Mean Variance Optimization with State Dependent Risk Aversion

Tomas Björk Agatha Murgoci Xunyu Zhou

Bachelier Finance Society
6^{th} World Congress
Toronto

The Classical Mean Variance Problem

- X_t - wealth process
- u_t - amount invested in the risky asset

$$\max E_{t,x}[X_T] - \frac{\gamma}{2} Var_{t,x}[X_T]$$

- time inconsistency

$$\max E_{t,x}[F(X_T)] + G\left(E_{t,x}[X_T]\right)$$

- the control does **not** depend on the current wealth

$$u(t,x) = h(t)$$

Why State Dependent Risk Aversion

- $u(t, x) = h(t)$ since the risk aversion parameter γ is constant
- it does not matter if the wealth is 100 USD, or 100 000 000 USD, we invest the **same** amount of dollars in the risky asset
- conceptual difference between the 1-period model and the multi-period model.
- make γ explicitely dependent on X_t

$$\max E_{t,x}[X_T] - \frac{\gamma(x)}{2} Var_{t,x}[X_T]$$

where $X_t = x$

The Problem

- riskless asset

$$dB_t = rB_t dt$$

- risky asset

$$dS_t = \alpha S_t dt + \sigma S_t dW_t$$

- wealth portfolio

$$dX_t = [rX_t + (\alpha - r)u_t]dt + \sigma u_t dW_t$$

- u_t - amount of money invested in the stock
- $J(t, x, \mathbf{u}) = E_{t,x}[X_T] - \frac{\gamma(x)}{2} Var_{t,x}[X_T]$

Time Inconsistency

$$E_{t,x}[X_T] - \frac{\gamma(x)}{2} Var_{t,x}[X_T] = E_{t,x}[F(x, X_T)] + G\left(x, E_{t,x}[X_T]\right)$$

- standard dynamic programming problem $\max E_{t,x}[F(X_T)]$
- here $\max E_{t,x}[F(x, X_T)] + G\left(x, E_{t,x}[X_T]\right)$
- conceptual problem - *what is optimal?*
- computational problem - *how we compute it?*

Possible ways out

- **Pre-commitment:** Solve (somehow) the problem at $0, x_0$ and ignore the fact that later on, your "optimal" control will no longer be viewed as optimal.
- **Game theory:** Take the time inconsistency seriously. View the problems as a game and look for a Nash equilibrium point.
    - Ekeland & Lazrak (2006); Ekeland & Pirvu (2007)
    - Basak & Chabakauri (2008)
    - Björk & Murgoci (2009)

The Game Theoretic Approach

- We view this as a game where there is one player for each t.
- Player No t chooses the control function $u(t, \cdot)$ at time t, and applies the control $u(t, X_t)$
- The value, to player No t, if all players use the control law u is

$$J(t, x; u) = E_{t,x}\left[x, F(X_T^u)\right] + G\left(x, E_{t,x}[X_T^u]\right)$$

Subperfect Nash Equilibrium

Definition

The strategy $\hat{u}$ is a **Nash subgame perfect equilibrium** if the following holds for all t:

- Assume that all players No s with $s > t$ use the control $\hat{u}(s, X_s)$.
- Then it is optimal for player No t also to use $\hat{u}(t, X_t)$.

Note!

- this leads to an extension of the HJB equation as a PDE system with an embedded fixed point problem.

Notation

$$
\begin{aligned}
V(T,x) &= F(x,x) + G(x,x) \\
F(x,y) &= y - \frac{\gamma(x)}{2} y^2, \\
G(x,y) &= \frac{\gamma(x)}{2} y^2.
\end{aligned}
$$

- Probabilistic interpretation

$$
\begin{aligned}
f(t,x,y) &= E_{t,x}\left[F(y, X_T^{\hat{\mathbf{u}}})\right] \\
g(t,x) &= E_{t,x}\left[X_T^{\hat{\mathbf{u}}}\right]
\end{aligned}
$$

Note! $V(t,x) = f(t,x,x) + \frac{\gamma(x)}{2} g^2(t,x)$

Fixed Point PDE System

$$
\begin{aligned}
\sup_{u \in \mathcal{U}} \{(\mathcal{A}^u V)(t,x) - (\mathcal{A}^u f)(t,x,x) + (\mathcal{A}^u f^x)(t,x) & \\
- \mathcal{A}^u G(x, g(t,x)) + G_y(x, g(t,x)) \cdot \mathcal{A}^u g(t,x)\} &= 0, \\
\mathcal{A}^{\hat{u}} f^y(t,x) &= 0, \\
\mathcal{A}^{\hat{u}} g(t,x) &= 0, \\
V(T,x) &= F(x,x) + G(x,x \\
f(T,x,y) &= F(y,x), \\
g(T,x) &= x.
\end{aligned}
$$

Remember! $V(t,x) = f(t,x,x) + \frac{\gamma(x)}{2} g^2(t,x)$

Solving the PDE system

Optimal control

$$\hat{u}(t,x) = -\frac{\alpha - r}{\sigma^2}\frac{f_x(t,x,x) + \gamma(x)g(t,x)g_x(t,x)}{f_{xx}(t,x,x) + \gamma(x)g(t,x)g_{xx}(t,x)}$$

New PDE system

$$\begin{aligned} f_t + [rx + (\alpha - r)\hat{u}]f_x + \frac{1}{2}\sigma^2 f_{xx} &= 0 \\ g_t + [rx + (\alpha - r)\hat{u}]g_x + \frac{1}{2}\sigma^2 g_{xx} &= 0 \end{aligned}$$

with f and g evaluated at (t,x,y) and

$$\begin{aligned} f(T,x,y) &= x - \frac{\gamma(y)}{2}x^2 \\ g(T,x,y) &= x \end{aligned}$$

One Possible Solution

for

$$\gamma(x) = \frac{\gamma}{x}$$

we show that

- $\hat{u}(t,x) = c(t)x$ is a solution to the PDE system
- $c(t) = \frac{\beta}{\gamma\sigma^2}\left[\frac{a(t)}{b(t)} + \gamma\left(\frac{a^2(t)}{b(t)} - 1\right)\right]$ where

$$\begin{aligned} \beta &= \alpha - r \\ a(t) &= e^{\int_t^T [r+\beta c(s)]ds} \\ b(t) &= e^{2\int_t^T [r+\beta c(s)+\frac{1}{2}\sigma^2 c^2(s)]ds} \end{aligned}$$

- $V(t,x) = \left\{a(t) + \frac{\gamma}{2}[a^2(t) - b(t)]\right\} x$

Existence for $c(t)$

$$
\begin{aligned}
c_0(t) &= 1 \\
c_{n+1}(t) &= \frac{\beta}{\gamma\sigma^2}\left[e^{-\int_t^T [r+\beta c_n(s)+\sigma^2 c_n^2(s)]ds} + \gamma e^{-\int_t^T \sigma^2 c_n^2(s)ds} - \gamma\right], \\
& \quad n = 0, 1, 2...
\end{aligned}
$$

- **Step 1.** $\{c_n(\cdot)\}$ uniformly bounded in $\mathcal{C}([0, T])$
- **Step 2.** $\{\dot{c}_n(\cdot)\}$ uniformly bounded in $\mathcal{C}([0, T])$.
- **Step 3.** For any $t_1,\ t_2 \in [0, T]$, we have

$$
\begin{aligned}
|c_n(t_2) - c_n(t_1)| &= \int_0^1 \dot{c}_n(t_1 + \theta(t_2 - t_1))d\theta(t_2 - t_1)| \\
&\leq k|t_2 - t_1| \quad \forall n
\end{aligned}
$$

where k is a constant independent of n.

Existence for $c(t)$

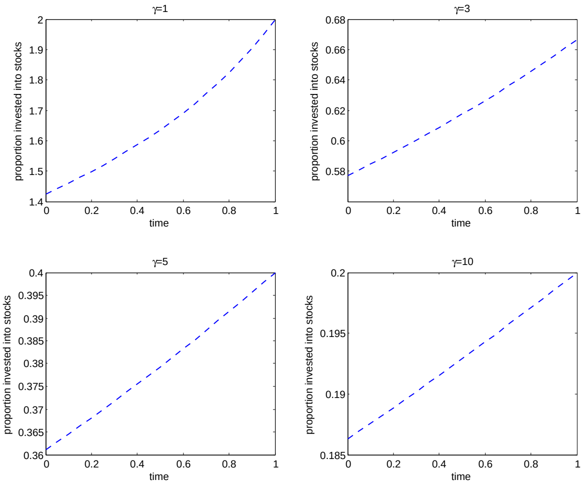
$$
\begin{aligned}
c_0(t) &= 1 \\
c_{n+1}(t) &= \frac{\beta}{\gamma\sigma^2}\left[e^{-\int_t^T [r+\beta c_n(s)+\sigma^2 c_n^2(s)]ds} + \gamma e^{-\int_t^T \sigma^2 c_n^2(s)ds} - \gamma\right], \\
& n = 0, 1, 2\ldots
\end{aligned}
$$

Step 1+2+3 $\Rightarrow$ there is a $c(\cdot) \in \mathcal{C}([0, T]$ such that

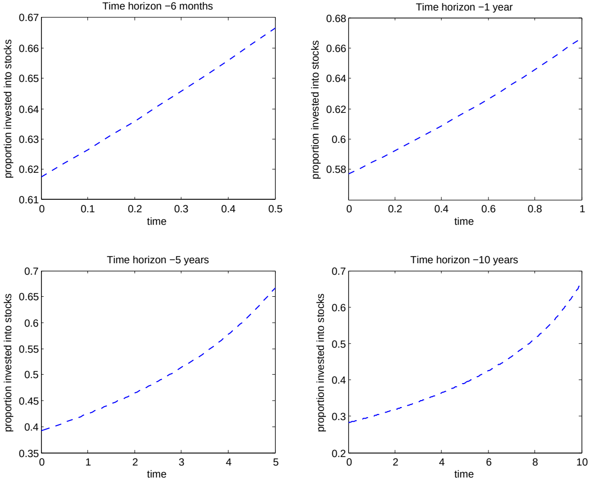
$$
c_{n_i}(\cdot) \overset{n\to\infty}{\rightrightarrows} c(\cdot) \in \mathcal{C}([0, T])
$$

Uniqueness can be proved easily

Proportion of Money invested in the Stock for Various γ



Proportion of Money invested in the Stock for Various Time Horizons



THANK YOU!