Optimal Investment for Worst-Case Crash Scenarios

A Martingale Approach

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Outline

Black-Scholes I: Review

In a Black-Scholes market consisting of a riskless bond

$$\mathrm{d}B_t = rB_t\mathrm{d}t$$

and a risky asset

$$\mathrm{d}P_t = P_t[(r+\eta)\mathrm{d}t + \sigma\mathrm{d}W_t]$$

the classical Merton optimal investment problem is to achieve

$$\max_{\pi} \ \mathbb{E}[u(X_T^{\pi})].$$

Here $X = X^{\pi}$ denotes the wealth process corresponding to the portfolio strategy π via

$$\mathrm{d}X_t = X_t[(r+\pi_t\eta)\mathrm{d}t + \pi_t\sigma\mathrm{d}W_t], \quad X_0 = x_0,$$

and u is the investor's utility for terminal wealth, which we assume to be of the CRRA form $u(x) = \frac{1}{\rho}x^{\rho}$, $x > 0$, for some $\rho < 1$.

Black-Scholes II: Critique

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Phenomenon: "Flight to Riskless Assets"

This strategy is not in line with real-world investor behavior or professional asset allocation advice: Towards the end of the time horizon, wealth should be reallocated from risky to riskless investment.

Black-Scholes III: Crashes

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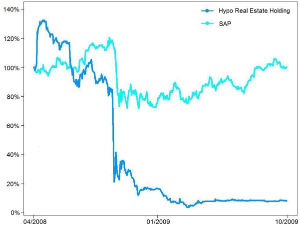

Investors are afraid of a large **market crash** that has the potential to destroy the value of their stock holdings.

Crash Modeling II: Jumps in Asset Dynamics

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However, for these models the optimal portfolio strategy remains *independent of the remaining investment time*: For instance, if

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with a compensated Poisson process $\tilde{N}$, then the optimal strategy is

$$\pi^\star = \frac{\eta}{(1-\rho)\sigma^2} + \textit{constant}\text{ correction term.}$$

Thus, the effect of a crash is only accounted for 'in the mean'.

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Unless market crashes depend on the investor's time horizon, a modification of the asset price dynamics does not resolve the problem.

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Following F. KNIGHT (1885-1972), let us distinguish two notions of 'risk':

- **risk**: quantifiable, susceptible of measurement, stochastic, statistical, modeled on $(\Omega, \mathfrak{F}, \mathbb{P})$
- **uncertainty**: 'true'/Knightian/pure uncertainty, no distributional properties, no statistics possible or available

Crash Modeling V: Crashes and Uncertainty

There is ample time series data on regular fluctuations of asset prices, but major crashes are largely unique events. Examples include

- economic or political crises and wars
- natural disasters
- bubble markets
- ... and more.

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Thus, while ordinary price movements are a matter of **risk**, market crashes are subject to **uncertainty**.

Optimal Investment Problem I: Crash Scenarios

We model a financial market **crash scenario** as a pair

$$(\tau, \ell)$$

where the $[0, T] \cup \{\infty\}$-valued stopping time τ represents the time when the crash occurs, and the $[0, \ell^\infty]$-valued $\mathfrak{F}_\tau$-measurable random variable ℓ is the relative crash height:

$$\mathrm{d}P_t = P_t[(r+\eta)\mathrm{d}t + \sigma \mathrm{d}W_t], \quad P_\tau = (1-\ell)P_{\tau-}.$$

Here $\ell^\infty \in [0,1]$ is the maximal crash height, and the event $\tau = \infty$ is interpreted as there being no crash at all.

Optimal Investment Problem II: Portfolio Strategies

The investor chooses a **portfolio strategy** π to be applied before the crash, and a strategy $\bar{\pi}$ to be applied afterwards.

Given the crash scenario (τ, ℓ), the dynamics of the investor's **wealth process** $X = X^{\pi,\bar{\pi},\tau,\ell}$ are given by

$$\begin{aligned}
\mathrm{d}X_t &= X_{t-}[(r + \pi_t \eta)\mathrm{d}t + \pi_t \sigma \mathrm{d}W_t] \text{ on } [0, \tau), \quad X_0 = x_0, \\
\mathrm{d}X_t &= X_{t-}[(r + \bar{\pi}_t \eta)\mathrm{d}t + \bar{\pi}_t \sigma \mathrm{d}W_t] \text{ on } (\tau, T], \\
X_\tau &= (1 - \pi_\tau)X_{\tau-} + (1 - \ell)\pi_\tau X_{\tau-} = (1 - \pi_\tau \ell)X_{\tau-}.
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'High' values of π lead to a high final wealth in the no-crash scenario, but also to a large loss in the event of a crash — 'low' values of π lead to small or no losses in a crash, but also to a low terminal wealth if no crash occurs.

Optimal Investment Problem III: Formulation

As above, the investor's attitude towards (measurable, stochastic) **risk** is modeled by a CRRA utility function

$$u(x) = \frac{1}{\rho} x^{\rho}, \; x > 0, \text{ for some } \rho < 1.$$

By contrast, he takes a worst-case attitude towards the (Knightian, 'true') **uncertainty** concerning the financial market crash, and thus faces the

Worst-Case Optimal Investment Problem

$$\max_{\pi,\bar{\pi}} \; \min_{\tau,\ell} \; \mathbb{E}[u(X_T^{\pi,\bar{\pi},\tau,\ell})]. \tag{P}$$

Problem (P) reflects an extraordinarily cautious attitude towards the threat of a crash. Note that there are *no distributional assumptions* on the crash time and height. Observe also that portfolio strategies are *not* compared scenario-wise.

Martingale Approach I: Idea and Motivation

The fundamental ideas underlying the martingale approach to worst-case optimal investment are:

- The worst-case investment problem can be regarded as a **game** between the investor and the market.
- The notion of **indifference** plays a fundamental role in this game.

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The martingale approach consists of 3 main components:

- the **Change-of-Measure Device**,
- the **Indifference-Optimality Principle**, and
- the **Indifference Frontier**.

Post-Crash Problem I: Change-of-Measure Device

To solve the post-crash portfolio problem, we use a well-known trick:

Theorem (Change-of-Measure Device)

Consider the classical optimal portfolio problem with random initial time τ and time-τ initial wealth ξ,

$$\max_{\bar{\pi}} \ \mathbb{E}[u(X_T^{\bar{\pi}}) \,|\, X_\tau^{\bar{\pi}} = \xi]. \qquad (\mathrm{P_{post}})$$

Then for any strategy $\bar{\pi}$ we have

$$u(X_T^{\bar{\pi}}) = u(\xi) \exp\left\{\rho \int_\tau^T \Phi(\bar{\pi}_s)\mathrm{d}s\right\} M_T^{\bar{\pi}}$$

with $\Phi(y) \triangleq r + \eta y - \frac{1}{2}(1-\rho)\sigma^2 y^2$ and a martingale $M^{\bar{\pi}}$ satisfying $M_\tau^{\bar{\pi}} = 1$. Thus the solution to (P_{post}) is the Merton strategy π^M.

Post-Crash Problem II: Reformulation

The Change-of-Measure Device allows us to reformulate the worst-case investment problem (P)

$$\max_{\pi,\bar{\pi}} \min_{\tau,\ell} \mathbb{E}[u(X_T^{\pi,\bar{\pi},\tau,\ell})]$$

as the

Pre-Crash Investment Problem

$$\max_{\pi} \min_{\tau} \mathbb{E}[V(\tau,(1-\pi_\tau \ell^\infty)X_\tau^\pi)]. \qquad (\mathrm{P}_{\mathrm{pre}})$$

Here V is the value function of the post-crash problem,

$$V(t,x) = \exp\{\rho\Phi(\pi^M)(T-t)\}u(x).$$

Controller-vs-Stopper I: Abstract Formulation

The formulation ($\mathrm{P_{pre}}$) takes the form of the abstract

Controller-vs-Stopper Game [Karatzas and Sudderth (2001)]

Consider a zero-sum stochastic game between player A (the controller) and player B (the stopper). Player A controls a stochastic process

$$W = W^{\lambda} \text{ on the time horizon } [0, T]$$

by choosing λ, and player B decides on the duration of the game by choosing a $[0, T] \cup \{\infty\}$-valued stopping time τ. The terminal payoff is W_{τ}^{λ}. Thus player A faces the problem

$$\max_{\lambda} \min_{\tau} \mathbb{E}[W_{\tau}^{\lambda}]. \qquad (\mathrm{P_{abstract}})$$

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$$\max_\lambda \min_\tau \mathbb{E}[W_\tau^\lambda]. \qquad (\mathrm{P_{abstract}})$$

In the worst-case investment problem,

$$W_t^\lambda = V(t, (1 - \pi_t \ell^\infty) X_t^\pi), \ t \in [0, T], \quad W_\infty^\lambda = V(T, X_T^\pi) = u(X_T^\pi).$$

Controller-vs-Stopper II: Indifference-Optimality Principle

If player A can choose his strategy $\hat{\lambda}$ in such a way that $W^{\hat{\lambda}}$ is a **martingale**, then player B's actions become irrelevant to him:

$$\mathbb{E}[W_\sigma^{\hat{\lambda}}] = \mathbb{E}[W_\tau^{\hat{\lambda}}] \text{ for all stopping times } \sigma, \tau.$$

Hence, we say that $\hat{\lambda}$ is an (abstract) **indifference strategy**.

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Proposition (Indifference-Optimality Principle)

If $\hat{\lambda}$ is an indifference strategy, and for all λ we have

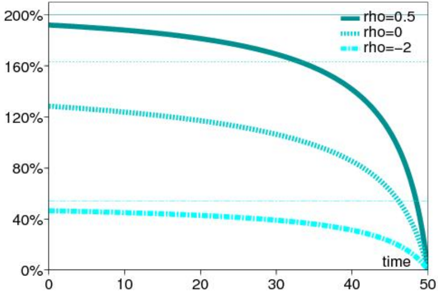
$$\mathbb{E}[W_\tau^{\hat{\lambda}}] \geq \mathbb{E}[W_\tau^{\lambda}] \text{ for just one stopping time } \tau,$$

then $\hat{\lambda}$ is optimal for player A in $(P_{abstract})$.

Indifference I: Indifference Strategy

The indifference strategy $\hat{\pi}$ for worst-case investment is given by the o.d.e.

$$\dot{\hat{\pi}}_t = -\frac{\sigma^2}{2\ell^\infty}(1-\rho)[1-\hat{\pi}_t\ell^\infty][\hat{\pi}_t - \pi^M]^2, \quad \hat{\pi}_T = 0. \qquad \text{(I)}$$



The indifference strategy is below the Merton line and satisfies $\hat{\pi}_t\ell^\infty \leq 1$.
It converges towards the Merton strategy if $\pi^M\ell^\infty \leq 1$.

Indifference II: Indifference Frontier

The indifference strategy represents a **frontier** which rules out too naïve investment.

Lemma (Indifference Frontier)

Let $\hat{\pi}$ *be determined from (I), and let* π *be any portfolio strategy. Then the worst-case bound attained by the strategy* $\tilde{\pi}$*,*

$$\tilde{\pi}_t \triangleq \pi_t \text{ if } t < \sigma, \quad \tilde{\pi}_t \triangleq \hat{\pi}_t \text{ if } t \geq \sigma,$$

where $\sigma \triangleq \inf\{t : \pi_t > \hat{\pi}_t\}$*, is at least as big as that achieved by* π*.*

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where $\sigma \triangleq \inf\{t : \pi_t > \hat{\pi}_t\}$, is at least as big as that achieved by π.

Proof.

Since $W_t^{\tilde{\pi}} = W_t^{\hat{\pi}}$ is a martingale for $t > \sigma$ and $W_t^{\tilde{\pi}} = W_t^{\pi}$ for $t \leq \sigma$,

$$\mathbb{E}[W_\tau^{\tilde{\pi}}] = \mathbb{E}[W_{\tau\wedge\sigma}^{\tilde{\pi}}] = \mathbb{E}[W_{\tau\wedge\sigma}^{\pi}] \geq \min_{\tau'} \mathbb{E}[W_{\tau'}^{\pi}]$$

for an arbitrary stopping time τ. $\square$

Solution I: Worst-Case Optimal Strategy

Combining the previous results, we arrive at the following

Theorem (Solution of the Worst-Case Investment Problem)

For the worst-case portfolio problem

$$\max_{\pi,\bar{\pi}} \min_{\tau,\ell} \mathbb{E}[u(X_T^{\pi,\bar{\pi},\tau,\ell})] \tag{P}$$

the optimal strategy in the pre-crash market is given by the indifference strategy $\hat{\pi}$. After the crash, the Merton strategy π^M is optimal.

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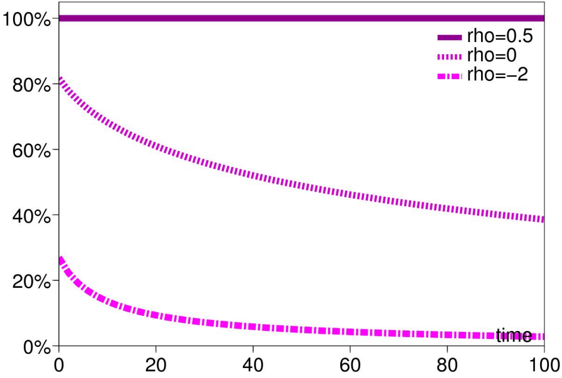
the optimal strategy in the pre-crash market is given by the indifference strategy $\hat{\pi}$*. After the crash, the Merton strategy* π^M *is optimal.*

Proof.

We need only consider pre-crash strategies below the Indifference Frontier. By the Indifference-Optimality Principle, the indifference strategy is optimal provided it is optimal in the no-crash scenario. This, however, follows immediately from the Change-of-Measure Device. □

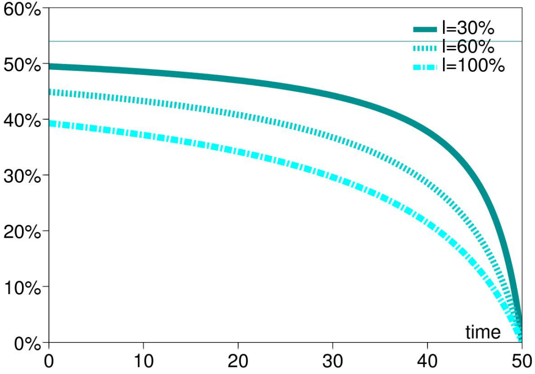
Solution III: Effective Wealth Loss

To illustrate the difference to traditional portfolio optimization, we determine the **effective wealth loss** of a Merton investor in his worst-case scenario.



Solution IV: Sensitivity to Crash Size

The solution to the worst-case investment problem is non-zero even for a maximum crash height $\ell^\infty = 100\%$.



Multi-Asset Markets I

The martingale approach generalizes directly to **multi-asset markets**. In this multi-dimensional setting, the indifference frontier is specified by

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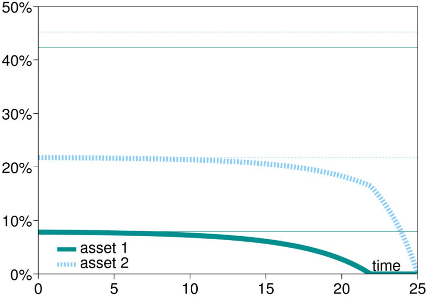

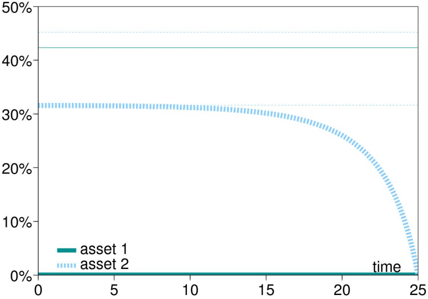

Alternative Dynamics I: Regular Jumps

Regular price jumps can be included in the stock price dynamics; thus the investor distinguishes regular jumps (risky) from crashes (uncertain).

$$\mathrm{d}P_t = P_{t-}\left[(r+\eta)\mathrm{d}t + \sigma.\mathrm{d}W_t - \int \xi \nu(\mathrm{d}t, \mathrm{d}\xi)\right], \quad P_\tau = (1-\ell)P_{\tau-}.$$

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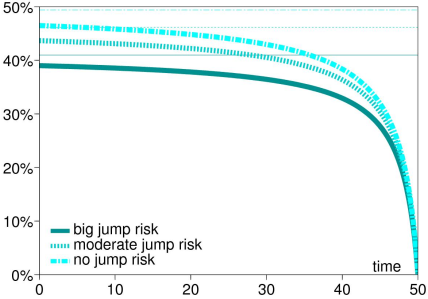
The effects are similar to the Black-Scholes case:



Alternative Dynamics II: Regime Shifts

We can model different market regimes by allowing the market coefficients to change after a possible crash:

$$\mathrm{d}P_t = P_{t-}\left[(r+\eta)\mathrm{d}t + \sigma.\mathrm{d}W_t\right] \text{ on } [0,\tau)$$
$$\mathrm{d}P_t = P_{t-}\left[(\bar{r}+\bar{\eta})\mathrm{d}t + \bar{\sigma}.\mathrm{d}\bar{W}_t\right] \text{ on } [\tau, T], \quad P_\tau = (1-\ell)P_{\tau}.$$

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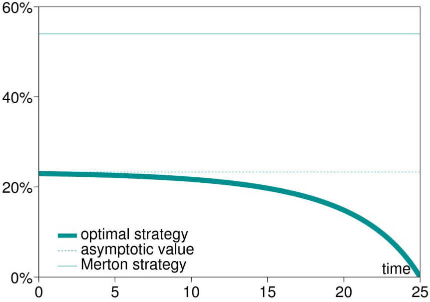
\end{aligned}
$$

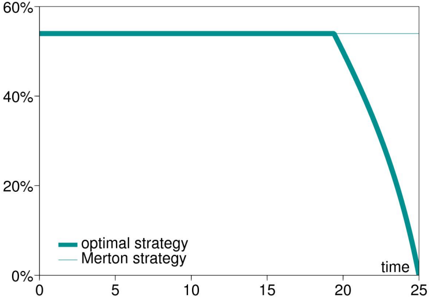
Now we need to distinguish between bull and bear markets:

If the post-crash market is worse than the pre-crash riskless investment, the investor perceives a **bear market**; in this case, it is optimal *not* to invest in risky assets.

Alternative Dynamics III: Bull Markets

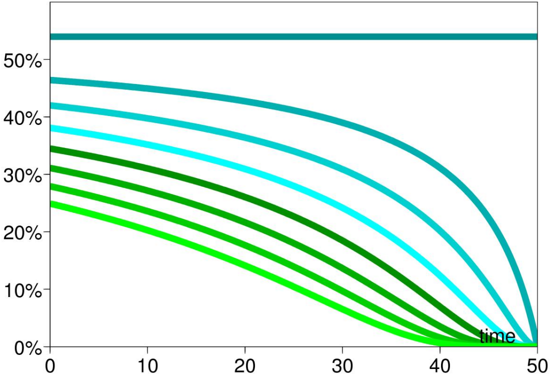
On the other hand, in a **bull market** it is optimal to use the indifference strategy as long as it is below the Merton line:





Alternative Dynamics V: Multiple Crashes

Finally the model can be extended to **multiple crashes**. The worst-case optimal strategy can be determined by backward recursion:

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