Dynamic Coherent Acceptability Indices

Igor Cialenco

Department of Applied Mathematics
Illinois Institute of Technology, USA
igor@math.iit.edu

joint work with T.R. Bielecki and Z. Zhang

6th World Congress of the Bachelier Finance Society
June 22-26, Toronto

Performance Measures $f(\text{return}, \text{risk})$

- Sharpe Ratio $SR(X) = \frac{\mathbb{E}(X)-r}{\mathrm{Std}(X)}$
- Gain-Loss Ratio $GLR(X) = \frac{\mathbb{E}(X)}{\mathbb{E}(X^-)}$
- Risk Adjusted Return on Capital $RAROC(X) = \frac{\mathbb{E}(X)}{\rho(X)}$
- Treynor Ratios $TR(X) = \frac{\mathbb{E}(X)-r}{\beta(X)}$
- Tilt Coefficient $TC(X) = \sup\{\lambda \in \mathbb{R}_+ \mid \mathbb{E}[Xe^{-\lambda X}] \geq 0\}$
- and more

General Desired Properties

Unitless, Monotone, Quasi-Concave

Objectives

Study these Measures of Performance from abstract and applied probability point of view

- establish a set of axioms (with financial meaning)
- describe all functions that satisfy these axioms (representations theorem)
- cover classical examples
- **do it consistently in time (process)**
- find new examples / maybe reject some classical ones

$$f(\text{return}, \text{risk}, t)$$

Definition (Cherny and Madan '08)

$\alpha : \mathcal{X} \to [0, +\infty]$ is called a coherent *acceptability index (AI)* if the following axioms are satisfied

(A1) **Monotonicity.** If $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$, then $\alpha(X) \leq \alpha(Y)$

(A2) **Scale invariance.** For every $X \in \mathcal{X}$ and $\lambda > 0$, $\alpha(\lambda X) = \alpha(X)$

(A3) **Quasi-concavity.** If $\alpha(X) \geq x$, $\alpha(Y) \geq x$, then $\alpha(\lambda X + (1-\lambda)Y) \geq x$ for all $\lambda \in [0,1]$

(A4) **Fatou.** If $|X_n| \leq 1$, $\alpha(X_n) \geq x$ and $X_n \to X$ in probability, then $\alpha(X) \geq x$

- SR - no; GLR - yes; RAROC - yes; TC - no; TVaRAI - yes; etc
- (A1) Y dominating X implies Y is more acceptable than X
- (A2) cash flows with same structure have same performance
- (A3) diversification does not decrease the performance level

Represenation Theorem (Cherny and Madan '08)

α is a coherent AI if and only if there exists a family $(\mathcal{Q}_x)_{x\in[0,+\infty]}$ of sets of probability measures, such that $\mathcal{Q}_x \subset \mathcal{Q}_y$ for $x \leq y$ and

$$\alpha(X) = \sup\left\{x \in \mathbb{R}_+ \ : \ \inf_{Q\in\mathcal{Q}_x} \mathbb{E}_Q[X] \geq 0\right\}$$

- $GLR(X) = \mathbb{E}[X]/\mathbb{E}[X^-]$ is coherent AI with representation

$$\mathcal{Q}_x = \{c(1+Y) : c \in \mathbb{R}_+,\ 0 \leq Y \leq x,\ \mathbb{E}[c(1+Y)] = 1\}$$

- Any coherent acceptability index can be characterized by a family of sets of probability measures

Performance measurements in dynamic market

- Probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- Finite time $\{0, 1, \ldots, T\}$
- Filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0}^T$
- $D = (D)_{t=0}^T$ cash flow
- $\mathcal{D}$ the set of all bounded processes

Definition

Dynamic acceptability index is a map

$$\alpha : \{0, \ldots, T\} \times \mathcal{D} \times \Omega \to [0, +\infty]$$

Definition

$\alpha : \{0, \ldots, T\} \times \mathcal{D} \times \Omega \to [0, +\infty]$ is called a **coherent dynamic acceptability index** if it satisfies the following axioms:

(D1) **Adapted:** $\alpha_t(D, \cdot)$ is $\mathcal{F}_t$-measurable

(D2) **Independence of the past:** If there exists $A \in \mathcal{F}_t$ such that $1_A D_s = 1_A D'_s$ for $s \geq t$, then $1_A \alpha_t(D) = 1_A \alpha_t(D')$

(D3) **Monotonicity:** If $D_s \geq D'_s$ for some $D, D' \in \mathcal{D}$ and all $s \geq t$, then $\alpha_t(D) \geq \alpha_t(D')$

(D4) **Scale invariance:** $\alpha_t(\lambda D, \omega) = \alpha_t(D, \omega)$ for all $\lambda > 0$

(D5) **Quasi-concavity:** If $\alpha_t(D, \omega) \geq x$, $\alpha_t(D', \omega) \geq x$, then $\alpha_t(\lambda D + (1 - \lambda) D', \omega) \geq x$ for all $\lambda \in [0, 1]$

Definition Continued

(D6) Translation Invariance:

$$\alpha_t(D + m1_t) = \alpha_t(D + m1_s)$$

for any $D \in \mathcal{D},\ s \geq t$ and $m\ -\ \mathcal{F}_t$-measurable

(D7) Dynamic consistency:

Let $D, D' \in \mathcal{D}$, and $X \geq 0$ be $\mathcal{F}_t$ measurable

(a) If $D_t \geq 0$ and $\alpha_{t+1}(D) \geq X$, then $\alpha_t(D) \geq X$

(b) If $D_t \leq 0$ and $\alpha_{t+1}(D) \leq X$, then $\alpha_t(D) \leq X$

Theorem (Representation/Duality Theorem)

A function $\alpha : \{0,1,\dots,T\} \times \mathcal{D} \times \Omega \to [0,+\infty]$ unbounded above is a dynamic coherent acceptability index if and only if there exists a sequence of non-decreasing dynamic coherent risk measures $(\rho^x)_{x\in\mathbb{R}_+}$, such that $\rho^x_t(D) \geq \rho^y_t(D)$ for $x \geq y$, and

$$\alpha_t(D,\omega) = \sup\{x \in \mathbb{R}_+ : \rho^x_t(D,\omega) \leq 0\}.$$

Theorem (Representation/Duality Theorem)

A function $\alpha : \{0, 1, \ldots, T\} \times \mathcal{D} \times \Omega \to [0, +\infty]$ *unbounded above is a dynamic coherent acceptability index if and only if there exists a sequence of non-decreasing dynamic coherent risk measures* $(\rho^x)_{x \in \mathbb{R}_+}$*, such that* $\rho^x_t(D) \geq \rho^y_t(D)$ *for* $x \geq y$*, and*

$$\alpha_t(D, \omega) = \sup\{x \in \mathbb{R}_+ : \rho^x_t(D, \omega) \leq 0\}.$$

The associated risk measures:

$$\rho^x_t(D) := \inf\{m \in \mathbb{R} \;:\; \alpha_t(D + m1_t) \geq x\}, \quad x \in \mathbb{R}^+,$$

where $\inf \emptyset = \infty$ and $\sup \emptyset = 0$.

Definition

Dynamic coherent risk measure is a function $\rho : \{0, \dots, T\} \times \mathcal{D} \times \Omega \to \mathbb{R}$ that satisfies the following axioms:

(A1) **Adapted:** For every $t \in \{0, \dots, T\}$, and every $D \in \mathcal{D}$, $\rho_t(D)$ is $\mathcal{F}_t$-measurable

(A2) **Independence of the past:** If there exists $A \in \mathcal{F}_t$ such that $1_A D_s = 1_A D'_s$ for all $s \geq t$, then $1_A \rho_t(D) = 1_A \rho_t(D')$

(A3) **Monotonicity:** If $D_s \geq D'_s$ for some $D, D' \in \mathcal{D}$, and for all $s \geq t$, then $\rho_t(D) \leq \rho_t(D')$

(A4) **Homogeneity:** $\rho_t(\lambda D) = \lambda \rho_t(D)$ for every $\lambda \geq 0$, $D \in \mathcal{D}$, $t \in \{0, \dots, T\}$

(A5) **Subadditivity:** $\rho_t(D + D') \leq \rho_t(D) + \rho_t(D')$ for every $D, D' \in \mathcal{D}$, $t \in \{0, \dots, T\}$

(A6) **Translation Invariance:** $\rho_t(D + m 1_s) = \rho_t(D) - m$ for every $D \in \mathcal{D}$, an $\mathcal{F}_t$-measurable random variable m, and for all $s \geq t$

Definition

cont

(A7) Dynamic consistency: For every $D \in \mathcal{D}$, we have

$$\min_{\omega} \rho_{t+1}(D,\omega) - D_t \leq \rho_t(D) \leq \max_{\omega} \rho_{t+1}(D,\omega) - D_t$$

(A7') Riedel 2004 :
If $D_t = D'_t$ and $\rho_{t+1}(D,\omega) = \rho_{t+1}(D',\omega)$ for all $\omega \in \Omega$, then $\rho_t(D,\omega) = \rho_t(D',\omega)$ for all $\omega \in \Omega$

(A7") superposition: $\rho_t(D) = \rho_t(D_t 1_t - \rho_{t+1}(D) 1_{t+1})$

- ▷ (A1)-(A6) and (A7') $\Rightarrow$ (A7)
- ▷ (A1)-(A6) and (A7") $\Rightarrow$ (A7)

Definition

$\mathcal{Q} \subset \mathcal{P}$ is called dynamic set of probability measures if

$$\inf_{Q\in\mathcal{Q}} \mathbb{E}_Q\Big[D\,|\,\mathcal{F}_t\Big] = \inf_{Q\in\mathcal{Q}} \mathbb{E}_Q\Big[\inf_{M\in\mathcal{Q}} \mathbb{E}_M\big[D\,|\,\mathcal{F}_{t+1}\big]\,|\,\mathcal{F}_t\Big] \ \forall t, D$$

- Two trivial examples: singleton set $\mathcal{Q} = \{\mathbb{Q}\}$ and $\mathcal{P}$
- Two non-trivial examples: for $x \geq 0$, define

$$\mathcal{Q}_x^u := \{\mathbb{Q} \in \mathcal{P} | \mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_j] \leq (1+x)\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_{j-1}] \ \forall j = 1, \ldots, T\}$$

$$\mathcal{Q}_x^l := \{\mathbb{Q} \in \mathcal{P} | \mathbb{E}_{\mathbb{Q}}[\frac{d\mathbb{P}}{d\mathbb{Q}}|\mathcal{F}_j] \leq (1+x)\mathbb{E}_{\mathbb{Q}}[\frac{d\mathbb{P}}{d\mathbb{Q}}|\mathcal{F}_{j-1}] \ \forall j = 1, \ldots, T\}$$

Theorem

Assume that $\{\mathcal{Q}_x\}_{x\in\mathbb{R}_+}$ *is a family of dynamic sets of probability measures, such that* $\mathcal{Q}_x \subset \mathcal{Q}_y$ *for* $x \leq y$ *and define*

$$\alpha_t(D) := \sup\left\{x \in \mathbb{R}_+ \ : \ \inf_{Q\in\mathcal{Q}_x} \mathbb{E}_Q\left[\sum_{s=t}^{T} D_t \,|\, \mathcal{F}_t\right] \geq 0\right\}$$

Then α *is a dynamic coherent acceptability index.*

- Static case is a particular case
- $\rho_t^x(D) := -\inf_{\mathbb{Q}\in\mathcal{Q}_x} \mathbb{E}_{\mathbb{Q}}[\sum_{i=t}^{T} D_i|\mathcal{F}_t]$ is a dynamic coherent risk measure
- The existence of family $\mathcal{Q}_x$ is guaranteed by $\{\mathcal{Q}_x^u\}$ and $\{\mathcal{Q}_x^l\}$

Dynamic Gain-Loss Ratio

$$\mathrm{dGLR}_t(D) = \begin{cases} \frac{\mathbb{E}[\sum_{s=t}^{T} D_s \mid \mathcal{F}_t]}{\mathbb{E}[(\sum_{s=t}^{T} D_s)^{-}|\mathcal{F}_t]} & \text{if } \mathbb{E}[\sum_{s=t}^{T} D_s \mid \mathcal{F}_t] > 0 \\ 0 & \text{otherwise} \end{cases}$$

Dynamic RAROC

Given a dynamic set of probability measures $\mathcal{Q}$ with $\mathbb{P} \in \mathcal{Q}$, define

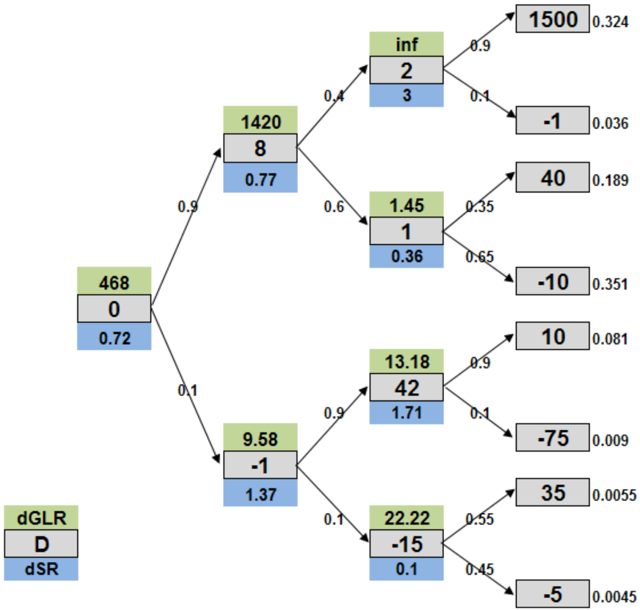
$$dRAROC_t(D) = 1_{\{\mathbb{E}[\sum_{s=t}^{T} D_s|\mathcal{F}_t]>0\}} \frac{\mathbb{E}[\sum_{s=t}^{T} D_s|\mathcal{F}_t]}{-\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\sum_{s=t}^{T} D_s|\mathcal{F}_t]}$$

(convention $dRAROC_t(D) = +\infty$ if $\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\sum_{s=t}^{T} D_s|\mathcal{F}_t] \geq 0$)

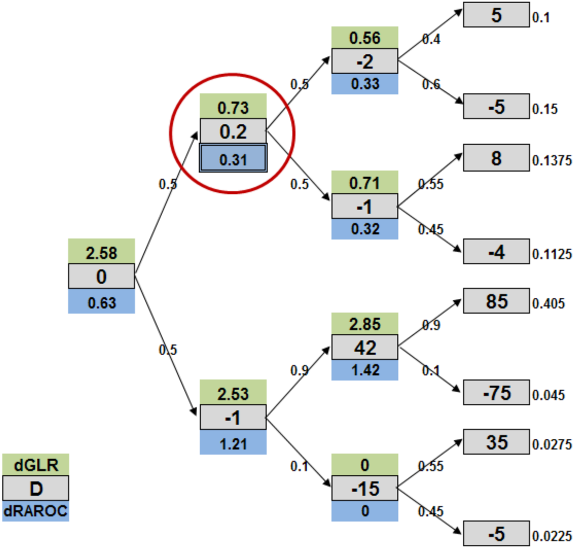
Dynamic Sharpe Ratio

$$\mathrm{dSR}_t(D) = \begin{cases} \frac{\mathbb{E}[\sum_{s=t}^{T} D_s \mid \mathcal{F}_t]}{\mathrm{Std}[\sum_{s=t}^{T} D_s|\mathcal{F}_t]} & \text{if } \mathbb{E}[\sum_{s=t}^{T} D_s \mid \mathcal{F}_t] > 0 \\ 0 & \text{otherwise} \end{cases}$$

Example 1: dGLR vs dSR



Example 2: dRAROC is not dynamic consistent



(a) If $D_t \geq 0$ and $\alpha_{t+1}(D) \geq X$, then $\alpha_t(D) \geq X$
(b) If $D_t \leq 0$ and $\alpha_{t+1}(D) \leq X$, then $\alpha_t(D) \leq X$