Introduction to immersed boundary method

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Review of the IB method

A general computational framework for simulations of problems involving the interaction of fluids and immersed elastic structures

Original example: blood interacts with valve leaflet
(Charles S. Peskin, 1972, flow patterns around heart valves)

Applications

- Computer-assisted design of prothetic valve (Peskin & McQueen)
- Platelet aggregation during blood clotting (Fogelson)
- Flow of particle suspensions (Fogelson & Peskin; Sulsky & Brackbill)
- Wave propagation in the cochlea (Beyer)

- Swimming organism (Fauci, Dillon, Cortez)
- Arteriolar flow (Arthurs, et. al.)
- Cell and tissue deformation under shear flow (Bottino, Stockie & Green; Eggleton & Popel)
- Flow around a circular cylinder (Lai & Peskin; Su, Lai & Lin), Interfacial flow with insoluble surfactant (Lai, Tseng & Huang)
- Valveless pumping (Jung & Peskin)
- Flapping filament in a flowing soap film (Zhu & Peskin), 2D dry foam (Kim, Lai & Peskin)
- Falling papers, sails, parachutes (Kim & Peskin), insect flight, (Jane Wang, Miller) ...
- Many others (Lim, ..)

IB features

Mathematical formulation:

- Treat the elastic material as a part of fluid
- The material acts force into the fluid
- The material moves along with the fluid

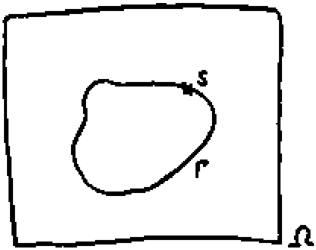
Numerical method:

- Finite difference discretization
- Eulerian grid points for the fluid variables
- Lagrangian markers for the immersed boundary
- The fluid-boundary are linked by a smooth version of Dirac delta function

Mathematical formulation

Consider a massless elastic membrane Γ immersed in viscous incompressible fluid domain Ω.

$$\begin{aligned} \Gamma : &\quad \boldsymbol{X}(s,t), \quad 0 \le s \le L_b \\ L_b : &\quad \text{unstressed length} \end{aligned}$$



Eg. Water balloon

Equations of motion

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right) + \nabla p = \mu\Delta\boldsymbol{u} + \boldsymbol{f}$$

$$\nabla\cdot\boldsymbol{u} = 0$$

$$\boldsymbol{f}(\boldsymbol{x},t) = \int_\Gamma \boldsymbol{F}(s,t)\delta(\boldsymbol{x}-\boldsymbol{X}(s,t))ds$$

$$\frac{\partial \boldsymbol{X}(s,t)}{\partial t} = \boldsymbol{u}(\boldsymbol{X}(s,t),t) = \int_\Omega \boldsymbol{u}(\boldsymbol{x},t)\delta(\boldsymbol{x}-\boldsymbol{X}(s,t))d\boldsymbol{x}$$

$$\boldsymbol{F}(s,t) = \frac{\partial}{\partial s}(T\boldsymbol{\tau}), \quad T = \sigma(\left|\frac{\partial \boldsymbol{X}}{\partial s}\right|; s,t), \quad \boldsymbol{\tau} = \frac{\partial\boldsymbol{X}/\partial s}{|\partial\boldsymbol{X}/\partial s|}$$

FLUID		IMMERSED BOUNDARY	
$\boldsymbol{u}(\boldsymbol{x},t)$:	velocity	$\boldsymbol{X}(\boldsymbol{x},t)$:	boundary configuration
$\boldsymbol{p}(\boldsymbol{x},t)$:	pressure	$\boldsymbol{F}(\boldsymbol{x},t)$:	boundary force
ρ :	density	T :	tension
μ :	viscosity	$\boldsymbol{\tau}$:	unit tangent

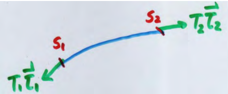
Why $\boldsymbol{F}(s,t) = \frac{\partial}{\partial s}(T\boldsymbol{\tau})$?

$$\text{total force} = \text{force of fluid on boundary seg.} + (T\boldsymbol{\tau})\Big|_{s_1}^{s_2}$$

$$\text{boundary is massless} \Rightarrow \text{total force} = 0$$

$$\text{force of boundary seg. on fluid} = (T\boldsymbol{\tau})\Big|_{s_1}^{s_2} = \int_{s_1}^{s_2} \frac{\partial}{\partial s}(T\boldsymbol{\tau})ds$$

$$\boldsymbol{F} = \frac{\partial T}{\partial s}\boldsymbol{\tau} + T\frac{\partial \boldsymbol{\tau}}{\partial s} = \frac{\partial T}{\partial s}\boldsymbol{\tau} + T\left|\frac{\partial \boldsymbol{X}}{\partial s}\right|\kappa\boldsymbol{n}$$



where $\kappa = \dfrac{|\partial\boldsymbol{\tau}/\partial s|}{|\partial \boldsymbol{X}/\partial s|}$ boundary curvature

$\boldsymbol{n} = \dfrac{\partial\boldsymbol{\tau}/\partial s}{|\partial\boldsymbol{\tau}/\partial s|}$ unit normal

$$T = \sigma_0\left(\left|\frac{\partial \boldsymbol{X}}{\partial s}\right| - 1\right)$$

σ_0 : the stiffness constant

The force density is singular

$$\boldsymbol{f}(\boldsymbol{x},t) = \int_\Gamma \boldsymbol{F}(s,t)\delta(\boldsymbol{x} - \boldsymbol{X}(s,t))ds.$$

$\boldsymbol{f}$ behaves like a one-dimensional delta function.

$$\begin{aligned}
\langle \boldsymbol{f}, \boldsymbol{w} \rangle &= \int_\Omega \boldsymbol{f}(s,t)\boldsymbol{w}(\boldsymbol{x},t)d\boldsymbol{x} \\
&= \int_\Omega \int_\Gamma \boldsymbol{F}(s,t)\delta(\boldsymbol{x} - \boldsymbol{X}(s,t))ds \cdot \boldsymbol{w}(\boldsymbol{x},t)d\boldsymbol{x} \\
&= \int_\Gamma \boldsymbol{F}(s,t) \int_\Omega \boldsymbol{w}(\boldsymbol{x},t)\delta(\boldsymbol{x} - \boldsymbol{X}(s,t))d\boldsymbol{x}ds \\
&= \int_\Gamma \boldsymbol{F}(s,t)\boldsymbol{w}(\boldsymbol{X}(s,t),t)ds
\end{aligned}$$

If $\boldsymbol{w}(\boldsymbol{x},t) = \boldsymbol{u}(\boldsymbol{x},t)$, the the total work done by the boundary is equal to the total work done on the fluid. Thus, the solution is NOT smooth. In fact the pressure and velocity derivatives are discontinuous across the boundary.

Theorem (1)

The pressure and the velocity normal derivatives across the boundary satisfy

$$[p] = \frac{\boldsymbol{F} \cdot \boldsymbol{n}}{|\partial \boldsymbol{X}/\partial s|},$$

$$\mu \left[\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \right] = - \frac{\boldsymbol{F} \cdot \boldsymbol{\tau}}{|\partial \boldsymbol{X}/\partial s|} \boldsymbol{\tau}.$$

Theorem (2)

The normal derivatives of the pressure across the boundary satisfies

$$\left[\frac{\partial p}{\partial \boldsymbol{n}} \right] = \frac{\frac{\partial}{\partial s} \left(\frac{\boldsymbol{F} \cdot \boldsymbol{\tau}}{|\partial \boldsymbol{X}/\partial s|} \right)}{|\partial \boldsymbol{X}/\partial s|}.$$

Physical meaning of the jump conditions.
Ref: Peskin & Printz 1993, LeVeque & Li 1997 (2D), Lai & Li 2001 (3D)

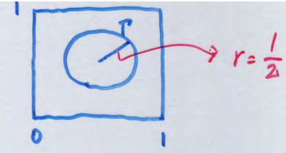
Test Example

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \Delta \boldsymbol{u} + \boldsymbol{f} + \boldsymbol{g}$$

$$\nabla\cdot\boldsymbol{u} = 0$$

$$\boldsymbol{f} = \int_\Gamma \boldsymbol{F}(s,t)\delta(\boldsymbol{x} - \boldsymbol{X}(s,t))ds$$

$$\frac{\partial \boldsymbol{X}(s,t)}{\partial t} = \boldsymbol{u}(\boldsymbol{X}(s,t),t)$$



$$\big(X(s),Y(s)\big) = \big(0.5\cos s, 0.5\sin s\big), \qquad \left|\frac{\partial \boldsymbol{X}}{\partial s}\right| = 0.5$$

$$\boldsymbol{n} = \big(\cos s, \sin s\big), \quad \boldsymbol{\tau} = \big(-\sin s, \cos s\big)$$

Test Example (cont.)

$$u(x,y,t) = \begin{cases} e^{-t}\left(2y - \frac{y}{r}\right) & r \geq 0.5 \\ 0 & r < 0.5 \end{cases}, \qquad \left[\frac{\partial u}{\partial \boldsymbol{n}}\right] = 2e^{-t}\sin s$$

$$v(x,y,t) = \begin{cases} e^{-t}\left(-2x + \frac{x}{r}\right) & r \geq 0.5 \\ 0 & r < 0.5 \end{cases}, \qquad \left[\frac{\partial v}{\partial \boldsymbol{n}}\right] = -2e^{-t}\cos s$$

$$p(x,y,t) = \begin{cases} 0 & r \geq 0.5 \\ 1 & r < 0.5 \end{cases}, \qquad [p] = -1$$

$$g_1(x,y,t) = \begin{cases} e^{-t}\left(\frac{y}{r} - 2y - \frac{y}{r^3}\right) + e^{-2t}\left(\frac{4x}{r} - 4x - \frac{x}{r^2}\right) & r \geq 0.5 \\ 0 & r < 0.5 \end{cases}$$

$$g_2(x,y,t) = \begin{cases} -e^{-t}\left(\frac{x}{r} - 2x - \frac{x}{r^3}\right) + e^{-2t}\left(\frac{4y}{r} - 4y - \frac{y}{r^2}\right) & r \geq 0.5 \\ 0 & r < 0.5 \end{cases}$$

$$F_{\boldsymbol{n}} = -0.5 \qquad F_1 = -0.5\cos s - e^{-t}\sin s \qquad [\boldsymbol{g}\cdot\boldsymbol{n}] = 0$$

$$F_{\boldsymbol{\tau}} = e^{-t} \qquad F_2 = -0.5\sin s + e^{-t}\cos s$$

Numerical Method

FLUID-Eulerian

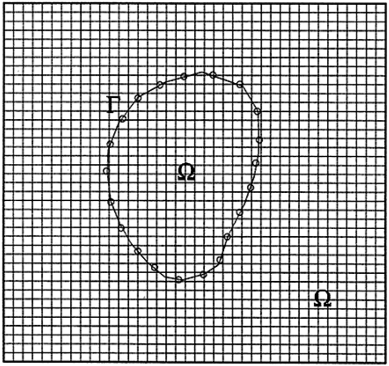
$N \times N$ lattice points $\boldsymbol{x} = (ih, jh)$

$$\boldsymbol{u}^n \approx \boldsymbol{u}(\boldsymbol{x}, n\Delta t),$$
$$\boldsymbol{f}^n \approx \boldsymbol{f}(\boldsymbol{x}, n\Delta t),$$
$$p^n \approx p(\boldsymbol{x}, n\Delta t),$$

IMMERSED BOUNDARY-Lagrangian

M moving points $\boldsymbol{X}_k$

$$\boldsymbol{U}_k^n \approx \boldsymbol{u}(\boldsymbol{X}_k, n\Delta t)$$
$$\boldsymbol{F}_k^n \approx \boldsymbol{F}(\boldsymbol{X}_k, n\Delta t)$$



Finite difference operators on lattice points and immersed boundary points

- The finite difference approximations to the derivative are

$$D_x^+ \phi_{i,j} = \frac{\phi_{i+1,j} - \phi_{i,j}}{h}, \qquad D_x^- \phi_{i,j} = \frac{\phi_{i,j} - \phi_{i-1,j}}{h},$$

$$D_x^0 \phi_{i,j} = \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2h}.$$

- The upwind difference approximation to the advection term is

$$\psi_{i,j} D_x^{\pm} \phi_{i,j} = \begin{cases} \psi_{i,j} D_x^- \phi_{i,j} & \text{if } \psi_{i,j} > 0, \\ \psi_{i,j} D_x^+ \phi_{i,j} & \text{if } \psi_{i,j} < 0. \end{cases}$$

- The centered difference approximation to the derivative on the Lagrangian boundary can be defined by

$$D_s \chi_k = \frac{\chi_{k+1/2} - \chi_{k-1/2}}{\Delta s},$$

where $\chi_{k+1/2}$ is some boundary quantity defined on $s = (k + \frac{1}{2})\Delta s$.

Numerical algorithm

How to march $(\boldsymbol{X}^n, \boldsymbol{u}^n)$ to $(\boldsymbol{X}^{n+1}, \boldsymbol{u}^{n+1})$?

Step1 Compute the boundary force

$$T^n_{k+1/2} = \sigma(|D_s \boldsymbol{X}^n_k|), \quad \boldsymbol{\tau}^n_{k+1/2} = \frac{D_s \boldsymbol{X}^n_k}{|D_s \boldsymbol{X}^n_k|}, \quad \boldsymbol{F}^n_k = D_s(T\boldsymbol{\tau})^n_{k+1/2},$$

where $T^n_{k+1/2}$ and $\boldsymbol{\tau}^n_{k+1/2}$ are both defined on $s = (k + \frac{1}{2})\Delta s$, and $\boldsymbol{F}^n_k$ is defined on $s = k\Delta s$.

Step2 Apply the boundary force to the fluid

$$\boldsymbol{f}^n(\boldsymbol{x}) = \sum_k \boldsymbol{F}^n_k \delta_h(\boldsymbol{x} - \boldsymbol{X}^n_k)\Delta s.$$

Step3 Solve the Navier-Stokes equations with the force to update the velocity

$$\rho \left(\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} + \sum_{i=1}^{2} u^n_i D^{\pm}_i \boldsymbol{u}^n \right) = -D^0 p^{n+1} + \mu \sum_{i=1}^{2} D^+_i D^-_i \boldsymbol{u}^{n+1} + \boldsymbol{f}^n,$$

$$D^0 \cdot \boldsymbol{u}^{n+1} = 0,$$

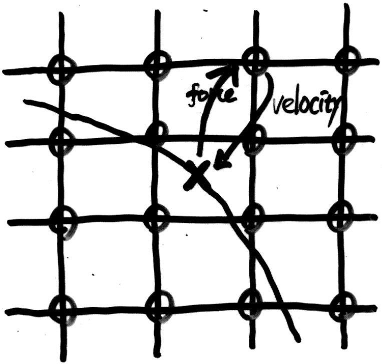
where $T^n_{k+1/2}$ and $\boldsymbol{\tau}^n_{k+1/2}$ are both defined on $s = (k + \frac{1}{2})\Delta s$, and $\boldsymbol{F}^n_k$ is defined on $s = k\Delta s$.

Numerical algorithm, cont.

Step4 Interpolate the new velocity on the lattice into the boundary points and move the boundary points to new positions.

$$U_k^{n+1} = \sum_{x} u^{n+1} \delta_h (x - X_k^n) h^2$$

$$X_k^{n+1} = X_k^n + \Delta t U_k^{n+1}$$



Discrete delta function δ_h

$\delta_h(\boldsymbol{x}) = \delta_h(x)\delta_h(y)$

1. δ_h is positive and continuous function.
2. $\delta_h(x) = 0$, for $|x| \geq 2h$.
3. $\sum_j \delta_h(x_j - \alpha)h = 1$ for all α.
   $\left(\sum_{j \text{ even}} \delta_h(x_j - \alpha)h = \sum_{j \text{ odd}} \delta_h(x_j - \alpha)h = \frac{1}{2} \right)$
4. $\sum_j (x_j - \alpha)\delta_h(x_j - \alpha)h = 0$ for all α.
5. $\sum_j \left(\delta_h(x_j - \alpha)h\right)^2 = C$ for all α.
   $\left(\sum_j \delta_h(x_j - \alpha)\delta_h(x_j - \beta)h^2 \leq C\right)$

Uniquely determined: $C = \frac{3}{8}$.

$$\delta_h(x) = \begin{cases} \frac{1}{8h}\left(3 - 2\left|x\right|/h + \sqrt{1 + 4\left|x\right|/h - 4(\left|x\right|/h)^2}\right) & |x| \leq h, \\ \frac{1}{8h}\left(5 - 2\left|x\right|/h - \sqrt{-7 + 12\left|x\right|/h - 4(\left|x\right|/h)^2}\right) & h \leq |x| \leq 2h, \\ 0 & \text{otherwise.} \end{cases}$$

Numerical issues of IB method

- Simple and easy to implement
- Embedding complex structure into Cartesian domain, no complicated grid generation
- Fast elliptic solver (FFT) on Cartesian grid can be applied
- Numerical smearing near the immersed boundary
- First-order accurate, accuracy of 1D IB model (Beyer & LeVeque 1992, Lai 1998), formally second-order scheme (Lai & Peskin 2000, Griffith & Peskin 2005)
- Adaptive IB method, (Roma, Peskin & Berger 1999, Griffith et. al. 2007)
- Immersed Interface Method (IIM, LeVeque & Li 1994), 3D jump conditions (Lai & Li 2001)
- High-order discrete delta function in 2D, 3D
- Numerical stability tests, different semi-implicit methods (Tu & Peskin 1992, Mayo & Peskin 1993, Newren, Fogelson, Guy & Kirby 2007, 2008)
- Stability analysis (Stockie & Wetton 1999, Hou & Shi 2008)
- Convergence analysis (Y. Mori 2008)

Review articles

- C. S. Peskin, The immersed boundary method, *Acta Numerica*, pp 1-39, (2002)
- R. Mittal & G. Iaccarino, Immersed boundary methods, *Annu. Rev. Fluid Mech.* 37:239-261, (2005), Flow around solid bodies

Accuracy, stability and convergence

- R. P. Beyer & R. J. LeVeque, Analysis of a one-dimensional model for the immersed boundary method, *SIAM J. Numer. Anal.*, 29:332-364, (1992)
- E. P. Newren, A. L. Fogelson, R. D. Guy & R. M. Kirby, Unconditionally stable discretizations of the immersed boundary equations, *JCP*, 222:702-719, (2007)
- Y. Mori, Convergence proof of the velocity field for a Stokes flow immersed boundary method, *CPAM*, vol LXI 1213-1263, (2008)

Application: Flow around a solid body

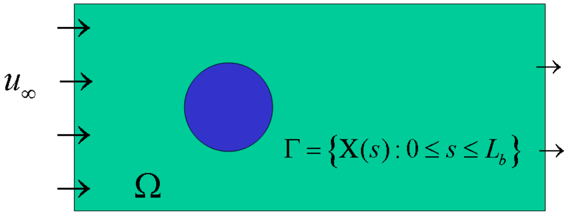


Figure: The fluid feels the force along the surface body to stop it.

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \frac{1}{Re}\Delta \boldsymbol{u} + \boldsymbol{f}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\boldsymbol{f}(\boldsymbol{x}, t) = \int_0^{L_b} \boldsymbol{F}(\boldsymbol{X}(s), t)\delta(\boldsymbol{x} - \boldsymbol{X}(s))ds$$

$$0 = \boldsymbol{u}_b(\boldsymbol{X}(s), t) = \int_\Omega \boldsymbol{u}(\boldsymbol{x}, t)\delta(\boldsymbol{x} - \boldsymbol{X}(s))d\boldsymbol{x}$$

$$\boldsymbol{u}(\boldsymbol{x}, t) \to \boldsymbol{u}_\infty \text{ as } |\boldsymbol{x}| \to \infty$$

$$
\begin{aligned}
\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^{n}}{\Delta t}+(\boldsymbol{u}^{n}\cdot\nabla)\boldsymbol{u}^{n} &= -\nabla p^{n+1}+\frac{1}{Re}\Delta\boldsymbol{u}^{n+1}+\boldsymbol{f}^{n+1},\\
\nabla\cdot\boldsymbol{u}^{n+1} &= 0,\\
\boldsymbol{f}^{n+1}(\boldsymbol{x}) &= \sum_{j=1}^{M}\boldsymbol{F}^{n+1}(\boldsymbol{X}_j)\delta_h(\boldsymbol{x}-\boldsymbol{X}_j)\Delta s\ ,\ \text{for all } \boldsymbol{x},\\
\boldsymbol{U}_b^{n+1}(\boldsymbol{X}_k) &= \sum_{\boldsymbol{x}}\boldsymbol{u}^{n+1}(\boldsymbol{x})\delta_h(\boldsymbol{x}-\boldsymbol{X}_k)h^2, \quad k=1,\ldots,M.
\end{aligned}
$$

One can solve the above linear system directly or approximately (A projection approach, Taira & Colonius, 2007).

Feedback forcing approach

Goldstein, et. al. 1993; Saiki & Biringen 1996; Lai & Peskin 2000

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \frac{1}{Re}\Delta \boldsymbol{u} + \int_\Gamma \boldsymbol{F}(s,t)\delta(\boldsymbol{x} - \boldsymbol{X}(s,t))ds$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\boldsymbol{F}(s,t) = K\int_0^t (\boldsymbol{U}_e(s,t') - \boldsymbol{U}(s,t'))dt' + R(\boldsymbol{U}_e(s,t) - \boldsymbol{U}(s,t))$$

$$\text{or} \qquad \boldsymbol{F}(s,t) = K(\boldsymbol{X}_e(s) - \boldsymbol{X}(s,t)), \quad K \gg 1$$

$$\frac{\partial \boldsymbol{X}(s,t)}{\partial t} = \boldsymbol{U}(s,t) = \int_\Omega \boldsymbol{u}(\boldsymbol{x},t)\delta(\boldsymbol{x} - \boldsymbol{X}(s,t))d\boldsymbol{x}$$

$$\boldsymbol{u}(\boldsymbol{x},t) \to \boldsymbol{u}_\infty \text{ as } |\boldsymbol{x}| \to \infty$$

- Treat the body surface as a nearly rigid boundary (stiffness is large) embedded in the fluid
- Allow the boundary to move a little but the force will bring it back to the desired location
- Be able to handle complicated geometry with known boundary velocity

Interpolating forcing approach

Su, Lai & Lin 2007

Idea: To compute the boundary force $\boldsymbol{F}^*(\boldsymbol{X}_k)$ accurately so the prescribed boundary velocity can be achieved in the intermediate step of projection method.

1. Distribute the force to the grid by the discrete delta function
$$\boldsymbol{f}^*(\boldsymbol{x}) = \sum_{j=1}^{M} \boldsymbol{F}^*(\boldsymbol{X}_j)\delta_h(\boldsymbol{x} - \boldsymbol{X}_j)\Delta s$$
2. $\dfrac{\boldsymbol{u}^{**} - \boldsymbol{u}^*}{\Delta t} = \boldsymbol{f}^*$, thus, $\boldsymbol{u}^{**}(\boldsymbol{X}_k) = \boldsymbol{u}_b(\boldsymbol{X}_k)$.

$$\begin{aligned}
\frac{\boldsymbol{u}_b(\boldsymbol{X}_k) - \boldsymbol{u}^*(\boldsymbol{X}_k)}{\Delta t} &= \sum_{\boldsymbol{x}} \boldsymbol{f}^*(\boldsymbol{x})\delta_h(\boldsymbol{x} - \boldsymbol{X}_k)h^2 \\
&= \sum_{\boldsymbol{x}} \left(\sum_{j=1}^{M} \boldsymbol{F}^*(\boldsymbol{X}_j)\delta_h(\boldsymbol{x} - \boldsymbol{X}_j)\Delta s \right) \delta_h(\boldsymbol{x} - \boldsymbol{X}_k)h^2 \\
&= \sum_{j=1}^{M} \left(\sum_{\boldsymbol{x}} \delta_h(\boldsymbol{x} - \boldsymbol{X}_j)\delta_h(\boldsymbol{x} - \boldsymbol{X}_k)h^2\Delta s \right) \boldsymbol{F}^*(\boldsymbol{X}_j)
\end{aligned}$$

Re	Su et al.	Lai & Peskin	Silva et al.	Ye et al.	Willimson(exp)
80	0.153	-	0.15	0.15	0.15
100	0.168	0.165	0.16	-	0.166
150	0.187	0.184	0.18	-	0.183

Table: The comparison of Strouhal number for different Reynolds numbers.