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Convex risk measures on Orlicz spaces: inf-convolution and shortfall

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1. Preliminaries

Consider an incomplete financial market with maturity $T > 0$ and zero interest rate.

$(\Omega, \mathcal{F}, P; \mathrm{F} = \{\mathcal{F}_t\}_{t\in[0,T]})$: a completed probability space, where F is a filtration satisfying the so-called usual condition

A left-continuous non-decreasing convex non-trivial function $\Phi : \mathrm{R}_+ \to [0, \infty]$ with $\Phi(0) = 0$ is called an Orlicz function, where Φ is non-trivial if $\Phi(x) > 0$ for some $x > 0$ and $\Phi(x) < \infty$ for some $x > 0$.

When Φ is an R_+-valued continuous, strictly increasing Orlicz function, we call it a strict Orlicz function.

Remark

Any polynomial function starting at 0 whose minimal degree is equal to or greater than 1, and all coefficients are positive, is a strict Orlicz function.

For example, cx^p for $c > 0$, $p \geq 1$, $x^2 + 3x^5$ and so forth.

Moreover, $e^x - 1$, $e^x - x - 1$, $(x+1)\log(x+1) - x$ and $x - \log(x+1)$ are strict Orlicz functions.

For any strict Orlicz function Φ, $\Phi(x) \in (0, \infty)$ for any $x > 0$ and $\lim_{x \to \infty} \Phi(x) = \infty$.

A strict Orlicz function Φ is differentiable a.e. and its left-derivative Φ' satisfies

$$\Phi(x) = \int_0^x \Phi'(u) du.$$

Φ' is left-continuous, and may have at most countably many jumps.

Define $I(y) := \inf\{x \in (0, \infty) | \Phi'(x) \geq y\}$,
which is called the generalized left-continuous inverse of Φ'.
Define $\Psi(y) := \int_0^y I(v) dv$ for $y \geq 0$,
which is an Orlicz function and called the conjugate function of Φ.

Definition

Orlicz space : $L^\Phi := \{X \in L^0 | E[\Phi(c|X|)] < \infty \text{ for some } c > 0\}$,

Orlicz heart : $M^\Phi := \{X \in L^0 | E[\Phi(c|X|)] < \infty \text{ for any } c > 0\}$.

Luxemburg norm : $\|X\|_\Phi := \inf \left\{\lambda > 0 | E\left[\Phi\left(\left|\frac{X}{\lambda}\right|\right)\right] \leq 1\right\}$,

Orlicz norm : $\|X\|^*_\Phi := \sup\{E[XY] | \, \|Y\|_\Phi \leq 1\}$.

Remark

$M^\Phi \subseteq L^\Phi$.

Both spaces L^Φ and M^Φ are linear.

In the case of the lower partial moments $\Phi(x) = x^p/p$ for $p > 1$, the Orlicz space L^Φ and the Orlicz heart M^Φ both are identical with L^p.

The conjugate function Ψ in this case is given by $\Psi(x) = x^q/q$, where $q = p/(p-1)$, and $M^\Psi = L^\Psi = L^q$.

In general, if $\limsup_{x\to\infty} \frac{x\Phi'(x)}{\Phi(x)} < \infty$, then M^Φ is identical with L^Φ.

For instance, $\Phi(x) = x - \log(x+1)$ other than the lower partial moments.

Otherwise, M^Φ would be a proper subset of L^Φ.

Example 1

In the case where $\Phi(x) = e^x - 1$ or $e^x - x - 1$, if a random variable X follows an exponential distribution with a positive parameter, then $X \in L^\Phi$ but $X \notin M^\Phi$.

Example 2

Set $\Phi(x) = e^{x^2} - 1$. Let X be a random variable following a normal distribution. Then $X \in L^\Phi$ but $X \notin M^\Phi$.

Example 3

It is natural that an aggregate insurance claim amount follows a compound distribution.

Denote by $(N_t)_{t\geq 0}$ the process which describes the number of claims during time period $[0, t]$.

Assume that $(N_t)_{t\geq 0}$ is a Poisson process with a positive parameter. The size of the i-th claim is denoted by R_i, which is a nonnegative-valued random variable.

We suppose that $(R_i)_{i\geq 1}$ is an i.i.d. sequence which is independent of $(N_t)_{t\geq 0}$.

The aggregate insurance claim amount in this model is given by

$$A_t := \begin{cases} \sum_{i=1}^{N_t} R_i, & \text{if } N_t > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Remark that $E[e^{cA_T}] = E[e^{N_T \log M(c)}]$ for any fixed time horizon $T > 0$ and any constant $c > 0$, where $M(c)$ is the moment generating function of R_i, that is, $M(c) := E[e^{cR_i}]$.

Taking an exponential type function as Φ, say $\Phi(x) = e^x - 1$, if there exist $c_1 > c_2 > 0$ such that $M(c_1) = \infty$ and $M(c_2) < \infty$, then $A_T \in L^\Phi$, but $A_T \notin M^\Phi$.

For instance, if each R_i follows an exponential distribution with parameter $\sigma > 0$, then

$$M(c) = \begin{cases} \frac{\sigma}{\sigma - c}, & \text{if } c < \sigma, \\ \infty, & \text{otherwise.} \end{cases}$$

Moreover, if Φ is a strict Orlicz function, the norm dual of $(M^\Phi, \|\cdot\|_\Phi)$ is given by $(L^\Psi, \|\cdot\|_\Phi^*)$.
The norm dual of $(L^\Phi, \|\cdot\|_\Phi)$ includes a singular part.
This fact would be crucial when we consider convex risk measures on Orlicz spaces.

Note that L^Φ becomes a Banach lattice under the usual pointwise ordering.

Throughout this paper, we fix a strict Orlicz function Φ.

Definition

A functional ρ defined on L^Φ is called a convex risk measure on L^Φ if it satisfies the following four conditions:

(1) Properness : $\rho(0) \in \mathbb{R}$ and ρ is $(-\infty, \infty]$-valued,

(2) Monotonicity : $\rho(X) \geq \rho(Y)$ for any $X, Y \in L^\Phi$ such that $X \leq Y$,

(3) Translation invariance : $\rho(X + m) = \rho(X) - m$ for $X \in L^\Phi$ and $m \in \mathbb{R}$,

(4) Convexity : $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ for any $X, Y \in L^\Phi$ and $\lambda \in [0, 1]$.

Moreover, if a convex risk measure ρ satisfies

(5) Positive homogeneity : $\rho(\lambda X) = \lambda\rho(X)$ for any $\lambda \geq 0$,

then ρ is called a coherent risk measure.

Biagini and Frittelli (2009) **asserts that, when we consider a robust representation of convex risk measures, a significant issue is whether or not the topology on which the convex risk measures are defined has the order continuity.**

In the case of order continuous, say L^p for $p \in [1, \infty)$ or Orlicz hearts, we do not have to care the order l.s.c. of convex risk measures. On the other hand, in the non-order continuous case, say L^∞ or Orlicz spaces, we have to check it.

Indeed, Corollary 28 of BF09 products a robust representation for convex risk measures defined on locally convex Fréchet lattices under the order l.s.c. and the C-property.

Now, we shall state a robust representation theorem for convex risk measures on L^Φ based on Corollary 28 of BF09.

Definition

A linear topology τ has C-property, if a net $\{X_\alpha\}$ converges to X in τ, then there exist a subsequence $\{X_{\alpha_n}\}_{n\geq 1}$ and convex combinations $Y_n \in \text{conv}(X_{\alpha_n}, X_{\alpha_{n+1}}, \ldots)$ such that Y_n is order convergent to X.

Note that the topology $(L^\Phi, \sigma(L^\Phi, L^\Psi))$ has the C-property, and, in this case, "$Y_n \to X$ in order" means "$Y_n \to X$ a.s. and there exists a $Y \in L^\Phi$ such that $|Y_n| \leq Y$ for any $n \geq 1$".

Let $\mathcal{P}^\Psi$ be the set of all probability measures being absolutely continuous with respect to P and having L^Ψ-density with respect to P, that is, $\mathcal{P}^\Psi := \{Q \ll P | dQ/dP \in L^\Psi\}$.

Theorem 1

Let ρ be a convex risk measure on L^Φ. We define

$$\alpha_\rho(Q) := \sup_{X \in \mathcal{A}_\rho} E_Q[-X]$$

for any $Q \in \mathcal{P}^\Psi$, where $\mathcal{A}_\rho := \{X \in L^\Phi | \rho(X) \leq 0\}$.
If ρ has the order l.s.c., then it is represented as follows:

$$\rho(X) = \sup_{Q \in \mathcal{P}^\Psi} \{E_Q[-X] - \alpha_\rho(Q)\}. \tag{1}$$

Remark

The functional α_ρ and the set $\mathcal{A}_\rho$ are called the penalty function and the acceptance set of ρ, respectively.
In order to prove that $\rho^*(-dQ/dP)$ and $\alpha_\rho(Q)$ coincide, we do not need the order l.s.c. of ρ.

Corollary

For a coherent risk measure ρ on L^Φ having the order l.s.c., there exists a convex subset $\mathcal{P}' \subseteq \mathcal{P}^\Psi$ such that $\rho(X) = \sup_{Q \in \mathcal{P}'} E_Q[-X]$.

2. Inf-convolution

Definition

(a) Suppose that $\mathcal{B} \subseteq L^{\Phi}$ is non-empty convex.
The inf-convolution of a convex risk measure ρ on L^{Φ} and $\mathcal{B}$ is defined as

$$\rho \square \mathcal{B}(X) := \inf_{Y \in \mathcal{B}} \rho(X - Y), \text{ for any } X \in L^{\Phi}.$$

(b) Let ρ_1 and ρ_2 be two convex risk measures on L^{Φ}.
The inf-convolution of ρ_1 and ρ_2 is defined as

$$\rho_1 \square \rho_2(X) := \inf_{Y \in L^{\Phi}} \{\rho_1(X - Y) + \rho_2(Y)\}, \text{ for any } X \in L^{\Phi}.$$

Proposition 1

Let ρ be a convex risk measure on L^Φ satisfying

$$\rho(X) < \infty \text{ for any } X \in L^\Phi \text{ such that } E[\Phi(|X|)] < \infty. \tag{2}$$

Moreover, let $\mathcal{B} \subseteq L^\Phi$ be a convex set including 0.
If $\rho\square\mathcal{B}(0) > -\infty$, then the following hold:

(a) $\rho\square\mathcal{B}$ is a convex risk measure on L^Φ.

(b) If ρ is coherent and $\mathcal{B}$ is cone, then $\rho\square\mathcal{B}$ is also coherent.

(c) If $\mathcal{B}$ is sequentially compact in $\sigma(L^\Phi, L^\Psi)$ and ρ is order l.s.c., then so is $\rho\square\mathcal{B}$.

Proposition 2

Let ρ_i, $i = 1, 2$, be convex risk measures on L^Φ. Denote their acceptance sets and penalty functions by $\mathcal{A}_i$ and α_i, respectively.
Assume that ρ_1 satisfies the condition (2), and $\mathcal{A}_2$ includes 0.
If $\rho_1 \Box \rho_2(0) > -\infty$, then the following hold:

(a) $\rho_1 \Box \rho_2$ is a convex risk measure on L^Φ, and represented as

$$\rho_1 \Box \rho_2(X) = \rho_1 \Box \mathcal{A}_2(X) = \inf_{Y \in \mathcal{B}} \{\rho_1(X - Y) + \rho_2(Y)\}, \text{ for any } X \in L^\Phi,$$

where $\mathcal{B}$ is a subset of L^Φ including $\mathcal{A}_2$.

(b) If ρ_1 and ρ_2 both are coherent, then so is $\rho_1 \Box \rho_2$.

(c) The penalty function of $\rho_1 \Box \rho_2$ is given by

$$\alpha_{1\Box 2}(Q) = \alpha_1(Q) + \alpha_2(Q) \text{ for any } Q \in \mathcal{P}^{\Psi}.$$

(d) Let $\mathcal{A}_2$ be sequentially compact in $\sigma(L^{\Phi}, L^{\Psi})$.
If ρ_1 is order l.s.c., then so is $\rho_1 \Box \rho_2$.
Moreover, the acceptance set of $\rho_1 \Box \rho_2$ satisfies

$$\mathcal{A}_{1\Box 2} = \overline{\mathcal{A}_1 + \mathcal{A}_2},$$

which is the closure of $\mathcal{A}_1 + \mathcal{A}_2$ in $\sigma(L^{\Phi}, L^{\Psi})$.

Remark
Under the condition of (d), ρ_2 is also order l.s.c.

Proposition 3

Let $\mathcal{B}$ be a convex subset of L^Φ including 0, ρ a convex risk measure on L^Φ satisfying (2).

Define $\rho^{\mathcal{B}}(X) := \inf\{x \in \mathrm{R} | x + X \in \mathcal{B}\}$ for any $X \in L^\Phi$.

Let $\alpha_{\mathcal{B}}$ be defined as, for any $Q \in \mathcal{P}^\Psi$,

$$\alpha_{\mathcal{B}}(Q) = \begin{cases} 0, & \text{if } E_Q[-X] \leq 0 \text{ for any } X \in \mathcal{A}_{\mathcal{B}}, \\ \infty, & \text{otherwise,} \end{cases}$$

where $\mathcal{A}_{\mathcal{B}}$ is the acceptance set of $\rho^{\mathcal{B}}$, that is,

$$\mathcal{A}_{\mathcal{B}} := \{X \in L^\Phi | \rho^{\mathcal{B}}(X) \leq 0\} = \{X \in L^\Phi | x + X \in \mathcal{B} \text{ for any } x > 0\}.$$

Assume that $-\mathcal{B}$ is solid, and $\rho\Box\rho^{\mathcal{B}}(0) > -\infty$.
(a) We have

$$\rho\Box\mathcal{B} = \rho\Box\rho^{\mathcal{B}}.$$

(b) If $\mathcal{B}$ is cone, then the penalty function $\alpha_{\rho\Box\mathcal{B}}$ of $\rho\Box\mathcal{B}$ is given by

$$\alpha_{\rho\Box\mathcal{B}} = \alpha_\rho + \alpha_{\mathcal{B}},$$

where α_ρ is the penalty function of ρ.

Solid

A is said to be solid if

$$X \in A, Y \in L^\Phi, Y \geq X \Longrightarrow Y \in A.$$

3. Shortfall risk measure

Let $\mathcal{C}$ be a convex subset of L^Φ including 0.
In this section, we regard $\mathcal{C}$ as the set of all attainable claims with zero initial endowment.

Furthermore, each element of $\mathcal{C}$ is interpreted as a hedging strategy.
Denote by X a contingent claim, which is a payoff at the maturity T.
Thus, X is an $\mathcal{F}_T$-measurable random variable.
In particular, we presume that X is in L^Φ.

Let l be a function from R to R_+ satisfying $l(x) = 0$ if $x \leq 0$, and $l(x) = \Phi(x)$ if $x > 0$.
We presume a risk-averse investor who intends to sell the claim X, and whose loss function is given by l.

When the price of X and the hedging strategy are given by $x \in \mathbb{R}$ and $U \in \mathcal{C}$, resp., the shortfall risk for the seller is expressed by

$$E[l(-x - U + X)].$$

Denote by $\delta > 0$ the threshold of the seller.
Note that the threshold δ determines the limit of the shortfall risk which she can endure.

Define, in addition, a subset of L^{Φ} as

$$\mathcal{A}_0 := \{X \in L^{\Phi} | E[l(-X)] \leq \delta\}.$$

We define, by using $\mathcal{A}_0$, a functional $\widehat{\rho}$ defined on L^Φ as

$\widehat{\rho}(X) := \inf\{x \in \mathbb{R} |$ there exists a $U \in \mathcal{C}$ such that $x + U + X \in \mathcal{A}_0\}$.

We call $\widehat{\rho}$ the shortfall risk measure.

Note that $\widehat{\rho}(-X)$ would give the least price which the seller can accept.

In other words, if the seller sells the claim X for a price more than $\widehat{\rho}(-X)$, then she could find a hedging strategy whose corresponding shortfall risk is less than or equal to the threshold δ.

We focus on a robust representation result for $\widehat{\rho}$.

As we have seen in Example 3, there are several examples of claims which are included in L^Φ, but not in M^Φ.

Since Orlicz hearts are order continuous, we do not need to get the order l.s.c. of $\widehat{\rho}$ to obtain its representation.

On the other hand, we have to investigate the order l.s.c. of $\widehat{\rho}$ for the Orlicz space case, since Orlicz spaces do not have the order continuity in general.

Assumption 1 $\widehat{\rho}(0) > -\infty$.

Denoting $\rho_0(X) := \inf\{x \in \mathrm{R} | x + X \in \mathcal{A}_0\}$,

Lemma 1 ρ_0 is a convex risk measure on L^Φ.

Proposition 4 $\widehat{\rho} = \rho_0 \Box (-\mathcal{C})$.

Lemma 2 ρ_0 is order l.s.c.

Reminder (Proposition 1 (c))
If $\mathcal{B}$ is sequentially compact in $\sigma(L^\Phi, L^\Psi)$ and ρ is order l.s.c., then so is $\rho \Box \mathcal{B}$.

Theorem 2

Under Assumption 1, if $\mathcal{C}$ is sequentially compact in $\sigma(L^\Phi, L^\Psi)$, then $\widehat{\rho}$ is a $(-\infty, +\infty]$-valued convex risk measure on L^Φ satisfying the following:

$$\widehat{\rho}(X) = \sup_{Q\in\mathcal{P}^\Psi} \left\{ E_Q[-X] - \sup_{X^1\in\mathcal{A}_1} E_Q[-X^1] - \inf_{\lambda>0} \frac{1}{\lambda} \left\{ \delta + E\left[\Psi\left(\lambda \frac{dQ}{dP} \right) \right] \right\} \right\},$$

where $\mathcal{A}_1 := \{X \in L^\Phi |$ there exists a $U \in \mathcal{C}$ such that $X + U \geq 0\}$.

Reminder (Theorem 1)

If a convex risk measure ρ has the order l.s.c., then

$$\rho(X) = \sup_{Q\in\mathcal{P}^\Psi} \{E_Q[-X] - \alpha_\rho(Q)\}.$$

Construction of $\mathcal{C}$ having the sequential compactness

From the view of Theorem 2, $\widehat{\rho}$ has a robust representation if $\mathcal{C}$ has the sequential compactness in $\sigma(L^\Phi, L^\Psi)$.
We construct an example of $\mathcal{C}$ being sequentially compact in $\sigma(L^\Phi, L^\Psi)$.

We consider an incomplete financial market being composed of one riskless asset and d risky assets.

The fluctuation of the risky assets is described by an $\mathbb{R}^d$-valued RCLL special semimartingale S, which is possibly non-locally bounded.

Instead, we suppose that S is locally in L^Φ in the following sense: there exists a localizing sequence $(\tau^n)_{n\geq 1}$ of stopping times such that, for any $n \geq 1$, the family $\{S_\tau | \tau\text{: stopping time}, \tau \leq \tau^n\}$ is a subset of L^Φ.

Now, we construct, by the same manner as *Xia and Yan (2006)*, **a $\sigma(L^\Phi, L^\Psi)$-closed set of stochastic integrals.**

Let K^s_Φ be the subspace of L^Φ spanned by the simple stochastic integrals of the form $h^{tr}(S_{\sigma_2} - S_{\sigma_1})$, where $\sigma_1 \leq \sigma_2$ are stopping times such that $\{S_\sigma | \sigma\text{: stopping time}, \sigma \leq \sigma_2\} \subseteq L^\Phi$ and $h \in L^\infty$ is $\mathcal{F}_{\sigma_1}$-measurable.

Denote the following:

$$\mathcal{M}^{\Psi,s} := \{Z \in L^{\Psi} | E[WZ] = 0 \text{ for any } W \in K^s_{\Phi} \text{ and } E[Z] = 1\},$$

$$\mathcal{M}^{\Psi,e} := \{Z \in \mathcal{M}^{\Psi,s} | Z > 0 \,\text{a.s.}\},$$

and $K^{\Phi} := \overline{K^s_{\Phi}}$, which is the closure of K^s_{Φ} in $\sigma(L^{\Phi}, L^{\Psi})$.

Θ^L denotes the set of all S-integrable predictable processes ϑ such that $\int_0^T \vartheta_s dS_s \in L^{\Phi}$ and $E[\int_0^T \vartheta_s dS_s \cdot Z] = 0$ for any $Z \in \mathcal{M}^{\Psi,s}$.

Moreover, we denote $G := \{\int_0^T \vartheta_s dS_s | \vartheta \in \Theta^L\}$.
Note that, if $\mathcal{M}^{\Psi,s} \neq \emptyset$, then $G \subseteq K^{\Phi}$.

Assumption 2 $\mathcal{M}^{\Psi,e} \neq \emptyset$.

Theorem 3
Under Assumption 2, we have $K^{\Phi} = G$, that is, G is closed in $\sigma(L^{\Phi}, L^{\Psi})$.

Assumption 3 $\lim_{k\to 0} k^{-1}E[\Phi(k|W|)] = 0$ uniformly in $W \in G$.

Under Assumptions 2 and 3, G is sequentially compact in $\sigma(L^{\Phi}, L^{\Psi})$ by Theorem IV.5.3 of *Rao and Ren (1991)* and Theorem 3.

Taking $G - A$ as the set $\mathcal{C}$ of all attainable claims with zero initial endowment, where A is a sequentially compact subset of L^Φ_+ in $\sigma(L^\Phi, L^\Psi)$, $\mathcal{C}$ is also sequentially compact.
Hence, we can conclude the following:

Theorem 4
Under Assumptions 1 − 3, $\widehat{\rho}$ is a $(-\infty, +\infty]$-valued convex risk measure on L^Φ satisfying

$$\widehat{\rho}(X) = \sup_{Q \in \mathcal{P}^\Psi} \left\{ E_Q[-X] - \sup_{X^1 \in \mathcal{A}_1} E_Q[-X^1] - \inf_{\lambda > 0} \frac{1}{\lambda} \left\{ \delta + E\left[\Psi\left(\lambda \frac{dQ}{dP} \right) \right] \right\} \right\}.$$

Thank you for your attention!!