Dual Representation of Quasiconvex Conditional Maps

Quasiconvex Dynamic Risk Measures

Marco Frittelli and Marco Maggis

Milano University

VI World Congress of the Bachelier Finance Society, Toronto, June 2010

On Quasiconvexity (QCO)

- $f : E \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ is quasiconvex (QCO) if

$$f(\lambda X + (1-\lambda)Y) \leq \max\{f(X), f(Y)\}, \ \lambda \in [0,1]$$

- Equivalently: f is (QCO) if all the lower level sets

$$\{X \in E \mid f(X) \leq c\} \quad \forall c \in \mathbb{R}$$

  are convex

- Findings on (QCO) real valued functions go back to De Finetti (1949), Fenchel (1949)...
- On (QCO) real valued functions and their dual representation: J-P Penot 1990 - 2007, Volle 1998, ...

Dual representation for real valued maps

As a straightforward application of the Hahn-Banach Theorem:

Proposition (Volle 98)

Let E be a locally convex topological vector space and E' be its topological dual space. If $f : E \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ is lsc and (QCO) then

$$f(x) = \sup_{x' \in E'} R(x'(x), x'),$$

where $R : \mathbb{R} \times E' \to \overline{\mathbb{R}}$ is defined by

$$R(m, x') := \inf \left\{ f(\xi) \mid \xi \in E \text{ such that } x'(\xi) \geq m \right\}.$$

An application of the above result leads to:

Dual representation of static (QCO) cash-subadditive risk measures

Proposition (Cerreia-Maccheroni-Marinacci-Montrucchio, 2009)

A function $\rho : L^\infty \to \overline{\mathbb{R}}$ is quasiconvex cash-subadditive decreasing if and only if

$$\begin{aligned} \rho(X) &= \max_{Q \in ba_+(1)} R(E_Q[-X], Q), \\ R(m, Q) &= \inf\{\rho(\xi) \mid \xi \in L^\infty \text{ and } E_Q[-\xi] = m\} \end{aligned}$$

where $R : \mathbb{R} \times ba_+(1) \to \overline{\mathbb{R}}$ and $R(m, Q)$ is the reserve amount required today, under the scenario Q, to cover an expected loss m in the future.

The conditional setting: let $\mathcal{G} \subseteq \mathcal{F}$

A map

$$\pi : L(\Omega, \mathcal{F}, P) \rightarrow L(\Omega, \mathcal{G}, P)$$

is quasiconvex (QCO) if $\forall X, Y \in L(\Omega, \mathcal{F}, P)$ and for all $\mathcal{G}$-measurable r.v. Λ, $0 \leq \Lambda \leq 1$,

$$\pi(\Lambda X + (1 - \Lambda) Y) \leq \pi(X) \vee \pi(Y);$$

or equivalently if all the lower level sets

$$\mathcal{A}(Y) = \{X \in L(\Omega, \mathcal{F}, P) \mid \pi(X) \leq Y\} \quad \forall\, Y \in L(\Omega, \mathcal{G}, P)$$

are conditionally convex, i.e. for all $X_1, X_2 \in \mathcal{A}(Y)$ one has that $\Lambda X_1 + (1 - \Lambda) X_2 \in \mathcal{A}(Y)$.

The problem

Let $\mathcal{G} \subseteq \mathcal{F}$ be an arbitrary sub sigma algebra.

Which is the dual representation of a (QCO) conditional map

$$\pi : L(\Omega, \mathcal{F}, P) \rightarrow L(\Omega, \mathcal{G}, P) \quad ?$$

The problem

Let $\mathcal{G} \subseteq \mathcal{F}$ be an arbitrary sub sigma algebra.

Which is the dual representation of a (QCO) conditional map

$$\pi : L(\Omega, \mathcal{F}, P) \to L(\Omega, \mathcal{G}, P) \quad ?$$

As in the convex case, the dual representation of a (QCO) conditional map turns out to have the same structure of the real valued case,

...but the proof is not a straightforward application of known facts.

Dynamic (QCO) Risk Measures

- Let Λ, $0 \leq \Lambda \leq 1$, be $\mathcal{G}$-measurable random variables
- The convexity of $\pi : L(\Omega, \mathcal{F}, P) \to L(\Omega, \mathcal{G}, P)$

$$\pi(\Lambda X + (1-\Lambda)Y) \leq \Lambda\pi(X) + (1-\Lambda)\pi(Y)$$

implies:

$$\pi(\Lambda X + (1-\Lambda)Y) \leq \Lambda\pi(X) + (1-\Lambda)\pi(Y) \leq \pi(X) \vee \pi(Y).$$

- Quasiconvexity alone:

$$\pi(\Lambda X + (1-\Lambda)Y) \leq \pi(X) \vee \pi(Y)$$

allows to control the risk of a diversified position.

Conditional Certainty Equivalent: CCE [F. Maggis 2010]

Consider a Stochastic Dynamic Utility (SDU) $u(x, t, \omega)$

$$u : \mathbb{R} \times [0, \infty) \times \Omega \to \mathbb{R} \cup \{-\infty\}$$

Definition

Let u be a SDU and X be a $\mathcal{F}_t$ measurable random variable. For each $s \in [0, t]$, the backward Conditional Certainty Equivalent $C_{s,t}(X)$ of X is the $\mathcal{F}_s$ measurable random variable solution of the equation:

$$u(C_{s,t}(X), s, \omega) = E\left[u(X, t, \omega) | \mathcal{F}_s\right].$$

This valuation operator $C_{s,t}(X) = u^{-1}\left(E\left[u(X, t, \omega)|\mathcal{F}_s\right], s, \omega\right)$ is the natural generalization to the dynamic and stochastic environment of the classical definition of the certainty equivalent, as given in Pratt 1964. Even if $u(., t, \omega)$ is concave the CCE is not a concave functional, but it is conditionally quasiconcave.

- Other applications of real valued quasiconvex maps in finance (static quasiconvex risk measures) can be found in the papers by:

- Cerreia–Voglio, Maccheroni, Marinacci and Montrucchio 2009
- Drapeau and Kupper 2010

- Dynamic quasiconvex risk measures are studied in:

- F. Maggis 2009 and 2010

Setting for the dual representation

$$\pi : L_{\mathcal{F}} \longrightarrow L_{\mathcal{G}}$$

We now state the assumptions on the spaces of random variables $L_{\mathcal{F}}$ and $L_{\mathcal{G}}$ and on the quasiconvex conditional map π in order to obtain the dual representation.

Notations

- $L^p_{\mathcal{F}} := L^p(\Omega, \mathcal{F}, P)$, $p \in [0, \infty]$.
- $L_{\mathcal{F}} := L(\Omega, \mathcal{F}, P) \subseteq L^0(\Omega, \mathcal{F}, P)$ is a lattice of $\mathcal{F}$ measurable random variables.
- $L_{\mathcal{G}} := L(\Omega, \mathcal{G}, P) \subseteq L^0(\Omega, \mathcal{G}, P)$ is a lattice of $\mathcal{G}$ measurable random variables.
- $L^c_{\mathcal{F}} = (L_{\mathcal{F}}, \geq)^c$ is the order continuous dual of $(L_{\mathcal{F}}, \geq)$, which is also a lattice.

Standing assumptions on the spaces

1. $L_{\mathcal{F}}$ (resp. $L_{\mathcal{G}}$) satisfies the property 1_F (resp 1_G):

$$X \in L_{\mathcal{F}} \text{ and } A \in \mathcal{F} \Longrightarrow (X\mathbf{1}_A) \in L_{\mathcal{F}}. \qquad (1_F)$$

2. $(L_{\mathcal{F}}, \sigma(L_{\mathcal{F}}, L^c_{\mathcal{F}}))$ *is a locally convex TVS.*

   This condition requires that the order continuous dual $L^c_{\mathcal{F}}$ is rich enough to separate the points of $L_{\mathcal{F}}$.

3. $L^c_{\mathcal{F}} \hookrightarrow L^1(\Omega, \mathcal{F}, P)$

4. $L^c_{\mathcal{F}}$ satisfies the property 1_F

Examples of spaces satisfying the assumptions

- The L^p spaces: $L_{\mathcal{F}} := L^p_{\mathcal{F}}$, with $p \in [1, \infty]$.
  Then: $L^c_{\mathcal{F}} = L^q_{\mathcal{F}} \hookrightarrow L^1_{\mathcal{F}}$ (with $q = 1$ when $p = \infty$).
- The Orlicz spaces $L_{\mathcal{F}} := L^{\Psi}_{\mathcal{F}}$, for any Young function Ψ.
  Then $L^c_{\mathcal{F}} = L^{\Psi^*} \hookrightarrow L^1_{\mathcal{F}}$, where Ψ^* is the conjugate of Ψ.
- The Morse subspace $L_{\mathcal{F}} := M^{\Psi}$ for any continuous Young function Ψ.
  Then $L^c_{\mathcal{F}} = L^{\Psi^*} \hookrightarrow L^1_{\mathcal{F}}$.

Conditions on $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$

Let $X_1, X_2 \in L_{\mathcal{F}}$

(MON) $X_1 \leq X_2 \Longrightarrow \pi(X_1) \leq \pi(X_2)$

Conditions on $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$

Let $X_1, X_2 \in L_{\mathcal{F}}$

(MON) $X_1 \leq X_2 \Longrightarrow \pi(X_1) \leq \pi(X_2)$

(τ-LSC) the lower level set

$$\mathcal{A}_Y = \{X \in L_{\mathcal{F}} \mid \pi(X) \leq Y\}$$

is τ closed for each $\mathcal{G}$-measurable Y

Conditions on $\pi : L_{\mathcal{F}} \rightarrow L_{\mathcal{G}}$

Let $X_1, X_2 \in L_{\mathcal{F}}$

(MON) $X_1 \leq X_2 \Longrightarrow \pi(X_1) \leq \pi(X_2)$

(τ-LSC) the lower level set

$$\mathcal{A}_Y = \{X \in L_{\mathcal{F}} \mid \pi(X) \leq Y\}$$

is τ closed for each $\mathcal{G}$-measurable Y

(REG) $\forall A \in \mathcal{G},\ \pi(X_1 \mathbf{1}_A + X_2 \mathbf{1}_A^C) = \pi(X_1)\mathbf{1}_A + \pi(X_2)\mathbf{1}_A^C$

On continuity from below (CFB)

(CFB) $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ is *continuous from below if*

$$X_n \uparrow X \quad P \text{ a.s.} \quad \Rightarrow \quad \pi(X_n) \uparrow \pi(X) \quad P \text{ a.s.}$$

Under a very weak assumption on $\sigma(L_{\mathcal{F}}, L^c_{\mathcal{F}})$, that is satisfied in all cases of interest, we have:

Proposition

If $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ satisfies (MON) and (QCO), then are equivalent:
(i) π is $\sigma(L_{\mathcal{F}}, L^c_{\mathcal{F}})$-(LSC)
(ii) π is (CFB)
(iii) π is order-(LSC) (i.e. the Fatou property)

Conclusion: in the following results, we may replace the condition $\sigma(L_{\mathcal{F}}, L^c_{\mathcal{F}})$-(LSC) with (CFB).

The dual representation of conditional quasiconvex maps

Theorem

If $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ *is (MON), (QCO), (REG) and* $\sigma(L_{\mathcal{F}}, L^c_{\mathcal{F}})$*-LSC then*

$$\pi(X) = ess \sup_{Q \in L^c_{\mathcal{F}} \cap \mathcal{P}} K(X, Q)$$

where

$$K(X, Q) := ess \inf_{\xi \in L_{\mathcal{F}}} \{\pi(\xi) \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]\}$$

$$\mathcal{P} =: \{Q << P \text{ and } Q \text{ probability}\}$$

Exactly the same representation of the real valued case, but with conditional expectations!

$Q = P$ on $\mathcal{G}$

Corollary

Suppose that the assumptions of the Theorem hold true.
If for $X \in L_{\mathcal{F}}$ there exists $\eta \in L_{\mathcal{F}}$ and $\varepsilon > 0$ such that $\pi(\eta) + \varepsilon < \pi(X)$, then

$$\pi(X) = ess \sup_{Q \in L^c_{\mathcal{F}} \cap \mathcal{P}_{\mathcal{G}}} K(X, Q),$$

where

$$\mathcal{P}_{\mathcal{G}} =: \{Q \in \mathcal{P} \text{ and } Q = P \text{ on } \mathcal{G}\}.$$

NOTE: The (weak) additional assumption allows us to replace $\mathcal{P} =: \{Q << P \text{ and } Q \text{ probability}\}$ with the same set $\mathcal{P}_{\mathcal{G}}$ that is used in the convex conditional case.

Cash additivity

- A map $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ is said to be

  (CAS) *cash additive if for all* $X \in L_{\mathcal{F}}$ *and* $\Lambda \in L_{\mathcal{G}}$

$$\pi(X + \Lambda) = \pi(X) + \Lambda.$$

Cash additivity

- A map $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ is said to be

  (CAS) *cash additive if for all* $X \in L_{\mathcal{F}}$ *and* $\Lambda \in L_{\mathcal{G}}$

$$\pi(X + \Lambda) = \pi(X) + \Lambda.$$

- Note: (CAS) and (QCO) implies Convexity.

Cash additivity

- A map $\pi : L_{\mathcal{F}} \rightarrow L_{\mathcal{G}}$ is said to be

  (CAS) *cash additive if for all* $X \in L_{\mathcal{F}}$ *and* $\Lambda \in L_{\mathcal{G}}$

$$\pi(X + \Lambda) = \pi(X) + \Lambda.$$

- Note: (CAS) and (QCO) implies Convexity.
- Next, we show that we recover the result of Detlefsen Scandolo 05 for convex conditional maps.

The conditional convex case

Corollary

Suppose that the assumptions of the Theorem hold true.
Suppose that for every $Q \in L^c_{\mathcal{F}} \cap \mathcal{P}_{\mathcal{G}}$ *and* $\xi \in L_{\mathcal{F}}$ *we have* $E_Q[\xi|\mathcal{G}] \in L_{\mathcal{F}}$.
If $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$ *satisfies in addition (CAS) then*

$$K(X,Q) = E_Q[X|\mathcal{G}] - \pi^*(Q)$$

and

$$\pi(X) = ess \sup_{Q \in L^c_{\mathcal{F}} \cap \mathcal{P}_{\mathcal{G}}} \{E_Q[X|\mathcal{G}] - \pi^*(Q)\}$$

where

$$\pi^*(Q) = ess \sup_{\xi \in L_{\mathcal{F}}} \{E_Q[\xi|\mathcal{G}] - \pi(\xi)\}.$$

Why the proofs of the real valued case and convex case do not work

- We cannot directly apply Hahn-Banach to $\pi : L_{\mathcal{F}} \to L_{\mathcal{G}}$, as it happened in the real case, since

$$\{\xi \in L_{\mathcal{F}} \mid \pi(\xi) \leq \pi(X) - \varepsilon\}^C$$

is not any more convex!

- Scalarization does not work! Convexity is preserved by the map:

$$\pi_0 : L_{\mathcal{F}} \to \mathbb{R} \quad \pi_0(X) := E[\pi(X)]$$

but not quasiconvexity!

Approximation argument

The idea is to approximate π with combinations of quasiconvex real valued functions π_A

$$\pi_A(X) := \operatorname*{ess\,sup}_{\omega \in A} \pi(X), \;\; A \in \mathcal{G}.$$

We consider finite partitions $\Gamma = \{A^\Gamma\}$ of $\mathcal{G}$ measurable sets A^Γ and

$$\pi^\Gamma(X) := \sum_{A^\Gamma \in \Gamma} \pi_{A^\Gamma}(X) \mathbf{1}_{A^\Gamma},$$

$$H^\Gamma(X) := \sup_{Q \in L^c_{\mathcal{F}} \cap \mathcal{P}} \inf_{\xi \in L_{\mathcal{F}}} \left\{ \pi^\Gamma(\xi) \mid E_Q[\xi|\mathcal{G}] \geq E_Q[X|\mathcal{G}] \right\}$$

Steps of the proof

I First we show $H^{\Gamma}(X) = \pi^{\Gamma}(X)$.

II Then it is a simple matter to deduce

$$\pi(X) = \inf_{\Gamma} \pi^{\Gamma}(X) = \inf_{\Gamma} H^{\Gamma}(X)$$

III Finally we prove that

$$\begin{aligned}\inf_{\Gamma} H^{\Gamma}(X) &= \inf_{\Gamma} \sup_{Q \in L_t^c \cap \mathcal{P}} \inf_{\xi \in L_t} \left\{ \pi^{\Gamma}(\xi) \middle| E_Q[\xi|\mathcal{F}_s] \geq E_Q[X|\mathcal{F}_s] \right\} \\ &= \sup_{Q \in L_t^c \cap \mathcal{P}} \inf_{\xi \in L_t} \{ \pi(\xi) | E_Q[\xi|\mathcal{F}_s] \geq E_Q[X|\mathcal{F}_s] \}\end{aligned}$$

that is based on a uniform approximation result.

Following [Filipovic, Kupper, Vogelpoth 2009-2010] we may consider maps

$$\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to \bar{L}^0(\mathcal{G})$$

where

$$L^p_{\mathcal{G}}(\mathcal{F}) = L^0(\mathcal{G})L^p(\mathcal{F}) = \{YX \mid Y \in L^0(\mathcal{G}),\ X \in L^p(\mathcal{F})\}$$

is an $L^0(\mathcal{G})$ normed module.

- We showed that the dual representation of a quasiconvex dynamic risk measure defined on $L^p_{\mathcal{G}}(\mathcal{F})$ also works in this setting.
- The proof is easier: it is similar to the real valued case, since it uses the conditional Hahn Banach Th., as developed in [FKV09]
- Quasiconvex dynamic risk measures defined on vector spaces $L^p_{\mathcal{F}}$ or on $L^0(\mathcal{G})$ normed module $L^p_{\mathcal{G}}(\mathcal{F})$ are different objects (satisfy different properties) and therefore the results in the two cases are different.

Thank you for your attention