Optimal Stopping for Non-linear Expectations

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A joint work with
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Outline

BSDEs and g-Expectations

Given a $B.M.$ $\mathbf{B}$ on a proba. space $(\Omega, \mathcal{F}, P)$, consider the Backward SDE:

$$Y_t = \xi + \int_t^T g(t, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad t \in [0, T]. \tag{1}$$

g-expectation via BSDE (Peng '93)

- Assume that the generator g is Lipschitz in (y, z). For any $\xi \in L^2(\mathcal{F}_T^B)$, (1) admits a unique solution (Y^ξ, Z^ξ).
- The solution mapping $\mathcal{E}_g : \xi \mapsto Y_0^\xi$ is called a *g-expectation*; And $\forall\, t \in [0, T]$, the *conditional g-expectation* of ξ w.r.t. $\mathcal{F}_t^B$ is defined by $\mathcal{E}_g[\xi|\mathcal{F}_t^B] \stackrel{\triangle}{=} Y_t^\xi$.

Note:

If g has quadratic growth in z, then one can define (condtional) g-expectation over $L^\infty(\mathcal{F}_T^B)$.

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Properties of g-Expectations

Assume that $g|_{z=0} = 0$ in (2)-(4) below. For any $\xi, \eta \in L^2(\mathcal{F}_T^B)$

1. **(Strict) Monotonicity:** If $\xi \leq \eta$, then $\mathcal{E}_g\left[\xi|\mathcal{F}_t^B\right] \leq \mathcal{E}_g\left[\eta|\mathcal{F}_t^B\right]$, $\forall\, t \in [0, T]$; Moreover, if "$=$" holds for some t, then $\xi = \eta$;
2. **Constant-preserving:** $\mathcal{E}_g\left[\xi|\mathcal{F}_t^B\right] = \xi$, if $\xi \in L^2(\mathcal{F}_t^B)$;
3. **Time-consistency:** $\mathcal{E}_g\left[\mathcal{E}_g\left[\xi|\mathcal{F}_t^B\right]\big|\mathcal{F}_s\right] = \mathcal{E}_g\left[\xi|\mathcal{F}_{t\wedge s}^B\right]$;
4. **"Zero-one law":** $\mathcal{E}_g[\mathbf{1}_A\xi|\mathcal{F}_t^B] = \mathbf{1}_A\mathcal{E}_g\left[\xi|\mathcal{F}_t^B\right], \ \forall\, A \in \mathcal{F}_t^B$;
5. **Translation invariance:** If g is independent of y, then
$$\mathcal{E}_g[\xi + \eta|\mathcal{F}_t^B] = \mathcal{E}_g\left[\xi|\mathcal{F}_t^B\right] + \eta, \quad \text{if } \eta \in L^2(\mathcal{F}_t^B).$$

Motivation: Optimal Stopping for g-expectations

Given a stopping time ν and an appropriate reward process Y, we are interested in finding a moment $\tau_*(\nu) \in \mathcal{S}^B_{\nu,T}$ such that

$$\mathcal{E}_g\left[Y_{\tau_*(\nu)} \middle| \mathcal{F}_\nu\right] = \operatorname*{esssup}_{\gamma \in \mathcal{S}^B_{\nu,T}} \mathcal{E}_g\left[Y_\gamma \middle| \mathcal{F}_\nu\right],$$

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Filtration-Consistent Nonlinear Expectations

- $\mathbf{F} = \{\mathcal{F}_t\}_{t \ge 0}$ — a generic right-continuous filtration on $(\Omega, \mathcal{F}, P)$;
- $\mathcal{S}_{\nu,\gamma} \overset{\triangle}{=} \{\mathbf{F}\text{-stopping times } \sigma : \nu \le \sigma \le \gamma\}$.

An **F**-consistent non-linear expectation (**F**-expectation for short) with domain $Dom(\mathcal{E}) = \Lambda \subset L^0(\mathcal{F}_T)$ is a family of operators $\{\mathcal{E}[\cdot|\mathcal{F}_\nu] : \Lambda \mapsto \Lambda_\nu \overset{\triangle}{=} \Lambda \cap L^0(\mathcal{F}_\nu)\}_{\nu \in \mathcal{S}_{0,T}}$ such that for any $\forall\, \xi, \eta \in \Lambda$

(A1) **(Strict) Monotonicity:** If $\xi \le \eta$, then $\mathcal{E}[\xi|\mathcal{F}_\nu] \le \mathcal{E}[\eta|\mathcal{F}_\nu]$, $\forall\, \nu \in \mathcal{S}_{0,T}$; Moreover, if "=" holds for some $\nu \in \mathcal{S}_{0,T}$, then $\xi = \eta$;

(A2) **Time Consistency:** $\mathcal{E}\big[\mathcal{E}[\xi|\mathcal{F}_\nu]\big|\mathcal{F}_\gamma\big] = \mathcal{E}[\xi|\mathcal{F}_{\nu\wedge\gamma}]$, $\forall\, \gamma \in \mathcal{S}_{0,T}$;

(A3) **"Zero-one Law":** $\mathcal{E}[\mathbf{1}_A \xi|\mathcal{F}_\nu] = \mathbf{1}_A \mathcal{E}[\xi|\mathcal{F}_\nu]$, $\forall\, A \in \mathcal{F}_\nu$;

(A4) **Translation Invariance :** $\mathcal{E}[\xi + \eta|\mathcal{F}_\nu] = \mathcal{E}[\xi|\mathcal{F}_\nu] + \eta$, if $\eta \in \Lambda_\nu$.

Note: (A3)+(A4) $\Longrightarrow$ "Constant-preserving".

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Algebraic requirements on domain $Dom(\mathcal{E}) = \Lambda$

Obviously, (A3) and (A4) entail that

- For any $\xi, \eta \in \Lambda$ and $A \in \mathcal{F}_T$, both $\xi + \eta$ and $\mathbf{1}_A \xi$ belong to Λ,

which implies that $\xi \vee \eta = \xi \mathbf{1}_{\{\xi > \eta\}} + \eta \mathbf{1}_{\{\xi \le \eta\}} \in \Lambda$, similarly, $\xi \wedge \eta \in \Lambda$;

Moreover, we assume that $\mathbb{R} \subset \Lambda$ and that

- Λ is positively solid : For any $\xi, \eta \in L^0(\mathcal{F}_T)$ with $0 \le \xi \le \eta$, if $\eta \in \Lambda$, then $\xi \in \Lambda$ as well.

Example

$L^p(\mathcal{F}_T)$, $0 \le p \le \infty$, are candidates for Λ described above.

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Assumptions on **F**-expectations

To extend Fatou's lemma, the Dominated Convergence Theorem and etc. to the **F**-expectation $\mathcal{E}$, we assume

(H0) For any $A \in \mathcal{F}_T$ with $P(A) > 0$, one has $\lim_{n\to\infty} \mathcal{E}[n\mathbf{1}_A] = \infty$;

(H1) For any $\xi \in Dom^+(\mathcal{E})$ and any $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_T$ with $\lim_{n\to\infty} \uparrow \mathbf{1}_{A_n} = 1$, one has $\lim_{n\to\infty} \uparrow \mathcal{E}[\mathbf{1}_{A_n}\xi] = \mathcal{E}[\xi]$;

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(H0)-(H2) are satisfied by the linear expectation E, Lipschitz and quadratic g-expectations.

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Basic Properties

Fatou's Lemma

Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence in $Dom^+(\mathcal{E})$ that converges a.s. to some $\xi \in Dom^+(\mathcal{E})$, then for any $\nu \in \mathcal{S}_{0,T}$

$$\mathcal{E}[\xi|\mathcal{F}_\nu] \le \varliminf_{n\to\infty} \mathcal{E}[\xi_n|\mathcal{F}_\nu].$$

Dominated Convergence Theorem

Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence in $Dom^+(\mathcal{E})$ that converges a.s. to some ξ. If there is an $\eta \in Dom^+(\mathcal{E})$ such that $\xi_n \le \eta$, $\forall\, n \in \mathbb{N}$, then $\xi \in Dom^+(\mathcal{E})$ and for any $\nu \in \mathcal{S}_{0,T}$

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- An **F**-adapted process X is said to be an $\mathcal{E}$-supermartingale (resp. $\mathcal{E}$-submartingale) if for any $0 \le s < t \le T$, $X_t \in Dom(\mathcal{E})$ and $\mathcal{E}[X_t|\mathcal{F}_s] \le$ (resp. $\ge$) X_s.

Proposition

Let X be a non-negative $\mathcal{E}$-supermartingale.

(1) $P\Big(X_t^+ \stackrel{\triangle}{=} \lim_{r\in\mathbb{Q}, r\downarrow t} X_r$ exists $\forall\, t \in [0,T]\Big) = 1$. To wit, X^+ defines an RCLL F-adapted process.

(2) If $X_t^+ \in Dom^+(\mathcal{E})$, $\forall\, t \in [0,T]$, then X^+ is an $\mathcal{E}$-supermartingale such that $X_t^+ \le X_t$, $\forall\, t \in [0,T]$. Moreover, if the function $t \mapsto \mathcal{E}[X_t]$ is right continuous, then X^+ is an RCLL modification of X.

Optional Sampling Theorem

Let X be a non-negative right-continuous $\mathcal{E}$-supermartingale (resp. $\mathcal{E}$-submartingale). If $X_\nu \in Dom^+(\mathcal{E})$, $\forall\, \nu \in \mathcal{S}_{0,T}$, then

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Pasting two **F**-Expectations

- Let $\mathcal{E}_i, \mathcal{E}_j$ be two **F**-expectations with the same domain Λ and satisfying (H1), (H2).

For any $\nu \in \mathcal{S}_{0,T}$, the pasting of $\mathcal{E}_i, \mathcal{E}_j$ at ν is defined by the following RCLL **F**-adapted process

$$\mathcal{E}^{\nu}_{i,j}[\xi|\mathcal{F}_t] \stackrel{\triangle}{=} \mathbf{1}_{\{\nu \le t\}}\mathcal{E}_j[\xi|\mathcal{F}_t] + \mathbf{1}_{\{\nu > t\}}\mathcal{E}_i\big[\mathcal{E}_j[\xi|\mathcal{F}_\nu]\big|\mathcal{F}_t\big], \quad t \in [0,T]$$

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Note:

- For any $\xi \in \Lambda^+$ and $\sigma \in \mathcal{S}_{0,T}$, $\mathcal{E}^{\nu}_{i,j}[\xi|\mathcal{F}_\sigma] = \mathcal{E}_i\big[\mathcal{E}_j[\xi|\mathcal{F}_{\nu\vee\sigma}]\big|\mathcal{F}_\sigma\big]$.
- Pasting may not preserve (H0). But, positive-convexity implies (H0).

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Stable Classes

A class $\mathscr{E} = \{\mathcal{E}_i\}_{i\in\mathcal{I}}$ of **F**-expectations is said to be *stable* if

(1) All $\mathcal{E}_i$, $i \in \mathcal{I}$ are positively-convex **F**-expectations with the same domain Λ and satisfying (H1), (H2);

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- We shall denote $Dom(\mathscr{E}) \stackrel{\triangle}{=} \Lambda^+ = Dom^+(\mathcal{E}_i)$, $\forall\, i \in \mathcal{I}$.

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Optimal Stopping with Multiple Priors

The stopper aims to find an optimal moment in a situation of multiple priors and the *Nature* is in cooperation with the stopper. More precisely, the stopper finds an optimal stopping time τ^* that satisfies

$$\sup_{(i,\gamma)\in\mathcal{I}\times\mathcal{S}_{0,T}} \mathcal{E}_i\left[Y^i_\gamma\right] = \sup_{i\in\mathcal{I}} \mathcal{E}_i\left[Y^i_{\tau^*}\right], \qquad (2)$$

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- $\mathscr{E} = \{\mathcal{E}_i\}_{i\in\mathcal{I}}$ is a stable class of **F**-expectations,
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$\mathscr{E}$-upper Snell Envelope

- $\forall\, \nu \in \mathcal{S}_{0,T}$, we define $Z(\nu) \overset{\triangle}{=} \underset{(i,\gamma)\in\mathcal{I}\times\mathcal{S}_{\nu,T}}{\text{esssup}} \; \mathcal{E}_i\left[Y_\gamma + \int_\nu^\gamma h_t^i dt \middle| \mathcal{F}_\nu\right] \geq Y_\nu$.
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Given $i \in \mathcal{I}$, $\forall\, \nu, \gamma \in \mathcal{S}_{0,T}$ with $\nu \leq \gamma$, one has $\mathcal{E}_i[Z^i(\gamma)|\mathcal{F}_\nu] \leq Z^i(\nu)$, which shows that $\left\{Z^i(t)\right\}_{t\in[0,T]}$ is an $\mathcal{E}_i$-supermartingale. Moreover the process $\left\{Z(t)\right\}_{t\in[0,T]}$ admits an RCLL modification Z^0.

We call Z^0 the $\mathscr{E}$-upper Snell envelope of Y: It is the smallest RCLL $\mathbf{F}$-adapted process dominating Y such that $\left\{Z_t^0 + \int_0^t h_s^i ds\right\}_{t\in[0,T]}$ is an $\mathcal{E}_i$-supermartingale for any $i \in \mathcal{I}$.

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Constructing an Optimal Stopping Time

- Given $\nu \in \mathcal{S}_{0,T}$, the stopping time $\tau_\delta(\nu) \stackrel{\triangle}{=} \inf\left\{t \in [\nu, T] : \; Y_t \geq \delta Z^0_t\right\} \wedge T$ is increasing in $\delta \in (0,1)$.
- Set $\overline{\tau}(\nu) \stackrel{\triangle}{=} \lim\limits_{\delta \nearrow 1} \tau_\delta(\nu)$. Then $\overline{\tau}(0)$ is an optimal stopping time for (2).

O.S.P.

Definition

The family $\{Y^i\}_{i \in \mathcal{I}}$ is called $\mathscr{E}$-uniformly-left-continuous if $\forall\, \nu, \gamma \in \mathcal{S}_{0,T}$ with $\nu \leq \gamma$ and for any sequence $\{\gamma_n\}_{n \in \mathbb{N}} \subset \mathcal{S}_{\nu,T}$ with $\gamma_n \nearrow \gamma$

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Theorem

Assume that $\{Y^i\}_{i\in\mathcal{I}}$ is "$\mathscr{E}$-uniformly-left-continuous".

- $\overline{\tau}(\nu) = \tau_1(\nu) \triangleq \inf\left\{t \in [\nu, T] : Z^0_t = Y_t\right\}$.
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Theorem

Assume that $\{Y^i\}_{i\in\mathcal{I}}$ is "$\mathscr{E}$-uniformly-left-continuous".

- $\overline{\tau}(\nu) = \tau_1(\nu) \overset{\triangle}{=} \inf\left\{t \in [\nu, T] :\, Z^0_t = Y_t\right\}$.
- *For any* $\nu \in \mathcal{S}_{0,T}$ *and* $\gamma \in \mathcal{S}_{\nu,\overline{\tau}(\nu)}$,

$$Z(\nu) = \underset{i\in\mathcal{I}}{\text{esssup}}\, \mathcal{E}_i\Big[Y_{\overline{\tau}(\nu)} + \int_\nu^{\overline{\tau}(\nu)} h^i_t dt \Big| \mathcal{F}_\nu\Big]$$
$$= \underset{i\in\mathcal{I}}{\text{esssup}}\, \mathcal{E}_i\Big[Z(\overline{\tau}(\nu)) + \int_\nu^{\overline{\tau}(\nu)} h^i_t dt \Big| \mathcal{F}_\nu\Big] = \underset{i\in\mathcal{I}}{\text{esssup}}\, \mathcal{E}_i\Big[Z(\gamma) + \int_\nu^{\gamma} h^i_t dt \Big| \mathcal{F}_\nu\Big].$$

In particular, when $\nu = 0$, $\overline{\tau}(0) = \inf\left\{t \in [0, T] :\, Z^0_t = Y_t\right\}$ satisfies

$$\sup_{(i,\gamma)\in\mathcal{I}\times\mathcal{S}_{0,T}} \mathcal{E}_i\big[Y^i_\gamma\big] = Z(0) = \sup_{i\in\mathcal{I}} \mathcal{E}_i\big[Y^i_{\overline{\tau}(0)}\big].$$

Conclusion: $\overline{\tau}(0)$, the first time the Snell envelope Z^0 meets Y after time $t = 0$, is an optimal stopping time for (2).