Forward Indifference Valuation of American Options

Tim S.T. Leung

Department of Applied Math & Statistics
Johns Hopkins University

6th World Congress of the Bachelier Finance Society, Toronto

June 23, 2010

The Merton Portfolio Optimization Problem

- On $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, consider two liquidly traded assets: a stock S & money market a.c. B with +ve interest rate $(r_t)_{t\geq 0}$.
- With initial wealth $x \in \mathbb{R}$, the investor dynamically rebalances his portfolio allocations in S and B. The discounted wealth is

$$X_t^\pi = x + \int_0^t \frac{\pi_u}{S_u}\, dS_u,$$

  where $(\pi_t)_{t\geq 0}$ is the amount invested in S.
- The *classical* Merton portfolio optimization:
  (i) investor's risk preference is modeled by a deterministic utility function $\hat{U}(x)$ defined at a fixed terminal time T;
  (ii) with wealth X_t, the Merton value function is

$$M_t(X_t) = \operatorname*{ess\,sup}_{\pi\in\mathcal{Z}_{t,T}} \mathbb{E}\left\{ \hat{U}(X_T^\pi) \middle| \mathcal{F}_t \right\}, \quad 0 \leq t \leq T.$$

- All $\hat{U}$, M, and optimal strategy $\hat{\pi}^*$ depend on T.

A Property of the Merton Value Process

- Goals: (i) specify the investor's utility $u_0(x)$ at time 0 (not T); (ii) utility evolves stochastically and consistently over time.
- Observation 1: M acts as the intermediate utility at time $t \leq T$.
- Observation 2: if the dynamic programming principle holds:

$$M_t(X_t) = \operatorname*{ess\,sup}_{\pi \in \mathcal{Z}_{t,s}} \mathbb{E}\left\{ M_s(X_s^\pi) \middle| \mathcal{F}_t \right\}, \qquad 0 \leq t \leq s \leq T,$$

  then $(M_t(X_t^\pi))_{0 \leq t \leq T}$ is a supermartingale for any admissible strategy π, and it is a martingale under some strategy $\hat{\pi}^*$.
- With Markovian prices, the optimal portfolio allocation can be found by solving the Hamilton-Jacobi-Bellman PDE (Merton ('69) and many others).
- Exponential Utility: DPP holds in semimartingale market (Mania-Schweizer '05); duality in terms of entropy minimization (Fritelli '00, Delbaen et al '02).

Forward Investment Performance Measurement

Definition

An $\mathcal{F}_t$-adapted process $(U_t(x))_{t\geq 0}$ is a forward performance process if:

1. $U_0(x) = u_0(x)$, *for* $x \in I\!R$, *where* $u_0 : I\!R \mapsto I\!R$ *is increasing and concave,*
2. *for each* $t \geq 0$, $x \mapsto U_t(x)$ *is increasing and concave in* x,
3. *for* $0 \leq t \leq s < \infty$, *we have*

$$U_t(X_t) = \operatorname*{ess\,sup}_{\pi \in \mathcal{Z}_{t,s}} I\!E\{U_s(X_s^\pi) | \mathcal{F}_t\}, \quad X_t \in \mathcal{F}_t. \tag{1}$$

- First introduced by Musiela-Zariphopoulou '08.
- (1) is called the horizon-unbiased cond'n in Henderson-Hobson '07, or the self-generating cond'n in Zitkovic '09.
- $(U_t(X_t^\pi))_{t\geq 0}$ is a $(\mathbb{P}, \mathcal{F}_t)$ supermartingale for any strategy π, and a martingale for some π^* (if it exists).

Forward Performance Indifference Valuation

- An investor holds an American option with a $\mathcal{F}_t$-adapted bounded payoff process $(g_t)_{0\leq t\leq T}$.
- The holder's value process at time $t \in [0,T]$ with wealth X_t is

$$V_t(X_t) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\pi\in\mathcal{Z}_{t,\tau}} \mathbb{E}\left\{U_\tau(X_\tau^\pi + g_\tau)\,|\,\mathcal{F}_t\right\}.$$

- The holder's forward indifference price process $(p_t)_{0\leq t\leq T}$ for the American option is defined by the equation

$$V_t(X_t) = U_t(X_t + p_t), \qquad t\in[0,T].$$

- Compare with the classical case:

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\pi\in\mathcal{Z}_{t,\tau}} \mathbb{E}\left\{M_\tau(X_\tau^\pi + g_\tau)\,|\,\mathcal{F}_t\right\},$$

  which corresponds to specifying that option proceeds received at exercise time τ are re-invested in the Merton portfolio up till time T.
- In contrast, the forward performance process U specifies utilities at all times, without reference to any specific horizon.

Construction of a Forward Performance

Let's model the discounted stock price as a continuous Itô process:

$$dS_t = S_t \sigma_t \left(\lambda_t \, dt + \, dW_t \right).$$

Theorem

Define the stochastic process $A_t = \int_0^t \lambda_s^2 \, ds$, $t \geq 0$. *Let the function* $u : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}$ *be* $\mathcal{C}^{3,1}$, *strictly concave and increasing in the spatial argument. Also, assume that it satisfies*

$$u_t = \frac{1}{2} \frac{u_x^2}{u_{xx}} \, ,$$

with initial condition $u(x, 0) = u_0(x)$, *where* $u_0 \in \mathcal{C}^3(\mathbb{R})$. *Then,*

$$U_t(x) = u\left(x, A_t\right), \qquad t \geq 0,$$

defines a forward performance process. Moreover, the optimal π^* *is*

$$\pi_t^* = -\frac{\lambda_t}{\sigma_t} \frac{u_x(X_t^{\pi^*}, A_t)}{u_{xx}(X_t^{\pi^*}, A_t)}.$$

Example: American Options with Stochastic Volatility

- The disc. stock price follows

$$dS_t = \mu(Y_t)S_t\,dt + \sigma(Y_t)S_t\,dW_t.$$

The drift and volatility coefficients $\mu(Y_t)$ and $\sigma(Y_t)$ are driven by a non-traded stochastic factor Y which evolves according to

$$dY_t = b(Y_t)\,dt + c(Y_t)\,(\rho dW_t + \sqrt{1-\rho^2}d\hat{W}_t),$$

with correlation coefficient $\rho \in (-1, 1)$.

- Consider the exponential risk preference function $u(x,t)$:

$$u(x,t) = -e^{-\gamma x + \frac{t}{2}},$$

with local risk aversion $\gamma > 0$.

- The exponential forward performance process is given by:

$$U_t^e(x) = -e^{-\gamma x + \frac{1}{2}\int_0^t \lambda(Y_s)^2 ds},$$

where $\lambda(y) = \mu(y)/\sigma(y)$.

The Holder's Forward Indifference Price

- The American option has a bounded and smooth payoff function $g(s,y,t)$.
- Non-tradability of Y renders the market incomplete.
- The holder's maximal expected forward performance is

$$V_t^e(X_t) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\pi \in \mathcal{Z}_{t,\tau}} I\!E \left\{ -e^{-\gamma(X_\tau^\pi + g(S_\tau, Y_\tau, \tau))} e^{\frac{1}{2}\int_0^\tau \lambda(Y_s)^2 ds} \,|\, \mathcal{F}_t \right\}$$
$$= e^{\frac{1}{2}\int_0^t \lambda(Y_s)^2 ds} V(X_t, S_t, Y_t, t),$$

where

$$V(x,s,y,t)$$
$$= \sup_{\substack{\tau \in \mathcal{T}_{t,T} \\ \pi \in \mathcal{Z}_{t,\tau}}} I\!E \left\{ -e^{-\gamma(X_\tau^\pi + g(S_\tau, Y_\tau, \tau))} e^{\frac{1}{2}\int_t^\tau \lambda(Y_s)^2 ds} \,|\, X_t = x, S_t = s, Y_t = y \right\}.$$

The HJB Variational Inequality

We write down the associated HJB variational inequality for V:

$$\begin{cases} V_t + \mathcal{L}_{SY}V + \mathcal{H}(V_{xx}, V_{xy}, V_{xs}, V_x) + \dfrac{\lambda(y)^2}{2}V \leq 0, \\ V(x,s,y,t) \geq -e^{-\gamma(x+g(s,y,t))}, \\ \Big(V_t + \mathcal{L}_{SY}V + \mathcal{H}(V_{xx}, V_{xy}, V_{xs}, V_x) + \dfrac{\lambda(y)^2}{2}V\Big)\cdot\Big(-e^{-\gamma(x+g)} - V\Big) = 0, \\ V(x,s,y,T) = -e^{-\gamma(x+g(s,y,T))}, \end{cases}$$

for $(x,s,y,t) \in \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times[0,T]$, where

$$\mathcal{L}_{SY}v = \frac{1}{2}\sigma(y)^2s^2v_{ss} + \rho c(y)\sigma(y)sv_{sy} + \frac{1}{2}c(y)^2v_{yy} + \lambda(y)\sigma(y)sv_s + b(y)v_y$$

is the infinitesimal generator of $(S_t, Y_t)_{t\geq 0}$ under $\mathbb{P}$, and

$$\mathcal{H}(v_{xx}, v_{xy}, v_{xs}, v_x) = \max_{\pi}\left(\frac{\pi^2\sigma(y)^2}{2}v_{xx} + \pi\left(\rho\sigma(y)c(y)v_{xy} + \sigma(y)^2sv_{xs} + \lambda(y)\sigma(y)v_x\right)\right).$$

The Forward Indifference Price

Then, the transformation $V(x,s,y,t) = -e^{-\gamma(x+p(s,y,t))}$ yields

$$\begin{cases} p_t + \mathcal{L}^0_{SY}p - \dfrac{1}{2}\gamma(1-\rho^2)c(y)^2p_y^2 \leq 0, \\ p(s,y,t) \geq g(s,y,t), \\ \left(p_t + \mathcal{L}^0_{SY}p - \dfrac{1}{2}\gamma(1-\rho^2)c(y)^2p_y^2\right) \cdot (g(s,y,t) - p(s,y,t)) = 0, \\ p(s,y,T) = g(s,y,T), \end{cases}$$

where $\mathcal{L}^0_{SY}v = \mathcal{L}_{SY}v - \rho c(y)\lambda(y)v_y - \lambda(y)\sigma(y)sv_s + \frac{1}{2}\sigma(y)^2s^2v_{ss} + \rho c(y)\sigma(y)sv_{sy} + \frac{1}{2}c(y)^2v_{yy} + (b(y) - \rho c(y)\lambda(y))v_y$.

- Note that $p(s,y,t)$ is the exponential forward indifference price and it is wealth independent.
- The optimal hedging strategy $\tilde{\pi}^*$ and exercise time τ_t^* are

$$\tilde{\pi}_t^* = \frac{\lambda(Y_t)}{\gamma\sigma(Y_t)} + \frac{S_t}{\gamma}p_s(S_t,Y_t,t) + \frac{\rho c(Y_t)}{\gamma\sigma(Y_t)}p_y(S_t,Y_t,t),$$
$$\tau_t^* = \inf\{t \leq u \leq T \,:\, p(S_u,Y_u,u) = g(S_u,Y_u,u)\}\,.$$

Dual Representation

- First, we define the set of equivalent local martingale measures $\mathcal{M}_f$. Define the local martingale $(Z_t^\phi)_{0\le t\le T}$ by

$$Z_t^\phi = \exp\left(-\frac{1}{2}\int_0^t \lambda(Y_s)^2 + \phi_s^2\,ds - \int_0^t \lambda(Y_s)\,dW_s - \int_0^t \phi_s\,d\hat{W}_s\right),$$

  where $(\phi_t)_{0\le t\le T}$ is an $\mathcal{F}_t$-adapted process such that $I\!E^{Q^\phi}\left\{\int_0^T \phi_t^2 dt\right\} < \infty$ and $I\!E\{Z_T^\phi\} = 1$. Then, a probability measure Q^ϕ defined by $\frac{dQ^\phi}{d\mathbb{P}} = Z_T^\phi$ is an ELMM w.r.t. $\mathbb{P}$ on $\mathcal{F}_T$.

- By Girsanov's Theorem, Q^ϕ, and $W_t^\phi = W_t + \int_0^t \lambda(Y_s)ds$ and $\hat{W}_t^\phi = \hat{W}_t + \int_0^t \phi_s\,ds$ are independent Q^ϕ-Brownian motions.

- The process ϕ is the risk premium for $\hat{W}$. When $\phi = 0$, we obtain the minimal martingale measure Q^0.

Forward Indifference Price via Entropy Minimization

- Treat Q^0 as the prior measure, and denote $Z_t^{\phi,0} = I\!E_t^{Q^0}\{\frac{dQ^\phi}{dQ^0}\}$.
- The conditional relative entropy $H_t^\tau(Q^\phi|Q^0)$ of Q^ϕ w.r.t. Q^0 over the interval $[t,\tau]$ as

$$H_t^\tau(Q^\phi|Q^0) = I\!E^{Q^\phi}\left\{\log \frac{Z_\tau^{\phi,0}}{Z_t^{\phi,0}}|\mathcal{F}_t\right\} = \frac{1}{2} I\!E^{Q^\phi}\left\{\int_t^\tau \phi_s^2\, ds|\mathcal{F}_t\right\}.$$

Proposition

The exponential forward indifference price can be represented as

$$p(S_t,Y_t,t) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{Q^\phi\in\mathcal{M}_f}\left(I\!E^{Q^\phi}\{g(S_\tau,Y_\tau,\tau)|\mathcal{F}_t\} + \frac{1}{\gamma}H_t^\tau(Q^\phi|Q^0)\right),$$

with the optimal risk premium $\phi_t^* = -\gamma c(Y_t)\sqrt{1-\rho^2}\, p_y(S_t,Y_t,t)$.

In the classical case, the entropy term is computed w.r.t the minimal entropy martingale measure Q^E, instead of Q^0.

Properties of the Forward Indifference Price

The dual representation allows us to deduce the following properties:

- If $\gamma_2 \geq \gamma_1 > 0$, then $p(s,y,t;\gamma_2) \leq p(s,y,t;\gamma_1)$ and $\tau^*(\gamma_2) \leq \tau^*(\gamma_1)$ almost surely.
- As γ increases to infinity, the penalty term vanishes, yielding

$$\lim_{\gamma\to\infty} p(s,y,t;\gamma) = \sup_{\tau\in\mathcal{T}_{t,T}} \inf_{Q^\phi\in\mathcal{M}_f} I\!E^{Q^\phi}\left\{ g(S_\tau, Y_\tau, \tau) \middle| S_t = s, Y_t = y \right\}.$$

  which is typically called the sub-hedging price (Karatzas-Kou '98).

- As $\gamma \downarrow 0$, it is optimal not to deviate from Q^0 (i.e. $\phi = 0$):

$$\lim_{\gamma\to 0} p(s,y,t;\gamma) = \sup_{\tau\in\mathcal{T}_{t,T}} I\!E^{Q^0}\left\{ g(S_\tau, Y_\tau, \tau) \middle| S_t = s, Y_t = y \right\}.$$

- In the classical expo. utility case, the zero risk-aversion limit leads to pricing under Q^E (Davis Price), not Q^0.

The Classical Marginal Utility Price

- The marginal utility price is the per-unit price that the investor is willing to pay for an infinitesimal position ($\delta \approx 0$) in the claim (see Davis '97, Kramkov-Sirbu '06):

$$\hat{h}_t = \frac{\mathbb{E}\left\{\hat{U}'(\hat{X}^*_T)\, C_T \,|\, \mathcal{F}_t\right\}}{M'_t(X_t)}, \qquad t \in [0,T],$$

where $\hat{X}^*_T$ is the optimal Merton portfolio wealth.

- We adapt this definition to the case with an American option:

$$h_t = \frac{\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left\{M'_\tau(\hat{X}^*_\tau)\, g_\tau \,|\, \mathcal{F}_t\right\}}{M'_t(X_t)}.$$

Marginal Utility Price

Proposition

In the stochastic vol. model, consider the Merton value function

$$M(x,y,t) = \sup_{\pi \in \mathcal{Z}_{t,T}} I\!E\left\{\hat{U}(X_T^\pi) \,|\, X_t = x, Y_t = y\right\}. \qquad (2)$$

If M *satisfies*

$$M_{xy}(x,y,t) = M_x(x,y,t)\, L(y,t), \qquad (3)$$

where $L : I\!R_+ \times [0,T] \mapsto I\!R$ *is a* C^1 *function such that the risk premium* $\varphi(y,t) = \sqrt{1-\rho^2}c(y)L(y,t)$, *defines an ELMM* Q^φ.
Then, the marginal utility price for the American option g *is*

$$h(s,y,t) = \sup_{\tau \in \mathcal{T}_{t,T}} I\!E^{Q^\varphi}\left\{g(S_\tau, Y_\tau, \tau) \,|\, S_t = s, Y_t = y\right\},$$

Note that $h(s,y,t)$ is wealth-independent, but depends on the choice of $\hat{U}$ (via L). When $\hat{U}(x) = -e^{-\gamma x}$, $Q^\varphi = Q^E$ (MEMM).

Marginal Forward Indifference Price

- Let the discounted stock price be a continuous Itô process:

$$dS_t = S_t \sigma_t \left(\lambda_t \, dt + \, dW_t \right).$$

- Let $U_t(x) = u(x, A_t)$ be the investor's forward performance process.
- The marginal forward indifference price process $(\tilde{p}_t)_{0 \le t \le T}$ for an American option g is defined as

$$\tilde{p}_t = \frac{\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E} \left\{ u_x \left(X_\tau^{\pi^*}, A_\tau \right) g_\tau \middle| \mathcal{F}_t \right\}}{u_x(X_t, A_t)},$$

where $A_t = \int_0^t \lambda_s^2 ds$.

- As it turns out, the marginal forward indifference price is given by

$$\tilde{p}_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{Q^0} \{ g_\tau \, | \, \mathcal{F}_t \},$$

where Q^0 is the minimal martingale measure ($\phi = 0$).

- Consequently (and surprisingly), $\tilde{p}_t$ is independent of both the holder's wealth and the choice of u.

Concluding Remarks

- Forward investment performance is applicable to pricing American options.
- Exponential forward performance yields a dual representation that involves relative entropy minimization.
- The MMM Q^0 also acts as the pricing measure for the marginal forward indifference price, which is *wealth-independent and risk-preference independent*.

Other Applications

- Other specifications of forward performance: alternative solution to the PDE $2u_t = (u_x^2/u_{xx})$.
- Application to (early exercisable) ESO valuation – optimal exercise timing under forward performance.