□): Δ-hedging, circles (○): Γ-hedging.

Previous results

Δ-Hedging

- Equidistant time grid, i.e. $t_i = i/n$
  - European options (Zhang, 1999): Order of convergence $1/\sqrt{n}$, i.e. $\lim_{n\to\infty} nE[\mathcal{R}^2(n)] = C$.
  - Digital options (Gobet and Temam, 2001): Order of convergence $1/n^{1/4}$.
- Nonuniform time grid
  - Digital options (Geiss, 2002): Order of convergence $1/\sqrt{n}$.

Γ-Hedging

- For the standard Black-Scholes model Gobet and Makhlouf (2009) gives non-sharp lower bounds for convergence rates for both equidistant and non-equidistant grids.

Results

Γ-hedging of an European option on an equidistant time grid (Brodén and Wiktorsson, 2009): Order of convergence $1/n^{3/4}$.

Recall that the assumptions A1-A3 are:
Let $\tilde{\sigma}(y) = \sigma(e^y)$.

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Results

Γ-hedging of an European option on an equidistant time grid (Brodén and Wiktorsson, 2009): Order of convergence $1/n^{3/4}$.

Theorem

If A1-A3 hold, then

$$\begin{aligned}\mathbb{E}[\mathcal{R}_\Gamma^2(n)] &= n^{-3/2}T_1^{3/2}C_{\frac{3}{2}}\lim_{t\uparrow T_1} g(t) + o\left(n^{-3/2}\right)\\ &= n^{-3/2}T_1^{3/2}C_{\frac{3}{2}}e^{-2rT_1}\frac{K_1^3\sigma^3(K_1)}{4\sqrt{\pi}}P_{X(T_1)|X(0)=x_0}(K_1) + o\left(n^{-3/2}\right),\end{aligned}$$

where

$$g(t) = (T_1 - t)^{3/2}\mathbb{E}\left[e^{-2rt}F^2_{1,xxx}(t, X_t)X_t^6\sigma^6(X_t)|X(0) = x_0\right], C_{3/2} \approx 0.62881,$$

and $P_{X(T_1)|X(0)=x_0}(K_1)$ is X:s transition density.

Detail

Results

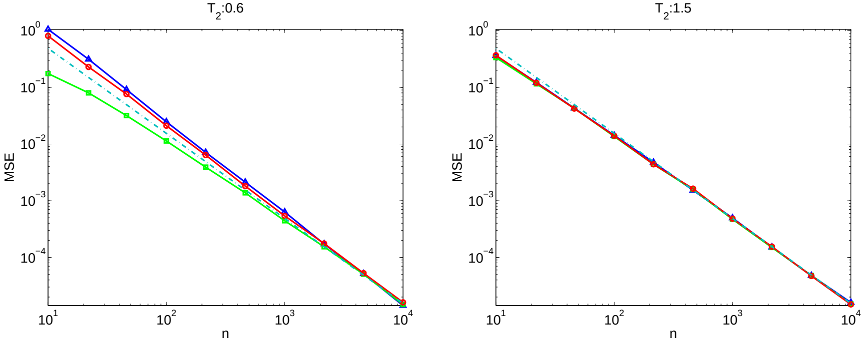

Figure: Log mean squared error as a function of the log number of re-balancings n, for the Black and Scholes model. Parameters $K_1 = 100$, $T_1 = 0.5$, $s_0 = 100$, $r = 0.03$, $\sigma = 0.3$ and $N_{MC} = 10^5$. Dash-dotted line: estimate from the Theorem, MC estimate with: squares: $K_2 = 80$, triangles: $K_2 = 100$ and circles: $K_2 = 120$.

Conclusions

- We have shown that when Γ-hedging a European option on an equidistant time grid the order of convergence is $1/n^{3/4}$.
- An explicit expression for the leading term of the second moment of the hedging error is derived.
- The expression serves as a good approximation of the real second moment of the hedging error also for $n < \infty$.

Further research

- Investigate higher order terms in the expansion of the hedging mean squared error in order to find an optimal choice of hedge instrument in a collection of possible hedge instruments.
- Hedging schemes using an arbitrary number of hedge instruments.
- More complicated market models.

References

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Thanks for the attention!

Questions ??

Supplementary

$$C_a = \sum_{k=1}^{\infty} \int_0^1 \int_0^x \int_0^w \frac{1}{(k-v)^a} \, \mathrm{d}v \, \mathrm{d}w \, \mathrm{d}x = \int_0^{\infty} \frac{e^t - 1 - t - \frac{t^2}{2}}{\Gamma(a) t^{a+1} (e^t - 1)} \, \mathrm{d}t \, .$$

which is well defined for $0 < a < 2$.

Back