When can a risk measure be updated consistently?

Berend Roorda[1] Hans Schumacher[2]

[1] FELab and School of Management and Governance
University of Twente, the Netherlands

[2] CentER and Department of Econometrics and Operations Research
Tilburg University, the Netherlands

Research sponsored in part by Netspar

BFS Toronto June 23 2010

Outline

- Intro to updating
- Weak time consistency
- Unique updating
- Concrete: extra power in weakly consistent entropic risk measures

Updating an expectation operator

0 t

$E^Q X$

Updating an expectation operator

0	t
$E^Q X$	$E_t^Q X$

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Law of iterated expectations:

$$E^Q X = E^Q E_t^Q X$$

Updating an expectation operator

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$E^Q X$	$E^Q_t X$

Law of iterated expectations:

$$E^Q X \quad = \quad E^Q E^Q_t X$$

$X \in L^\infty(\Omega, \mathcal{F}, P)$, $Q \ll P$

$E^Q_t X$ also often written as $E^Q[X|\mathcal{F}_t]$

Updating a coherent risk measure

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$$\phi_0(X) = \inf_{Q \in \mathcal{Q}} E^Q X$$

Updating a coherent risk measure

$0 \qquad\qquad\qquad t$

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Strongly time consistent:

$$\phi_0(X) = \phi_0(\phi_t(X)) \text{ iff } \mathcal{Q} \text{ has pasting property}$$

Delbean 2003
extension to convex class Föllmer and Penner 2006

Strong time consistency???

$$\phi_0(X) = \phi_0(\phi_t(X))$$

What if the risk measure resembles a capital charge?

Strong time consistency???

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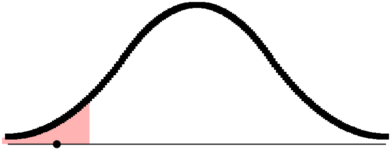
What if the risk measure resembles a capital charge?



Strong time consistency requires that (at time 0) you don't discriminate between the depicted payoff distribution (in some state, at time t, say) and its risk level $\phi_t(X)$ indicated by the dot . . .

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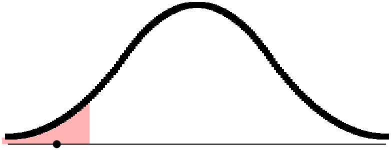

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Strong time consistency is inappropriate for risk measures that (which?) are much more conservative than pricing measures

Weak time consistency

Sequential consistency is the combination of

$$\begin{cases} \text{Acceptance consistency:} & \phi_s(X) \geq 0 \Leftarrow \phi_t(X) \geq 0 \\ \text{Rejection consistency:} & \phi_s(X) \leq 0 \Leftarrow \phi_t(X) \leq 0 \end{cases}$$

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On weak time consistency: Burgert 2005, Tutsch 2006, Weber 2006, Föllmer & Penner 2006, R&S 2007

The refinement update

ϕ_0

$\Updownarrow$

$\mathcal{A}$

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$\mathcal{A} \longrightarrow \mathcal{A}^t$

$\{X \mid \forall\, F \in \mathcal{F}_t : \ 1_F X \in \mathcal{A}\}$

The refinement update

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$\mathcal{A} \longrightarrow \mathcal{A}^t \quad \subseteq \quad \mathcal{A}_t$

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The refinement update

$$\begin{array}{ccccc} \phi_0 & & & & \phi_t \\ \Updownarrow & & & & \Updownarrow \\ \mathcal{A} & \longrightarrow & \mathcal{A}^t & \subseteq & \mathcal{A}_t \end{array}$$

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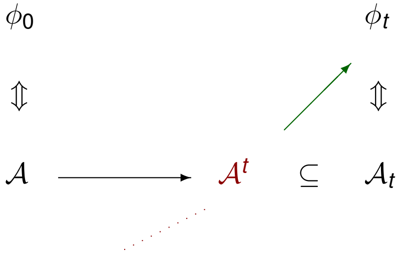

'conditional capital requirement'

Detlefsen & Scandolo 2005;

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and can only be consistent if equality holds
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$$
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\phi_0 & & & & & & \phi_t \\
& & & & & \nearrow & \\
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Extra conditions for sequential consistency in paper

Answer to the main question

When can a risk measure be updated consistently?

(i) Determine the refinement update, given by

$$\phi_0^t(X) = \operatorname{ess\,sup}\{Y \in L_t^\infty \mid \phi_0(1_F(X-Y)) \geq 0 \text{ for all } F \in \mathcal{F}_t\}$$

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- Most results in paper rely on (strong) sensitivity
- Compatibility:
  update to s, then update to t = update to t at once
- Time consistency can be seen as a property of ϕ_0 itself

Coherent case revised

0 t

$$\phi_0 = \inf_{Q \in \mathcal{Q}} E^Q X$$

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Coincides with the refinement update

Strong: pasting $\forall\, Q \in \mathcal{Q} \quad \forall\, Q' \in \mathcal{Q} \quad : \quad Q' Q_t \in Q$

Coherent case revised

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Strong:	pasting	$\forall\, Q \in \mathcal{Q}$	$\forall\, Q' \in \mathcal{Q}$	:	$Q' Q_t \in Q$
Sequential:	juncted	$\forall\, Q \in \mathcal{Q}$	$\exists\, Q' \in \mathcal{Q}$	:	$Q' Q_t \in Q$

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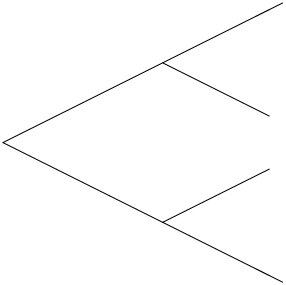
In R&S 2005 in a simple setting

Paper on dual characterizations convex risk measures (L^∞ setting) in preparation

Example Strong versus Weak



Entropic:

$-\frac{1}{\beta}\log E[e^{-\beta X}]$

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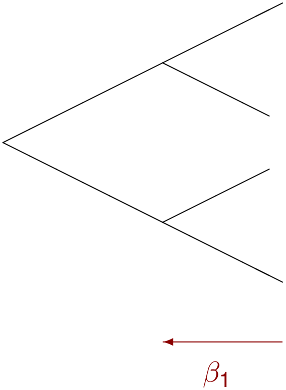

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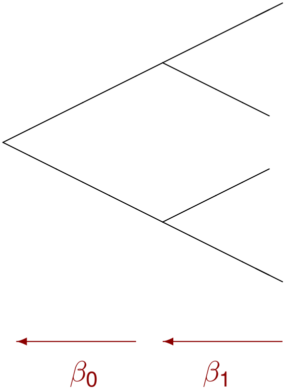

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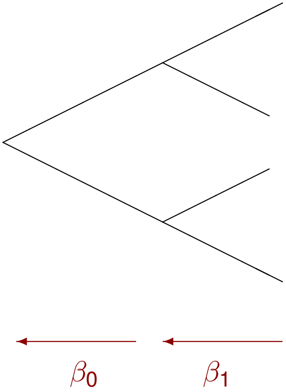

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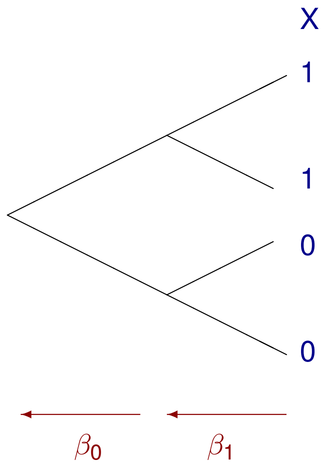

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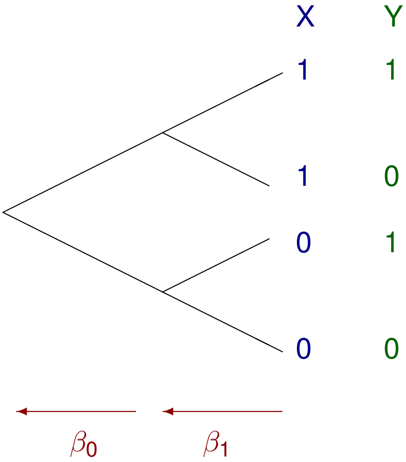

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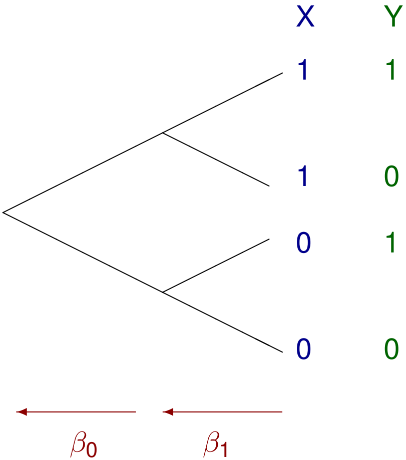

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Strong:
(i) choose β_0 and β_1
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Weak:
(i) limit $\beta_0 + \beta_1 = b$
(ii) *given* position, apply worst pair

Sequentially consistent entropic risk measures

Now $\beta = (\beta_0, \ldots, \beta_{T-1})$, $\beta_t \geq 0$ and $\mathcal{F}_t$-measurable

$$\phi^b(X) = \operatorname{ess\,inf}\{\phi^\beta \mid \Sigma_{t=0}^{T-1} \beta_t = b\}$$

One parameter b: overall level of conservatism
β: *pattern* of conservatism

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(a *compound risk measure* Cheridito & Kupper 2006)

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Easy to compute!

Example: $b = 50$, 100 grid points b_k for beta in (0,50]
Backw. recursively keep track of $\phi_t^{b_k}(X)$ for all grid points b_k

First use of conservatism (%)

Consistent updating

										.	0
									.		
								.		.	0
							.		.		
						.		.		.	0
					20		.		.		
				12		25		.		.	0
			0		0		33		.		
		0		0		0		50		.	0
	0		0		0		0		100		
0		0		0		0		0		.	-10
	0		0		0		0		100		
		0		0		0		50		.	0
			0		0		33		.		
				12		25		.		.	0
					20		.		.		
						.		.		.	0
							.		.		
								.		.	0
									.		

Next use of conservatism after 12% is used

										.	0
									.		
								.		.	0
							.		.		
						.		.		.	0
					17		.		.		
				12		22		.		.	0
			0		0		29		.		
		0		0		0		44		.	0
	0		0		0		0		88		
0		0		0		0		0		.	-10
	0		0		0		0		88		
		0		0		0		44		.	0
			0		0		29		.		
				12		22		.		.	0
					17		.		.		
						.		.		.	0
							.		.		
								.		.	0
									.		

(My) Conclusions

1. Risk dynamics $\neq$ Price dynamics
   - far away from strong time consistency
   - strong should be weakened to sequential consistency
   - then still unambiguous updates
   - computations may remain fairly simple:
     backward recursion in *risk profiles* $\{\phi_t^c(X)\}_{c \in \mathcal{C}}$

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