Lectures on the Foundations of Mathematical Finance

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The aim of these lectures is to give an introduction to the mathematical foundations of finance, rather than to mathematical finance per se. The reader is assumed to know the basics of stochastic differential equations and mathematical finance (at the level of Shreve's textbooks [5], [6]).

1 No-arbitrage for Finite Probability Spaces

The notion of *arbitrage* will be one of the main themes of the course.

We will start the course by examining models based on finite probability spaces with discrete time. By studying this toy model we can introduce the necessary ideas and language of functional analysis at a relatively non-technical level. Later we will study models based on arbitrary probability spaces with continuous time.

In this lecture we shall follow sections 2.1–2 of Delbaen and Schachermayer [2] very closely.

1.1 Finite Model of a Financial Market ([2], §2.1)

The financial market model that we shall consider here is based on a finite probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = \omega_1, \ldots, \omega_N$ is a finite set, P is a probability measure with $P(\omega) > 0$ for all $\omega \in \Omega$, and $\mathcal{F}$ is the σ-algebra of all subsets of Ω. In addition, we consider a filtration $\mathcal{F}_0 \subset \cdots \subset \mathcal{F}_T = \mathcal{F}$ of sub-σ-algebras of $\mathcal{F}$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Definition 1.1.1. *Asset prices* are given by an $\mathbb{R}^{d+1}$–valued adapted process $\hat{S} = (\hat{S}_t^0, \ldots, \hat{S}_t^d)_{t=0}^T$. We will assume that $\hat{S}_0^0 = 1$ and $\hat{S}_t^0 > 0$ for all $t = 1, \cdots, T$.

The requirement of being an adapted process, that is $\hat{S}_t$ is $\mathcal{F}_t$-measurable, simply means that the prices at t are *known* at time t, despite being uncertain at any earlier time. The first component S_t^0 will play the role of a *numeraire*, which is a fancy way to call the units used to express the value of the other assets. In the simplest case we have $S_t^0 \equiv 1$ for all t, so we can think of it as a fixed currency amount, say one Canadian dollar. More generally, S_t^0 can represent the value of a bank account accumulating interest as time goes by.

Definition 1.1.2. A *trading strategy* is an $\mathbb{R}^{d+1}$–valued predictable process $\hat{H} = (\hat{H}_t^0, \ldots, \hat{H}_t^d)_{t=1}^T$.

The components of a trading strategy H_t are the number of units of an asset being held from time $t-1$ until time t. The holding decision is made at $t-1$, which explains why H_t needs to be predictable, that is $\hat{H}_t$ is $\mathcal{F}_{t-1}$-measurable.

Given an asset $\hat{S}_t$ and a trading strategy $\hat{H}_t$, the inner product

$$\hat{V}_t = \hat{H}_t \hat{S}_t := \sum_{i=0}^{d} \hat{H}_t^i \hat{S}_t^i \tag{1}$$

is called the *portfolio value* at time t.

Definition 1.1.3. The trading strategy $\hat{H}$ is said to be *self–financing* if

$$\hat{H}_{t+1}\hat{S}_t = \hat{H}_t\hat{S}_t, \tag{2}$$

for every $t = 1, \ldots, T-1$.

The left–hand side of (2) corresponds the amount of money necessary to form a portfolio to be held from time t until time $t+1$ at the market prices prevailing at time t, whereas the right–hand side is the amount of money obtained from the portfolio held from time $t-1$ until time t. In other words, a trading strategy is self–financing provided there is no injection or withdraw of funds at any given time.

Given asset prices $\hat{S}_t = (\hat{S}_t^0, \ldots, \hat{S}_t^d)$, let $S_t = (S_t^1, \ldots, S_t^d)$ be the $\mathbb{R}^d$–valued adapted process of *discounted* prices with components

$$S_t^j = \frac{\hat{S}_t^j}{\hat{S}_t^0}, \quad j = 1, \ldots, d.$$

Accordingly, given an $\mathbb{R}^{d+1}$–valued trading strategy $\hat{H}_t = (\hat{H}_t^0, \ldots, \hat{H}_t^d)$, let $H_t = (H_t^1, \ldots, H_t^d)$ be the $\mathbb{R}^d$-valued trading strategy with components

$$H_t^j = \hat{H}_t^j, \quad j = 1, \ldots, d.$$

Notice that, for every predictable $\mathbb{R}^d$–valued process H_t we can construct a unique $\mathbb{R}^{d+1}$–valued self–financing trading strategy $\hat{H}_t$ such that $\hat{H}_t^j = H_t^j$ for all $j = 1, \ldots, d$ and all $t = 1, \ldots, T$ simply by setting $H_1^0 = 0$ and finding H_t^0 inductively for $j = 2, \ldots, T$ using (2). Together with the next proposition, this construction shows that, as long as we are interested in discounted portfolio values only, there is no loss of information when we consider the restricted $\mathbb{R}^d$–valued strategy H_t instead of the $\mathbb{R}^{d+1}$–value strategy $\hat{H}_t$.

Proposition 1.1.4. Let $\hat{H}_t$ be the unique $\mathbb{R}^{d+1}$–value *self–financing* trading strategy associated with an arbitrary $\mathbb{R}^d$–value strategy H_t through the construction above. Then the discounted portfolio value $V_t = \hat{V}_t/S_t^0$ is independent of the scalar process $\hat{H}_t^0$.

Proof. Since $\hat{S}_0^0 = 1$ and $H_1^0 = 0$, we have that

$$V_0 = \hat{V}_0 = \hat{H}_1^1\hat{S}_1^1 + \cdots + \hat{H}_1^d\hat{S}_1^d.$$

Using the self-financing condition $\hat{H}_t\hat{S}_t = \hat{H}_{t+1}\hat{S}_t$, we find that the change in V_t is

$$\begin{aligned}
\Delta V_{t+1} &:= V_{t+1} - V_t \\
&= \hat{V}_{t+1}/\hat{S}_{t+1}^0 - \hat{V}_t/\hat{S}_t^0 \\
&= \hat{H}_{t+1}\hat{S}_{t+1}/\hat{S}_{t+1}^0 - \hat{H}_{t+1}\hat{S}_t/\hat{S}_t^0 \\
&= \hat{H}_{t+1}^0(1-1) + \sum_{j=1}^d \hat{H}_{t+1}^j(\hat{S}_{t+1}^j/\hat{S}_{t+1}^0 - \hat{S}_t^j/\hat{S}_t^0) \\
&= \sum_{j=1}^d H_{t+1}^j \Delta S_{t+1}^j \\
&= H_{t+1}\Delta S_{t+1}.
\end{aligned}$$

Thus $V_t = V_0 + H_1\Delta S_1 + \cdots + H_t\Delta S_t$, which is manifestly independent of $\hat{H}_t^0$. $\square$

In particular, using the following standard notation for stochastic integrals (see [3])

$$(H \cdot S)_T = (H \cdot S)_{t=1}^T := \sum_{t=1}^T H_t\Delta S_t$$

we have that

$$V_T = V_0 + (H \cdot S)_T. \tag{3}$$

1.2 No Arbitrage and the FTAP

Let $\mathcal{H}$ be the space of predictable $\mathbb{R}^d$-valued processes H_t for a financial market with discounted asset prices S_t. Let $L^0(\Omega, \mathcal{F}, P)$ denote the space of all measurable functions on Ω, which for finite sample spaces is canonically isomorphic to $\mathbb{R}^N$. Similarly, let $L^\infty(\Omega, \mathcal{F}, P) = \mathbb{R}^N$ denote the space of bounded measurable functions on Ω.

Definition 1.2.1. The subspace $K \subset L^0(\Omega, \mathcal{F}, P)$ defined by

$$K = \{(H \cdot S)_T \mid H \in \mathcal{H}\},$$

is called the set of *attainable claims at price* 0, whereas the convex cone $C \subset L^\infty(\Omega, \mathcal{F}, P)$ defined by

$$C = \{g \in L^\infty(\Omega, \mathcal{F}, P) \mid g \leq f \text{ for some } f \in K\}$$

is called the set of *super-replicable claims at price* 0. (We say "super-replicable" since such claims are dominated by attainable claims — the terminology is not ideal!) For $a \in \mathbb{R}$, the sets $K_a = a + K$ and $C_a = a + C$ are respectively the sets of *attainable* and *super–replicable claims at price* a.

Remark 1.2.2. Observe that we can write $C = K + L^0_-$. It then follows that C is a closed set, since K is closed (being a linear subspace of $\mathbb{R}^N$) and L^0_- is a closed polyhedral cone. This is one of the many instances where working with finite–dimensional vector spaces simplifies the analysis tremendously.

Definition 1.2.3. A financial market S satisfies the *no arbitrage condition (NA)* if

$$K \cap L^0_+(\Omega, \mathcal{F}, P) = \{0\}.$$

or equivalently

$$C \cap L^\infty_+(\Omega, \mathcal{F}, P) = \{0\}.$$

Because this is the central concept in this notes, it deserves further explanation. In view of (3), the set K consists of random variables that coincide with the discounted terminal values of self–financing trading strategies starting at zero initial value. On the other hand, L^0_+ is the set of non–negative vectors in $\mathbb{R}^N$. Thus, an arbitrage is a self–financing trading strategy starting with zero initial value and with terminal value given by a random variable that is non–negative and not identically equal to zero. That, an arbitrage is a strategy that starts at zero, never loses money, and has a strictly positive probability of making money.

Proposition 1.2.4. The condition (NA) implies that $C \cap (-C) = K$.

Proof. Clearly $K \subset C \cap (-C)$. For the reverse inclusion, consider an element $g \in C \cap (-C)$. It then follows from Remark 1.2.2 that we can write $g = f_1 - h_1 = f_2 + h_2$ for some elements $f_1, f_2 \in K$ and $h_1, h_2 \in L^0_+$. But then $f_1 - f_2 = h_1 + h_2$ is in $K \cap L^0_+$, which is 0 by (NA). Thus $h_1 = h_2 = 0$ and so $g \in K$. □

Definition 1.2.5. A probability measure Q on $(\Omega, \mathcal{F})$ is an *equivalent martingale measure* (EMM) for S if $Q \sim P$ (that is, $Q[\omega_n] > 0$ for all n) and S is a Q-martingale, that is

$$E_Q[S_{t+1} \mid \mathcal{F}_t] = S_t, \qquad t = 0, 1, \ldots, T-1.$$

The set of EMM's for S is denoted $\mathcal{M}^e(S)$.

Lemma 1.2.6. For a probability measure Q on $(\Omega, \mathcal{F})$, the following are equivalent:

(i) S is a Q–martingale.

(ii) $E_Q[f] = 0$ for all $f \in K$.

(iii) $E_Q[g] \leq 0$ for all $g \in C$.

Proof. We shall prove only that $(i) \Leftrightarrow (ii)$, since $(ii) \Leftrightarrow (iii)$ is obvious. First observe that for a Q–martingale S and a predictable trading strategy H_t we have

$$\begin{aligned} E_Q[H_t \Delta S_t | \mathcal{F}_{t-1}] &= E_Q[\sum_{j=1}^{d} H_t^j (S_t^j - S_{t-1}^j | \mathcal{F}_{t-1}] \\ &= \sum_{j=1}^{d} H_t^j E_Q[S_t^j - S_{t-1}^j | \mathcal{F}_{t-1}] = 0. \end{aligned}$$

But this shows that $(H \cdot S)_t$ is also a Q–martingale, since

$$E_Q[(H \cdot S)_t | \mathcal{F}_{t-1}] = E_Q[\sum_{s=1}^{t-1} H_s \Delta S_s + H_t \Delta S_t | \mathcal{F}_{t-1}] = (H \cdot S)_{t-1}.$$

In particular, $E_Q[(H \cdot S)_T] = (H \cdot S)_0 = 0$, which shows that (1) $\Rightarrow$ (2). Conversely, let A be an arbitrary $\mathcal{F}_{t-1}$–measurable set and consider the strategy $H(\omega, s) = \mathbf{1}_A(\omega)\mathbf{1}_{(t-1,t]}(s)$. Then $(H \cdot S)_T = \mathbf{1}_A(S_t - S_{t-1})$ and (ii) implies that

$$E_Q[\mathbf{1}_A(S_t - S_{t-1})] = 0$$

which is equivalent to

$$E_Q[S_t | \mathcal{F}_{t-1}] = S_{t-1},$$

which in turn means that S is a Q–martingale. □

We are now in a position to prove the following theorem, known as the Fundamental Theorem of Asset Pricing.

Theorem 1.2.7 (FTAP)**.** For a financial market modeled on a finite probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, the following are equivalent:

1. S satisfies (NA).
2. $\mathcal{M}^{\mathrm{e}}(S) \neq \emptyset$.

Proof. (2) $\Rightarrow$ (1) (easy part): Suppose $Q \in \mathcal{M}^{\mathrm{e}}(S)$. By Lemma 1.2.6, $E_Q[g] \leq 0$ for all $g \in C$. On the other hand, if there were a non-zero element $g \in C \cap L^\infty_+$, then we would have $E_Q[g] > 0$, since $Q \sim P$. So necessarily S must satisfy (NA).

(1) $\Rightarrow$ (2) (interesting part): By the condition (NA), $K \cap L^\infty_+ = \{0\}$, and so K and L^∞_+ are disjoint convex sets. Let $B = \{\sum_n \mu_n 1_{\omega_n} \mid \mu_n \geq 0,\ \sum_n \mu_n = 1\}$. Then $B \subset L^\infty_+$ is a convex compact set which is disjoint from K. Now, by the separating hyperplane theorem (take the proof in [4, Theorem V.4] and eliminate the use of Hahn-Banach), there is a linear functional $Q \in (L^\infty)^* = L^1$ separating B and K. This means that we can find numbers $\alpha < \beta$ such that

$$Q[f] \leq \alpha < \beta \leq Q[g], \quad \text{for all } f \in K,\, g \in B.$$

Since K is linear, we have $\alpha \geq 0$, and without loss of generality we can take it to be 0, which implies that $\beta > 0$. Let e_n be the n-th canonical basis vector of $\mathbb{R}^N$. Since $e_n \in B$ we have that $Q(e_n) > 0$. Moreover, let $I = (1, \ldots, 1)$. Then by linearity $Q[I] > 0$. Normalizing so that $Q[I] = 1$, we can associate Q with a probability measure equivalent to P satisfying property (ii) of Lemma 1.2.6. Then $Q \in \mathcal{M}^{\mathrm{e}}(S)$. □

Corollary 1.2.8. Let S satisfy (NA) and let $f = a + (H \cdot S)_T$ for some $H \in \mathcal{H}$ and $a \in \mathbb{R}$. Then a and H are uniquely determined by this expression and, moreover, $a = E_Q[f]$ and $a + (H \cdot S)_t = E_Q[f \mid \mathcal{F}_t]$.

Proof. For uniqueness, suppose that $f = a_1 + (H_1 \cdot S)_T = a_2 + (H_2 \cdot S)_T$, and say $a_1 > a_2$. Then $\big((H_2 - H_1) \cdot S\big)_T = a_1 - a_2 > 0$ is an arbitrage. But since we are assuming that S satisfy (NA), we must have $a_1 = a_2$. Next suppose that $H_1 \neq H_2$ and define

$$A = \{\omega \mid (H_1 \cdot S)_t - (H_2 \cdot S)_t > 0\},$$

for some t. Then $H := (H_{1,t} - H_{2,t})1_A 1_{(0,t]}$ is an arbitrage trading strategy because $(H \cdot S)_T = 0$ outside A, while $(H \cdot S)_T = (H_1 \cdot S)_t - (H_2 \cdot S)_t > 0$ on A. But again, since we are assuming that S satisfy (NA), we must have $H_1 = H_2$.

The last part follows from the fact (already established) the stochastic integral $(H \cdot S)_t$ is a Q-martingale.

□

1.2.1 Convex cones and polar sets

Using a standard definition in convex analysis, let the *polar* set of our cone C of super-replicable claims be given by

$$C^\circ = \{ f \in L^1(\Omega, \mathcal{F}, P) \mid E[fg] \leq 0, \forall f \in C \}.$$

According to the bipolar theorem (for a very general version, see [1]), the bipolar set $C^{\circ\circ} := (C^\circ)^\circ$ coincides with the closed convex hull of C. But by virtue of Remark 1.2.2, we know that C is already a closed set, from which we conclude that $C^{\circ\circ} = C$.

Now denote by $\mathcal{M}^{\mathrm{a}}S$ the set *absolutely continuous martingale measures* for S, that is probability measures Q which are absolutely continuous with respect to P and such that S is a Q–martingale. Consider the cone generated by $\mathcal{M}^{\mathrm{a}}S$, that is,

$$\mathrm{cone}(\mathcal{M}^{\mathrm{a}}S) := \left\{ f = \lambda \frac{dQ}{dP}, \lambda \geq 0, Q \in \mathcal{M}^{\mathrm{a}}S \right\}$$

The next proposition establishes a perfect polar relation between C and $\mathcal{M}^{\mathrm{a}}S$.

Proposition 1.2.9. Suppose S satisfies (NA). Then $C^\circ = \mathrm{cone}\,\mathcal{M}^{\mathrm{a}}(S)$, and $\mathcal{M}^{\mathrm{e}}(S)$ is dense in $\mathcal{M}^{\mathrm{a}}(S)$.

Proof. Since S satisfies (NA), $\mathcal{M}^{\mathrm{e}}(S) \neq \emptyset$ by the FTAP. Pick any $Q^* \in \mathcal{M}^{\mathrm{e}}(S)$. Then for all $Q \in \mathcal{M}^{\mathrm{a}}(S)$ and $0 < \mu \leq 1$, we have $\mu Q^* + (1-\mu)Q \in \mathcal{M}^{\mathrm{e}}(S)$, since $(1-\mu)Q$ is absolutely continuous with respect to $\mathcal{M}^{\mathrm{e}}(S)$. In particular, $\mathcal{M}^{\mathrm{a}}(S)$ is arbitrarily close to $\mathcal{M}^{\mathrm{e}}(S)$, which proves the density statement.

Now let $Q \in \mathcal{M}^{\mathrm{a}}(S)$ and let $\lambda > 0$. Then by Lemma 1.2.6, we have

$$E\left[\lambda \frac{Q}{dP} g\right] = \lambda E_Q[g] \leq 0, \quad \forall g \in C.$$

This shows that $\mathrm{cone}(\mathcal{M}^{\mathrm{a}}(S)) \subset C^\circ$. For the converse, observe first that $L^\infty_- \subset C$ because 0 is achievable, which implies that $C^\circ \subset L^1_+$. This means that $f \in C^\circ \subset L^1_+$ can be written as $f = \lambda \frac{dQ}{dP}$ for some $\lambda \geq 0$ and some probability measure Q. But then

$$0 \geq E[fg] = \lambda E_Q[g], \quad \forall g \in C$$

which means that $Q \in \mathcal{M}^{\mathrm{a}}(S)$, by Lemma 1.2.6. Thus $C^\circ \subset \mathrm{cone}(\mathcal{M}^{\mathrm{a}}(S))$. □

In view of Lemma 1.2.6, another way to formulate the proposition is as follows.

Proposition 1.2.10. For all $g \in L^\infty(\Omega, \mathcal{F}, P)$, the following are equivalent:

1. $g \in C$.
2. $E_Q[g] \leq 0$ for all $Q \in \mathcal{M}^{\mathrm{a}}(S)$.
3. $E_Q[g] \leq 0$ for all $Q \in \mathcal{M}^{\mathrm{e}}(S)$.

2 Utility Maximization in Finite Probability Spaces

Consider

$$U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\},$$

satisfying

(i) U is increasing on $\mathbb{R}$;

(ii) U is continuous on $dom(U) = \{x/U(x) > -\infty\}$;

(iii) U is strictly concave on the interior of $dom(U)$;

(iv) $U'(\infty) = 0$.

Regarding negative wealth, we assume:

- Case 1
  - ✓ $U(x) = -\infty$, $x < 0$;
  - ✓ $U(x) > -\infty$, $x > 0$;
  - ✓ $U'(0) = +\infty$;
  - ✓ Examples: $U(x) = log(x)$, $U(x) = \frac{x^p}{p}, p \in (-\infty, 1)\backslash\{0\}$.
- Case 2
  - ✓ $U(x) > -\infty$, $x \in \mathbb{R}$;
  - ✓ $U'(-\infty) = +\infty$;
  - ✓ Example: $U(x) = -\frac{e^{-\gamma x}}{\gamma}$, $\gamma > 0$.

The utility maximization we are interested in is:

$$u(x) := \sup_{H \in \mathcal{H}} E[U(x + (H \cdot S)_T)], \quad x \in dom(U) \tag{4}$$

Before tackling this problem, it is convenient to define the conjugate function V by

$$V(y) = \sup_{x \in \mathbb{R}} (U(x) - xy), \quad y > 0,$$

which can be seen as the *Legendre* transform of $-U(-x)$. It follows that V satisfies the following properties:

(i) $V : \mathbb{R} \to \mathbb{R}$ is finite valued;

(ii) V is differentiable and strictly convex on $(0, +\infty)$;

(iii) $V'(0) = -\infty$.

Moreover,

- Case 1 $V(\infty) = \lim_{x \mapsto 0} U(x)$ and $V'(\infty) = 0$;
- Case 2 $V(\infty) = +\infty$ and $V'(\infty) = +\infty$.

In addition, $U(x) = \inf_{y>0}[V(y) + yx]$ and $-V'(U'(x)) = x$. In other words, $I := (U')^{-1} = -V'$.

2.1 The complete market case

Assume that $\mathcal{M}^e(S) = \{Q\}$ and consider the Arrow-Debreux securities $1_{\{\omega_n\}}$ so that $E_Q[1_{\{\omega_n\}}] = Q(\omega_n) := q_n$ and because the market is complete, $1_{\{\omega_n\}} = q_n + (H^n \cdot S)$ for some $H^n \in \mathcal{H}$. It follows from the previous lemmas that a random variable $f \in L^\infty$ satisfies $f \leq x + (H \cdot S)_T$ for some $H \in \mathcal{H}$ iff $E_Q(f) \leq x$. Therefore, in this finite, complete case, we can rewrite (4) as the following concave optimization with a linear constraint:

$$\begin{aligned} u(x) &= \sup_{\substack{f \in \mathbb{R}^N \\ E_Q[f] \leq x}} E[U(f)] \\ &= \sup_{\substack{(f_1, ..., f_n) \\ \sum_{n=1}^N f_n q_n \leq x}} \sum_{n=1}^N p_n U(f_n). \end{aligned} \tag{5}$$

To solve this problem, let us introduce the Lagrangian:

$$\begin{aligned} L(f_1, ..., f_n, y) &= \sum_{n=1}^N p_n U(f_n) - y(\sum_{n=1}^N f_n q_n - x) \\ &= \sum_{n=1}^N p_n (U(f_n) - y\frac{q_n}{p_n} f_n) + xy. \end{aligned}$$

It follows from the saddle point theorem that a solution to (5) is given by a saddle point $(\widehat{f}_1, ..., \widehat{f}_N, \widehat{y})$ of L, that is,

$$L(f_1, ..., f_N, \widehat{y}) \leq L(\widehat{f}_1, ..., \widehat{f}_N, \widehat{y}) \leq (\widehat{f}_1, ..., \widehat{f}_N, y), \quad \forall f \in \mathbb{R}^N, \quad y > 0.$$

To see this, define

$$\phi(f_1, .., f_N) = \inf_{y \geq 0} L(f_1, ..., f_N, y), \quad f_n \in dom(U)$$

and

$$\psi(y) = \inf_{f \in \mathbb{R}^N} L(f_1, ..., f_N, y), \quad y \geq 0.$$

Now notice that if $f = (f_1, .., f_N)$ satisfies $E_Q[f] \leq x$, then $\phi(f_1, .., f_N) = L(f_1, ..., f_N, 0) = \sum_{n=1}^N p_n U(f_n)$. Conversely, if $E_Q(f) > x$, then $\phi(f_1, .., f_N) = -\infty$. Therefore,

$$\sup_{f \in \mathbb{R}^N} \phi(f_1, .., f_N) = \sup_{\substack{f \in \mathbb{R}^N \\ E_Q[f] \leq x}} \sum_{n=1}^N p_n U(f_n) = u(x).$$

Moreover, observe that for fixed $y > 0$, the optimization over $\mathbb{R}^N$ appearing in the definition of $\psi(y)$ splits into N separate one dimensional optimization problems. Explicitly, using the definition of V, we see that:

$$\begin{aligned} \psi(y) &= \sum_{n=1}^N p_n V(y\frac{q_n}{p_n}) + xy \\ &= E[V(y\frac{dQ}{dP})] + xy \\ &:= v(y) + xy. \end{aligned}$$

Observe that v(y) inherits all the proprieties of V. In particular, for $x \in dom(U)$, there exists a unique $\widehat{y} = \widehat{y}(x) > 0$ such that $v'(\widehat{y}(x)) = -x$, which is therefore the unique optimizer for $\psi(y)$.

Fixing $\widehat{y}(x)$, we see that the function $(f_1,..,f_N)\mapsto L(f_1,..,f_N,\widehat{y})$ achieves its maximum at $(\widehat{f}_1,...,\widehat{f}_N)$ satisfying $U'(\widehat{f}_n)=\widehat{y}(x)\frac{q_n}{p_n}\Leftrightarrow \widehat{f}_n=I(\widehat{y}(x)\frac{q_n}{p_n})$ which implies that

$$\begin{aligned}
\inf_{y>0}\psi(y) &= \inf_{y>0}(v(y)+xy)\\
&= v(\widehat{y}(x))+x\widehat{y}(x)\\
&= \sum_{n=1}^{N} p_n V(\widehat{y}(x))+x\widehat{y}(x)\\
&= \sum_{n=1}^{N} p_n(U(\widehat{f_n})-x\widehat{y}(x)\frac{q_n}{p_n})+x\widehat{y}(x)\\
&= L(\widehat{f}_1,...,\widehat{f}_N,\widehat{y}).
\end{aligned}$$

Notice that $\widehat{f_n}$ is in the interior of $dom(U)$, which means that L is continuously differentiable at $(\widehat{f}_1,...,\widehat{f}_N,\widehat{y})$ and $\frac{\partial L}{\partial y}\mid_{\widehat{f},\widehat{y}}=0$ so that the constraint is binding, that is $\sum_{n=1}^{N} q_n\widehat{f}_n=x$.

Finally, it is clear that $\sum_{n=1}^{N} p_n U(\widehat{f}_n)\leq u(x)$. Conversely, for all $(f_1,..,f_n)$ satisfying $E_Q[f]\leq x$, we have

$$\sum_{n=1}^{N} p_n U(f_n)\leq L(f_1,...,f_N,\widehat{y})\leq L(\widehat{f}_1,...,\widehat{f}_N,\widehat{y})=\sum_{n=1}^{N} p_n U(\widehat{f}_n).$$

Therefore, $u(x)=v(\widehat{y}(x))+x\widehat{y}(x)\Rightarrow u'=\widehat{y}(x)$. So that u inherits all the properties from U.

Theorem 2.1.1. For a finite complete case, define $u(x)=\sup_{H\in\mathcal{H}}E[U(X_T)]$, $x\in dom(u)$, $X_T=x+(H\cdot S)_T$, and $v(x)=E[V(y\frac{dQ}{dP})]$, $y>0$. Then,

(i) u and v are conjugates and inherit the proprieties of U and V;

(ii) $\widehat{X}_T(\omega_n)=I(y\frac{dQ}{dP})$ (or equivalent $U'(\widehat{X}_T(\omega_n))=y\frac{dQ}{dP}(\omega_n)$) is optimal wealth, where y satisfies $u'(x)=y$ (or equivalently $v'(y)=-x$).

Notes:

(1) $U'(\widehat{X}_T(\omega_n))=y\frac{q_n}{p_n}$, where U' represents the marginal utility, $\widehat{X}_T(\omega_n)$ is the optimal wealth, and q_n is the price of the Arrow-Debrew security $1_{\{\omega_n\}}$ with a probability of its success p_n.

(2) Observe that $u'(x)=y$ and $U'(\widehat{X}_T)=y\frac{dQ}{dP}$ implies that $u'(x)=E[U'(\widehat{X}_T)]$.

Consider an agent with $x+\epsilon$ as an initial endowment who uses x to finance $\widehat{X}_T^x=x+(\widehat{H}\cdot S)_T$ for some $\widehat{H}\in\mathcal{H}$ and ϵ to buy the numeraire. Thereby ending with $\widehat{X}_T+\epsilon$ at T. Comparing this by the optimal wealth $\widehat{X}_T^{x+\epsilon}$ gives:

$$\begin{aligned}
\lim_{\epsilon\to 0^+}\frac{u(x+\epsilon)-u(x)}{\epsilon} &= \lim_{\epsilon\to 0^+}\frac{E[U'(\widehat{X}_T^{x+\epsilon})-U'(\widehat{X}_T^{x})]}{\epsilon}\\
&\geq \lim_{\epsilon\to 0^+}\frac{E[U'(\widehat{X}_T^{x}+\epsilon)-U'(\widehat{X}_T^{x})]}{\epsilon}\\
&= E[U'(\widehat{X}_T^{x})].
\end{aligned}$$

The argument with $\epsilon<0$ implies that $u'(x)=E[U'(\widehat{X}_T^x)]\Rightarrow$ the agent is indifferent. Similarly, we can prove that $xu'(x)=E[\widehat{X}_T^x U'(\widehat{X}_T^x)]$.

3 The Dalang–Morton–Willinger Theorem

3.1 The closedness of C

Let $X = \Delta S_1 = S_1 - S_0$ and its corresponding subspaces:

$$E^X = \{H : \Omega \to \mathbb{R}^d, \mathfrak{F}_0\text{-measurable and } X \cdot H = 0 \ \ a.s\} \text{ and}$$
$$\mathcal{H}^X = \{f : \Omega \to \mathbb{R}^d, \mathfrak{F}_0\text{-measurable and } Pf = f\},$$

where $P = I - P'$ and P' being the projection associated to E^X. Observe that $H \cdot X = H \cdot (S_1 - S_0) = (H.S)_1$. Define:

$$\begin{aligned} I : \qquad \mathcal{L}^0(\Omega, \mathfrak{F}_0, P; \mathbb{R}^d) &\longrightarrow \mathcal{L}^0(\Omega, \mathfrak{F}_0, P) \\ H &\longmapsto (H \cdot S)_1 \end{aligned}$$

Definition 3.1.1. We say H is a canonical form for S if $H \in \mathcal{H}^X$, where $X = \Delta S$.

Lemma 3.1.2. The kernel of I is E^X. The restriction of I to $\mathcal{H}^X$ is injective, linear and has full range.

Proof. The first statement follows from the definition of E^X. For the second, let H and $H' \in \mathcal{H}^X$ with $I(H) = I(H')$. Then $X \cdot (H - H') = 0$ a.s $\Rightarrow (H - H') \in E^X$. But $(H - H') \in \mathcal{H}^X$, then $(H - H') \in E^X \cap \mathcal{H}^X = \{0\} \Rightarrow H = H'$ a.s. $\square$

Proposition 3.1.3. Let $(S_t)_{t=0}^1$ be adapted to $(\Omega, (\mathfrak{F}_t)_{t=0}^1, P)$ and let $(H^n)_{n=1}^\infty$ be a sequence in $\mathcal{L}^0(\Omega, \mathfrak{F}_0, P; \mathbb{R}^d)$ in canonical form. Then:

(i) $(H^n)_{n=1}^\infty$ is bounded iff $(H^n \cdot \Delta S)_{n=1}^\infty$ is.

(ii) $(H^n)_{n=1}^\infty$ converges a.s iff $(H^n \cdot \Delta S)_{n=1}^\infty$ does.

If, in addition, S satisfies (NA), then

(i') $(H^n)_{n=1}^\infty$ is bounded iff $((H^n \cdot \Delta S)_-)_{n=1}^\infty$ is.

(ii') $(H^n)_{n=1}^\infty$ converges to zero a.s iff $((H^n \cdot \Delta S)_-)_{n=1}^\infty$ does.

Proof. We consider just the "if" part of each statement.

(i) and (i'): Suppose that $(H^n)_{n=1}^\infty$ is **not** bounded. Let $K = \mathbb{R}^d \cup \{\infty\}$ and take $x_0 = \infty \in K$. Since $(H^n)_{n=1}^\infty$ diverges to ∞ on a set B of positive measure, there exists a subsequence $(L^k)_{k=1}^\infty = (H^{\tau_k})_{k=1}^\infty$ such that $(L^k(\omega))_{k=1}^\infty$ diverges to ∞ on B.

Now put $\widehat{L}^k = \frac{L^k}{|L^k|} \mathbb{I}_{B \cap \{|L^k| \geq 1\}}$ so that $|\widehat{L}^k(\omega)| = 1$ for $\omega \in B$ and k sufficiently large.

By passing to a subsequence again, we may suppose that $(\widehat{L}^k)_{k=1}^\infty$ converges to $\widehat{L}$, which is in canonical form and satisfies $|\widehat{L}| = 1$ on B. Therefore we assume that $(H^n \cdot \Delta S)_{n=1}^\infty$ is bounded. Then $(\widehat{L}^n \cdot \Delta S)_{n=1}^\infty$ necessarily goes to zero a.s. But then $\widehat{L} \cdot \Delta S = \lim_{k\to\infty} \widehat{L} \cdot \Delta S = 0$ a.s and since $\widehat{L}$ is in canonical form, this implies that $\widehat{L} = 0$ a.s (contradiction).

In addition, suppose that S satisfies (NA). Then if we assume that $((H^n \cdot \Delta S)_-)_{n=1}^\infty$ is bounded. So, $\widehat{L} \cdot \Delta S_- = \lim_{k\to\infty} \widehat{L}^k \cdot \Delta S_- = 0$ a.s. (NA) implies $\widehat{L} \cdot \Delta S_- = 0$ a.s $\Rightarrow \widehat{L} = 0$ a.s (contradiction).

(ii) and (ii'): Suppose that $(H^n)_{n=1}^\infty$ does **not** converge a.s, but $(H^n \cdot \Delta S)_{n=1}^\infty$ does. Then, $(H^n \cdot \Delta S)_{n=1}^\infty$ is bounded a.s. Therefore $(H^n)_{n=1}\infty$ is also bounded a.s from the previous item. Using again the compact set $K = \mathbb{R}^d \cup \{\infty\}$; we can find a subsequence $(H^{\tau_k})_{k=1}^\infty$ comparing to some $H^0 \in \mathcal{H}^X$. Since $(H^n)_{n=1}^\infty$ itself does not converge, we can find another subsequence $(H^{\sigma_k})_{k=1}^\infty$ converging to some $\widehat{H}^0$ with $P[H^0 \neq \widehat{H}^0] > 0$. But since $(H^n \cdot \Delta S)_{n=1}^\infty$ converge, we must have $(H^0 - \widehat{H}^0) \cdot \Delta S = \lim_{k\to\infty} H^{\tau_k} \cdot \Delta S - \lim_{k\to\infty} H^{\sigma_k} \cdot \Delta S = 0$ a.s. Therefore, $H^0 = \widehat{H}^0$.

Assume (NA) and also that $(H^n)_{n=1}^\infty$ does not converge to zero but $((H^n \cdot \Delta S)_-)_{n=1}^\infty$ does. We can again find a convergent subsequence $(H^{\sigma_k})_{k=1}^\infty$ converging to some $\widehat{H}^0$ such that $P(\widehat{H}^0 \neq 0) > 0$. But $\widehat{H}^0 \cdot \Delta S_- = \lim_{k\to\infty} H^{\sigma_k} \cdot \Delta S_- = 0$ a.s. which, together with (NA), means that $\widehat{H}^0 = 0$ a.s. $\square$

Theorem 3.1.4. Let $(S_t)_{t=0}^1$ be a one step process.

(i) $K = \{H \cdot \Delta S / H \in L^0(\Omega, \mathfrak{F}_0, P; \mathbb{R}^d)\}$ is closed in $L^0(\Omega, \mathfrak{F}_1, P)$.

(ii) If S satisfies (NA), then C=K-$L^0_+(\Omega, \mathfrak{F}_1, P)$ is also closed.

Proof. (i) let $(f_n) = (H^n \cdot \Delta S)_{n=1}^\infty$ be a sequence in K converging to $f_0 \in L^0(\Omega, \mathfrak{F}_1, P)$, with each H^n in canonical form. By passing to a subsequence, we can suppose that $(f_n)_{n=1}^\infty$ converges to f_0 a.s. Then, $(H^n)_{n=1}^\infty$ converges a.s to some $H^0 \in L^0(\Omega, \mathfrak{F}_0, P; \mathbb{R}^d)$ so that $f_0 = H^0 \cdot \Delta S \in K$.

(ii) Let $f_n = g_n - h_n$ be a sequence in C converging to $f_0 \in L^0(\Omega, \mathfrak{F}_1, P)$, where $g_n = H^n \cdot \Delta S$ for H^n in canonical form and $h_n \in L^0_+(\Omega, \mathfrak{F}_1, P)$.

Again by passing to a subsequence, we can assume that (f_n) converges to f_0 a.s. Since $g_n \leq f_n$ we have that g_{n-} is bounded. Because of (NA), we then have that $(H^n)_{n=1}^\infty$ is also bounded a.s. By passing to a convergent subsequence $(H^{\tau_k})_{k=1}^\infty$, we may suppose that $g_{\tau_k} = H^{\tau_k} \cdot \Delta S$ converges a.s to $g_0 = H^0 \cdot \Delta S$, $H^0 = L^0(\Omega, \mathfrak{F}_0, P; \mathbb{R}^d)$. Since (f_{τ_k}) still converge a.s to f_0, we have that $h_{\tau_k} = g_{\tau_k} - f_{\tau_k}$ converges a.s to $h_0 \geq 0$. Thus $f_0 = g_0 - h_0 \in C$. □

3.2 The DMW Theorem for $T = 1$

Theorem 3.2.1. Let $(S_t)_{t=0}^1$ be a one-step price process adapted to $(\Omega, (\mathfrak{F}_t)_{t=0}^1, P)$ satisfying the (NA) condition. Then, $\exists$ an equivalent probability measure Q such that:

(i) S_0, $S_1 \in L^1(\Omega, \mathfrak{F}_1, Q; \mathbb{R}^d)$;

(ii) $E_Q[S_1 \mid \mathfrak{F}_0] = S_0$;

(iii) $\frac{dQ}{dP}$ is bounded.

Proof. First construct P_1 given by:

$$\frac{dP_1}{dP} = c e^{-\|S_1\| - \|S_0\|},$$

so that $P_1 \sim P$, $\frac{dP_1}{dP}$ is bounded, S_0, $S_1 \in L^1(\Omega, \mathfrak{F}_1, P_1; \mathbb{R}^d)$. Next, take $C_1 = C \cap L^1(\Omega, \mathfrak{F}_1, P_1; \mathbb{R}^d)$. Then it is easy to show that C_1 is closed in $L^1(\Omega, \mathfrak{F}_1, P_1; \mathbb{R}^d)$, because C is closed in $L^0(\Omega, \mathfrak{F}_1, P; \mathbb{R}^d) = L^0(\Omega, \mathfrak{F}_1, P_1; \mathbb{R}^d)$. Moreover, C_1 is a convex cone because C is a convex cone and by (NA), $C_1 \cap L^1_+(\Omega, \mathfrak{F}_1, P_1; \mathbb{R}^d) = \{0\}$. It then follows from the Hahn-Banach Theorem (see the general version next lecture!) that we can find an equivalent probability measure Q such that $\frac{dQ}{dP_1}$ is bounded and $E_Q[f] \leq 0$ for all $f \in C_1$. We then have that S_0, $S_1 \in L^1(\Omega, \mathfrak{F}_1, Q; \mathbb{R}^d)$ and that $\frac{dQ}{dP} = \frac{dQ}{dP_1} \frac{dP_1}{dP}$ is bounded.

For the martingale property, observe that for each component $j = i, .., d$ and each $A \in \mathfrak{F}_0$, we have that $\mathbb{I}_A(S_1^j - S_0^j) \in C_1$ and $-\mathbb{I}_A(S_1^j - S_0^j) \in C_1$. Therefore, $E_Q[\mathbb{I}_A(S_1^j - S_0^j) \mid \mathfrak{F}_0] = 0$ and so $E_Q[\mathbb{I}_A(S_1 - S_0) \mid \mathfrak{F}_0] = 0$. □

3.3 Proof of the DMW theorem for $T \geq 1$

Let us use induction on the number of intervals necessary to reach T.

For $T = 1$, the result holds.

Suppose that it holds for $n = T - 1$, that is, consider $t = 1, ..., T$ and the process $(S_t)_{t=1}^T$ adapted to $(\Omega, (\mathfrak{F}_t)_{t=1}^T, P; \mathbb{R}^d)$ for which there exists on equivalence probability measure Q^1 on $\mathfrak{F}_T$ such that:

(i) $\frac{dQ^1}{dP}$ is bounded;

(ii) $S_1, ..., S_T \in L^1(\Omega, \mathfrak{F}_T, Q^1; \mathbb{R}^d)$;

(iii) $(S_t)_{t=1}^T$ is a Q^1-martingale.

Using the one-step DMW Theorem for $(S_t)_{t=1}^1$ and $(\Omega, (\mathfrak{F}_t)_{t=0}^1, Q^1; \mathbb{R}^d)$, we can find a bounded function f_1 such that f_1 is $\mathfrak{F}_1$-measurable, $f_1 > 0$, $E_{Q^1}[f_1] = 1$, $E_{Q^1}[\| S_1 \mid f_1] < \infty$, $E_{Q^1}[\| S_0 \mid f_1] < \infty$ and for all $A \in \mathfrak{F}_0$:

$$\int_A S_0 f_1 dQ^1 = \int_A S_1 f_1 dQ^1.$$

Now define Q on $\mathfrak{F}_T$ by $Q[A] = \int_A f_1 dQ^1$, $A \in \mathfrak{F}_T$. That is, $\frac{dQ}{dP} = f_1 \frac{dQ^1}{dP}$ is bounded and $\frac{dQ}{dP} > 0$. So $Q \sim P$.

Moreover, for $t = 0, ..., T$, we have $\int_\Omega |S_t| dQ = \int_\Omega |S_t| f_1 dQ^1 < \infty$.

Finally, for the martingale property, observe that for all $A \in \mathfrak{F}_0$, we have:

$$\int_A S_0 dQ = \int_A S_0 f_1 dQ^1 = \int_A S_1 f_1 dQ^1 = \int_A S_1 dQ.$$

So that $E_Q[S_1 \mid \mathfrak{F}_0] = S_0$, for $t \geq 1$, let $A \in \mathfrak{F}_t$, then

$$\begin{aligned} \int_A S_t dQ &= \int_A S_t f_1 dQ^1 \\ &= \int_A S_{t+1} f_1 dQ^1 \\ &= \int_A S_{t+1} dQ. \end{aligned}$$

So, $E_Q[S_{t+1} \mid \mathfrak{F}_t] = S_t$.

4 No-arbitrage in continuous time

4.1 Stochastic integrals for semimartingales

Recall that we have defined the stochastic integral $H \mapsto (H.W)_t$ pathwise for bounded simples strategies and used the isometry

$$\| H \|_{L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, P \otimes \lambda)} = \| (H.S)_\infty \|_{L^2(\Omega, \mathfrak{F}, P)}$$

to extend it by continuity to the entire space $L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, P \otimes \lambda)$, in such way that $(H.W)_t$ is an L^2-bounded martingale, that is, $\sup_t \| (H.W)_t \|_{L^2(\Omega, \mathfrak{F}, P)} < \infty$. When H is locally in $L^2(P \otimes \lambda)$ (which is equivalent to $\int_0^t H_s^2 ds < \infty$), the same construction yields a local martingale $(H.W)_t$ which is locally L^2-bounded.

Recall also that for an L^2-bounded martingale S_t, we define the quadratic variation measure on $\mathcal{P}$ as:

$$d[S](]|\tau, \sigma]|) := E(|S_\sigma - S_\tau|^2)$$

and the following isometry holds for bounded simple integrand H:

$$\| H \|_{L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, d[S])} = \| (H.S)_\infty \|_{L^2(\Omega, \mathfrak{F}, P)}$$

we can then extend $H \mapsto (H.S)_t$ to the entire space $L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, d[S])$ by continuity in such a way that $(H.S)_t$ is also an L^2 bounded martingale. We can use again localization to extend this to a locally L^2 integrand H and locally L^2 bounded local martingale. Since every continuous local martingale S is automatically locally L^2-bounded, this is the right degree of generality for this class. To include integrators with jumps, we will extend the theory even beyond local martingales.

Suppose first that S is a càdlàg adapted process of bounded variation, that is,

$$|S|_t := \sup_{0 \leq t_0 \leq t_1 \leq \ldots \leq t_n \leq t} \sum_{i=0}^n |S_{t_{i+1}} - S_{t_i}| < \infty \text{ a.s for all } t < \infty.$$

Then for almost all $\omega \in \Omega$, the path $(S_t(\omega))_{0\leq t<\infty}$ is of bounded variation on compact subsets of $\mathbb{R}_+$ by $dS(\omega)(]a,b]) = S_b(\omega) - S_a(\omega)$. So that the stochastic integral $(H.S)_t(\omega) = \int_0^t H_u(\omega)dS_u(\omega)$ is well defined as a Lebesgue-Sieltjes integral for each process H such that $(H_u(\omega))_{0\leq u\leq t}$ is $dS(\omega)$-integrable. We have then led to investigate process of the form

$$S = M + A \tag{6}$$

where M is a bounded martingale and A is locally of bounded variation.

For that, let us define $\mathcal{S}$ as the class of bounded simple integrands H with the topology of uniform convergence which is given by the norm:

$$\| H \|_\infty = \sup\{\| H_t \|_{L^\infty(\Omega,\mathfrak{F}_t,P)} \backslash t \in \mathbb{R}+\} \tag{7}$$

For this class, we can define the stochastic integral as before:

$$I(H) = (H.S)_\infty = \sum_{i=1}^{n} f_{i-1}(S_{\tau_i} - S_{\tau_{i-1}}) \tag{8}$$

for any càdlàg process S.

Definition 4.1.1. S is a strict semi-martingale if the map:

$$I: \quad \mathcal{S} \to L^0(\omega, \mathfrak{F}_\infty, P) \tag{9}$$

$$H \mapsto I(H) = (H \cdot S)_\infty \tag{10}$$

is continuous for the topologies of $\|\|_\infty$ on $\mathcal{S}$ and convergence in probability on L^0.

(i) S is a semi-martingale if it is locally a strict semi-martingale.

Theorem 4.1.2 (Bicheteler-Dellacherie)**.** S is a semi-martingale in the sense of the definition above *if and only if* it can be decomposed as $S = M + A$ as in (6).

We say that S is a **special** semi-martingale if in addition the process A is predictable.

It is relatively easy to show that $(H.S)_t$ is a semi-martingale, even when H is only locally in $\mathcal{S}$. To extend for H beyond L^∞ -bounded simple, consider the semi-martingale topology induced on the set of one-dimensional semi-martingale by the distance:

$$D[S] = \sum_{n=1}^{\infty} 2^{-n} sup\{E[|\ (K - S)_n\ |] \wedge 1/K \leq 1\}$$

where K is a predictable process. This means that $S^n \to 0$ iff $(K \cdot S^n)_t \to 0$ uniformly in t and K. One can show that this space is complete. We then say that an $\mathbb{R}^d$-valued process H is S-integrable $(L(S))$ w.r.t a semi-martingale S if $(H1_{H\leq n} \cdot S)_{n=1}^\infty$ is a cauchy sequence. We define $(H \cdot S)_t$ as the limit of the sequence.

Notice $(H \cdot S)_t$ is a semi-martingale and can be decomposed to a local martingale $\widetilde{M}_t + \widetilde{A}_t$ that are different from $(H \cdot M)_t$ and $(H \cdot .A)_t$ resp.

Remark 4.1.3. One can construct examples where $S = M + A$ is a special semi-martingale, H is $S-$integral, so that $(H \cdot S)_t$ exists, but $(H \cdot A)$ does not exist.

Lemma 4.1.4. Let S be a special semi-martingale with decomposition $S = M + A$ and H be an $\mathbb{R}^d-$valued predictable process. If the stochastic integral $(H \cdot S)$ is itself special, then $(H \cdot A)$ exists as a Lebesgue-Stieltjes integral.

One can find examples of a martingale M_t and an $M-$integrable process H such that $(H \cdot M)_t$ is not a local martingale.

Lemma 4.1.5. Let M be an $\mathbb{R}^d-$valued local martingale and let H be an $\mathbb{R}^d-$valued $M-$integarble process. Then $(H \cdot M)_t$ is a local martingale if there exists a sequence of stopping times $\tau_n \nearrow \infty$ and integrable function $\varphi_n \in L^1$ with $\varphi_n \leq 0$ s.t $< H, \triangle M >^{\tau}_m \geq \varphi_n$.

Theorem 4.1.6. If S is a special semi-martingale with canonical decomposition $S = M + A$ and if H is $S-$integrable then $(H.S)$ is special martingale iff:

i $(H \cdot M)$ is defined as an integral in local martingale sense

ii $(H \cdot A)$ is defined as a Lesbegue-Stieltjes integral.

Proof. Let H be $S-$integrable. If $(H \cdot S)$ is special then $(H \cdot A)$ exists as a L-S integrable by Lemma 1, which gives (ii). Moreover, $(H \cdot S)$ is a special, it must be locally integrable, that is, there is a sequence $\tau_n \to \infty$ and $\varphi_n \in L^1$ s.t $(H \cdot S)^{\tau_n} \geq \varphi_n$. Now let σ_n be stopping times such that $(\int_0^{\sigma_n} \mid H_s \mid dA_s) \in L^1$ (which must exist since $(H \cdot A)$ is a regular L-S integral). Then for each n, $(H \cdot M)^{\tau_n \wedge \sigma_n} \geq \varphi_n - \int_0^{\sigma_n} \mid H_s \mid dA_s$ and Lemma 2 shows that $(H \cdot M)$ is a local martingale. Conversely, if (i) and (ii) hold, then $(H \cdot S)$ is the sum of a local martingale and a predictable bounded variation process $(H \cdot A)$ and therefore special. $\square$

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