Electrical impedance tomography with two electrodes

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joint work with H. Hakula, M. Hanke, L. Harhanen, B. Harrach and S. Hollborn.

Outline of the talk

1. Backscatter data, sweep data and their motivation.
2. Localization of inhomogeneities.
   (a) Analytic continuation of the data.
   (b) Numerical examples.

1. Backscatter and sweep data

General form of the considered data

Let $D \subset \mathbb{R}^2$ be the open unit disk with a strictly positive conductivity $\sigma \in L^\infty(D)$ such that $\Omega := \operatorname{supp}(\sigma - 1)$ is a compact subset of D. We consider the Neumann problem

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } D, \qquad \frac{\partial u}{\partial \nu} = f \quad \text{on } \partial D$$

where $f \in H^s_\diamond(\partial D)$, $s \in \mathbb{R}$, is the input current density. These equations define the potential $u \in H^{\min\{1,s+3/2\}}(D)/\mathbb{C}$ uniquely.

We denote the reference potential, i.e., the solution for $\sigma \equiv 1$, by $u_0 \in H^{s+3/2}(D)/\mathbb{C}$.

It follows from the regularity theory for elliptic partial differential equations that the difference Neumann-to-Dirichlet map

$$\Lambda - \Lambda_0 : f \mapsto (u - u_0)|_{\partial D}$$

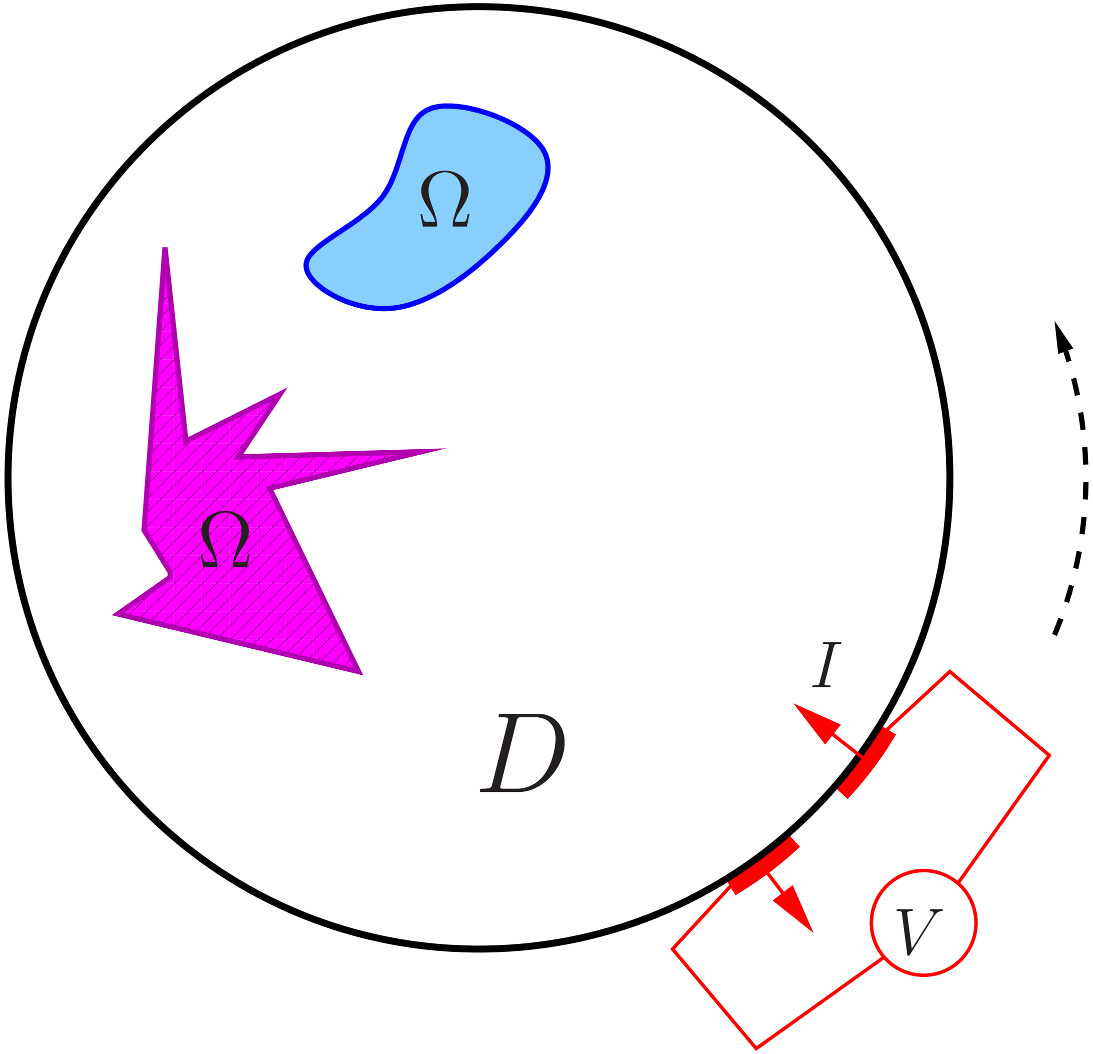
is bounded (and compact) between $H^s_\diamond(\partial D)$ and $H^r(\partial D)/\mathbb{C}$ for any $s, r \in \mathbb{R}$.

In what follows, we consider two types of EIT boundary measurements that can be presented in the form

$$\mathrm{data}(\theta) \;=\; \langle f_\theta, (\Lambda - \Lambda_0) f_\theta \rangle_{\partial D},$$

for suitable families of distributional boundary currents $\{f_\theta\}$ parametrized by θ.

Backscatter measurement



Backscatter data

Let $\delta'_\theta \in H_\diamond^{-3/2-\epsilon}(\partial D)$, $\epsilon > 0$, be a dipole boundary current applied at $z_\theta := (\cos\theta, \sin\theta) \in \partial D$, i.e.,

$$\langle \delta'_\theta, g \rangle_{\partial D} = -\frac{\partial g}{\partial \tau}(z_\theta) \qquad \text{for } g \in H^{3/2+\epsilon}(\partial D),$$

where τ is the arc length parameter of ∂D.

We define the *backscatter data* of electric impedance tomography to be the function

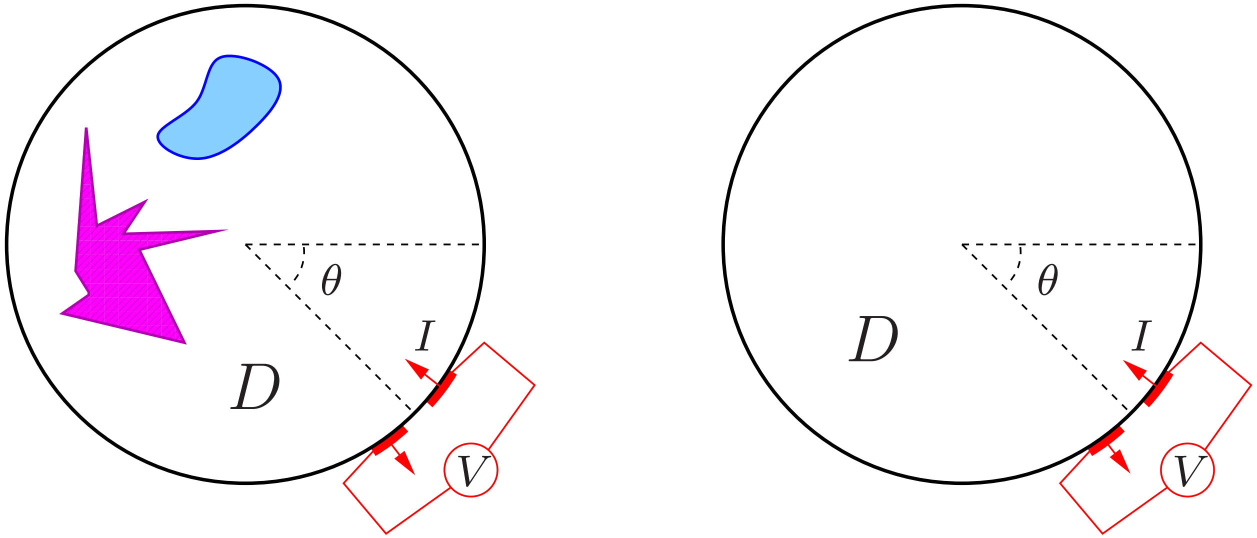
$$b : z_\theta \mapsto \langle \delta'_\theta, (\Lambda - \Lambda_0)\delta'_\theta \rangle_{\partial D}, \quad \partial D \to \mathbb{R},$$

or in other words,

$$b(z_\theta) = -\frac{\partial w_\theta|_{\partial D}}{\partial \tau}(z_\theta),$$

where $w_\theta := u - u_0$ is the relative potential corresponding to the dipole boundary current $f = \delta'_\theta$ at z_θ.

Motivation of the backscatter data

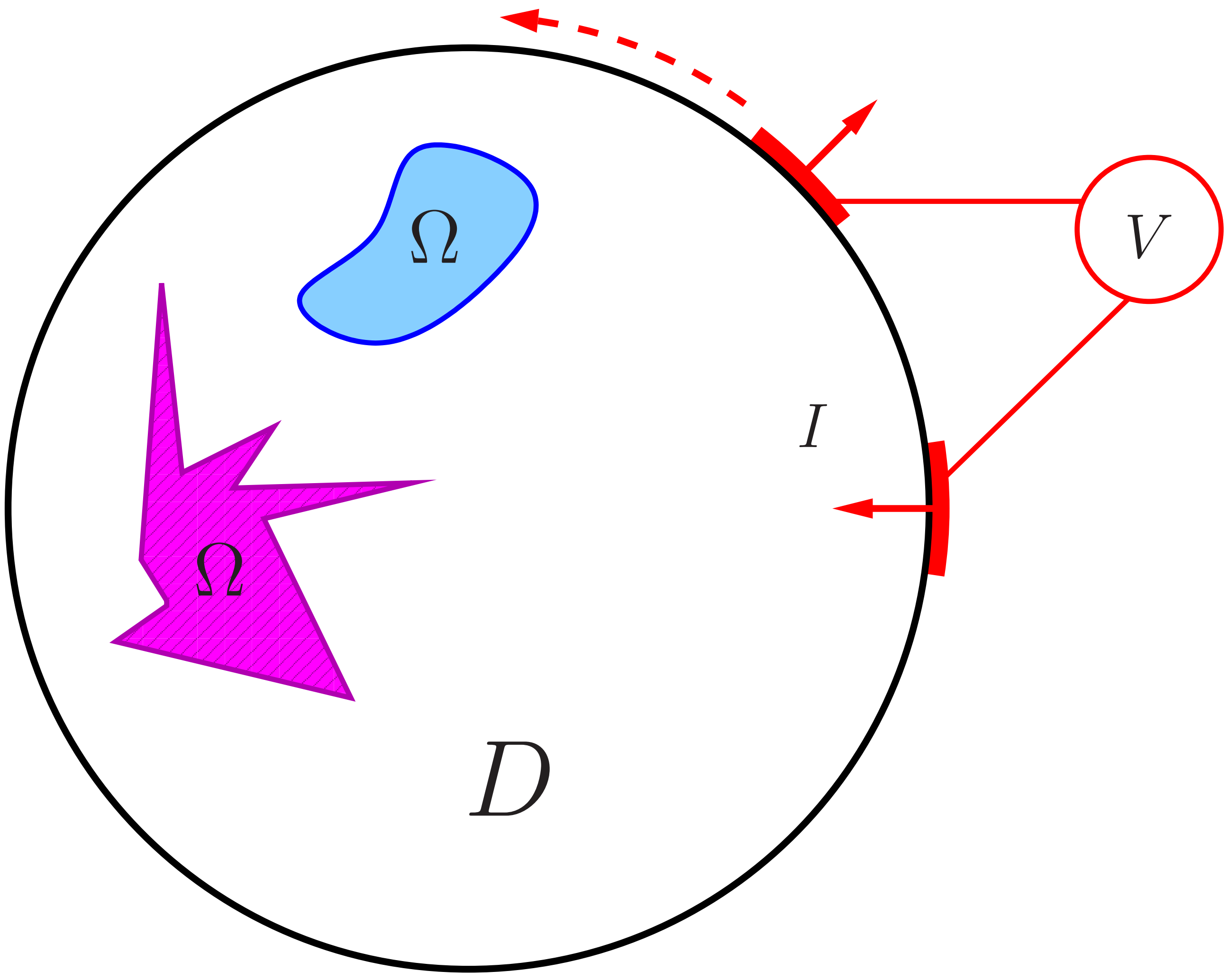


Suppose that the available measurement $M(z_\theta)$ is the reading of the voltmeter on the left minus that on the right. According to the so-called *complete electrode model*, it holds that

$$M(z_\theta) = 4h^2 b(z_\theta) + O(h^3).$$

Hence, the backscatter data may be approximated by real-world electrode measurements — at least to a certain extent.

Sweep measurement



Sweep data

Let $\delta_\theta - \delta_0 \in H_\diamond^{-1/2-\epsilon}(\partial D)$, $\epsilon > 0$, be difference of two point currents at $z_\theta, z_0 \in \partial D$, respectively, i.e.,

$$\langle \delta_\theta - \delta_0, g \rangle_{\partial D} = g(z_\theta) - g(z_0) \qquad \text{for } g \in H^{1/2+\epsilon}(\partial D).$$

We define the *sweep data* of electric impedance tomography to be the function

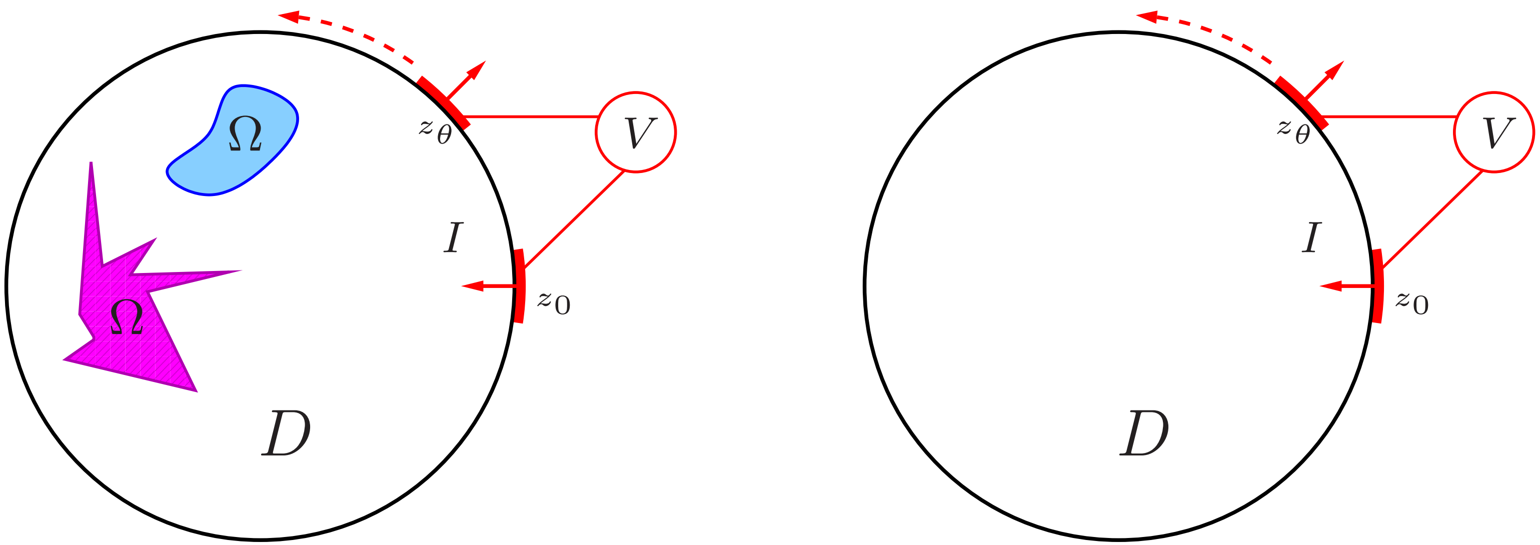
$$\varsigma : z_\theta \mapsto \langle \delta_\theta - \delta_0, (\Lambda - \Lambda_0)(\delta_\theta - \delta_0) \rangle_{\partial D}, \quad \partial D \to \mathbb{R},$$

or in other words,

$$\varsigma(z_\theta) = w_\theta(z_\theta) - w_\theta(z_0),$$

where $w_\theta := u - u_0$ is the relative potential corresponding to the boundary current $f = \delta_\theta - \delta_0$.

Motivation of the sweep data



Suppose that the available measurement $M(z_\theta)$ is the reading of the voltmeter on the left minus that on the right. According to the so-called *complete electrode model* of electrical impedance tomography, it holds that

$$M(z_\theta) = \varsigma(z_\theta) + O(h^2),$$

where $h > 0$ is the width of the electrodes.

Differences/similarities between the two data types

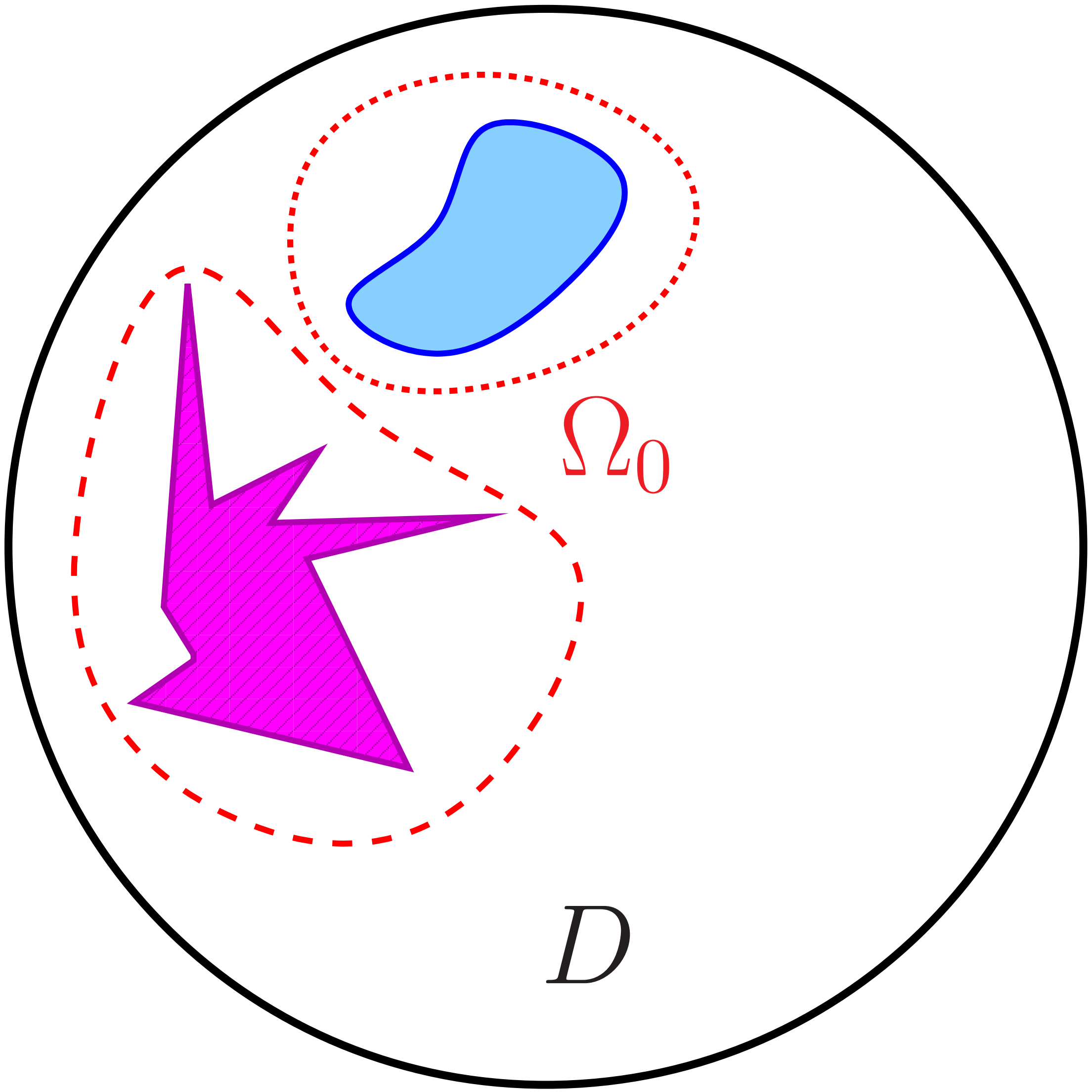
- The backscatter data uniquely determines a simply connected insulating cavity within D (but not an ideally conducting inclusion!). There are currently no analogous results for the sweep data.

- It can be shown that both the backscatter data and the sweep data are boundary values of holomorphic functions living in the exterior of the conductivity inhomogeneity.

- As sweep data arguably corresponds to a more practical measurement setting, we will consider it in the following.

2. Localization of inhomogeneities

(a) Analytic continuation of the data



Ω_0
D

A factorization of $\Lambda - \Lambda_0$

Let $\Omega_0 \subset \mathbb{R}^2$ consist of m smooth, well separated and simply connected components and be such that $\Omega = \mathrm{supp}(\sigma - 1) \subset \Omega_0$ and $\overline{\Omega}_0 \subset D$. We define an auxiliary operator

$$B : f \mapsto u_0|_{\partial\Omega_0}, \quad H^s_\diamond(\partial D) \to H^r(\partial\Omega_0)/\mathbb{C}^m, \qquad s, r \in \mathbb{R},$$

where u_0 is the reference potential corresponding to the boundary current f.

It turns out that $\Lambda - \Lambda_0$ obeys the factorization

$$\Lambda - \Lambda_0 = B^* G B,$$

where $G : H^r(\partial\Omega_0)/\mathbb{C}^m \to H^{-r}_{\diamond\diamond}(\partial\Omega_0)$ is bounded for any $r \in \mathbb{R}$ and coincides with it own dual operator.

Analytic continuation of $B(\delta_\theta - \delta_0)$

The reference potential corresponding to the current density $\delta_\theta - \delta_0$ can be given explicitly, which results in the representation

$$(B(\delta_\theta - \delta_0))(x) \;=\; \frac{1}{\pi}(\log|x - z_0| - \log|x - z_\theta|), \qquad x \in \partial\Omega_0.$$

By introducing the complex numbers $\xi(x) = x_1 + \mathrm{i}x_2$ and $\zeta = e^{\mathrm{i}\theta}$, this can be written as

$$(B(\delta_\theta - \delta_0))(x) \;=\; \frac{1}{2\pi}\left(\log\frac{|1-\xi|^2}{1-\bar{\xi}\zeta} + \log\frac{\zeta}{\zeta - \xi}\right), \qquad x \in \partial\Omega_0,$$

where $\log$ denotes the principal value of the complex logarithm.

Taking advantage of the fact that we are allowed to consider $B(\delta_\theta - \delta_0)$ as an element of $H^r(\partial\Omega_0)/\mathbb{C}^m$, we may add a suitable function of ζ to $B(\delta_\theta - \delta_0)$ on each component of $\partial\Omega_0$ in order to move the branch cut of the latter logarithm of the above expression entirely inside Ω_0. (This is actually an oversimplification of the employed procedure.)

This results in the representation ($\zeta = e^{\mathrm{i}\theta}$)

$$(B(\delta_\theta - \delta_0))(x) \;=\; g(x,\zeta), \qquad (x,\zeta) \in \partial\Omega_0 \times \partial D,$$

which extends as a continuous function to $\partial\Omega_0 \times \overline{D} \setminus \overline{\Omega}_0$. Moreover, $g(x,\zeta)$ is complex differentiable with respect to its second variable.

Analytic continuation of the sweep data

Due to the above material, we have

$$\varsigma(\zeta) = \langle B(\delta_\theta - \delta_0), GB(\delta_\theta - \delta_0)\rangle_{\partial\Omega_0} = \int_{\partial\Omega_0} g(x,\zeta)[Gg(\cdot,\zeta)](x)\, ds_x,$$

where $\zeta = e^{\mathrm{i}\theta}$. It thus follows 'easily' from basic results on (complex) line integrals that ς extends as a holomorphic function to $D \setminus \overline{\Omega}_0$.

Since Ω_0 is an (rather) arbitrary set enclosing $\Omega = \mathrm{supp}(\sigma - 1)$, it is straightforward to conclude that ς actually extends as a univalent holomorphic function to $D \setminus \Omega$, under only mild topological conditions on Ω.

Non-complex interpretation

By considering the real part of the extension of ς to $D \setminus \Omega$ and noting that the corresponding imaginary part (and thus its tangential derivative) vanishes on ∂D, we obtain the following theorem.

Theorem. *There exists a solution to the Cauchy problem*

$$\Delta u = 0 \quad \text{in } D \setminus \Omega, \qquad u = \varsigma \quad \text{on } \partial D, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D,$$

if $\Omega = \operatorname{supp}(\sigma - 1)$ *is regular enough.* (Otherwise, we may consider some slightly larger set instead of Ω, e.g., its convex hull.)

This result generalizes for a general smooth and simply connected domain $D \subset \mathbb{R}^2$ since conformal maps can be used to transfer sweep data between boundaries of different domains.

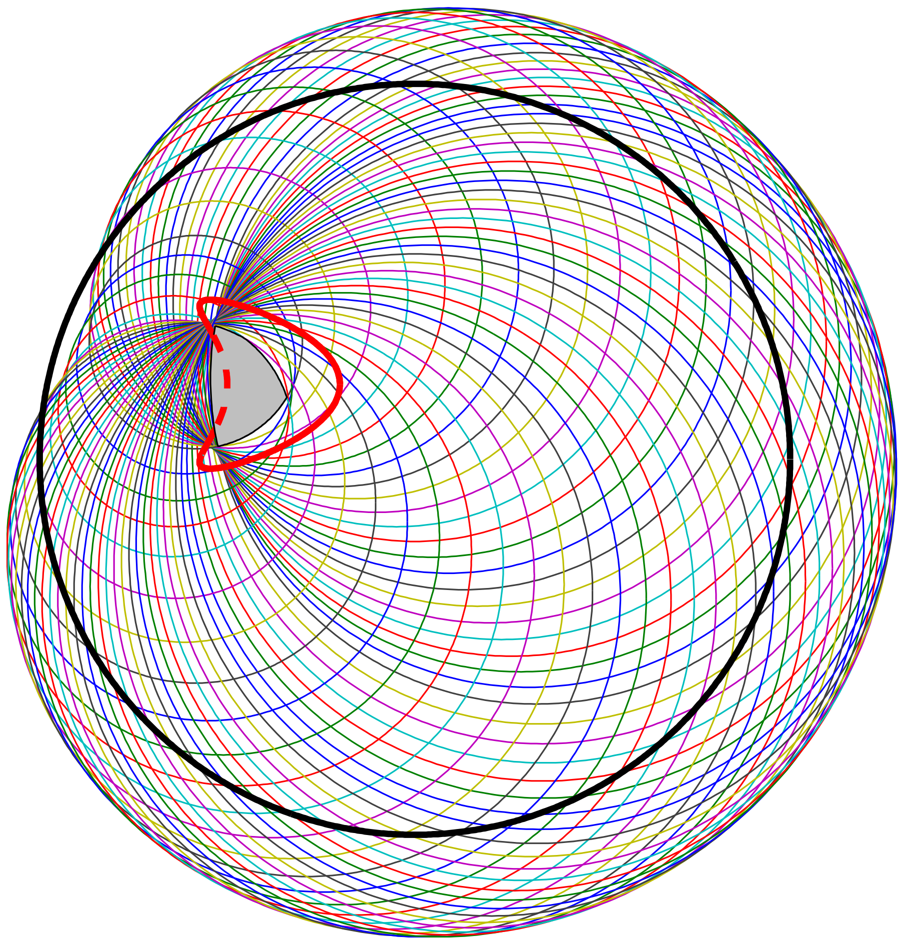
(b) Numerical examples

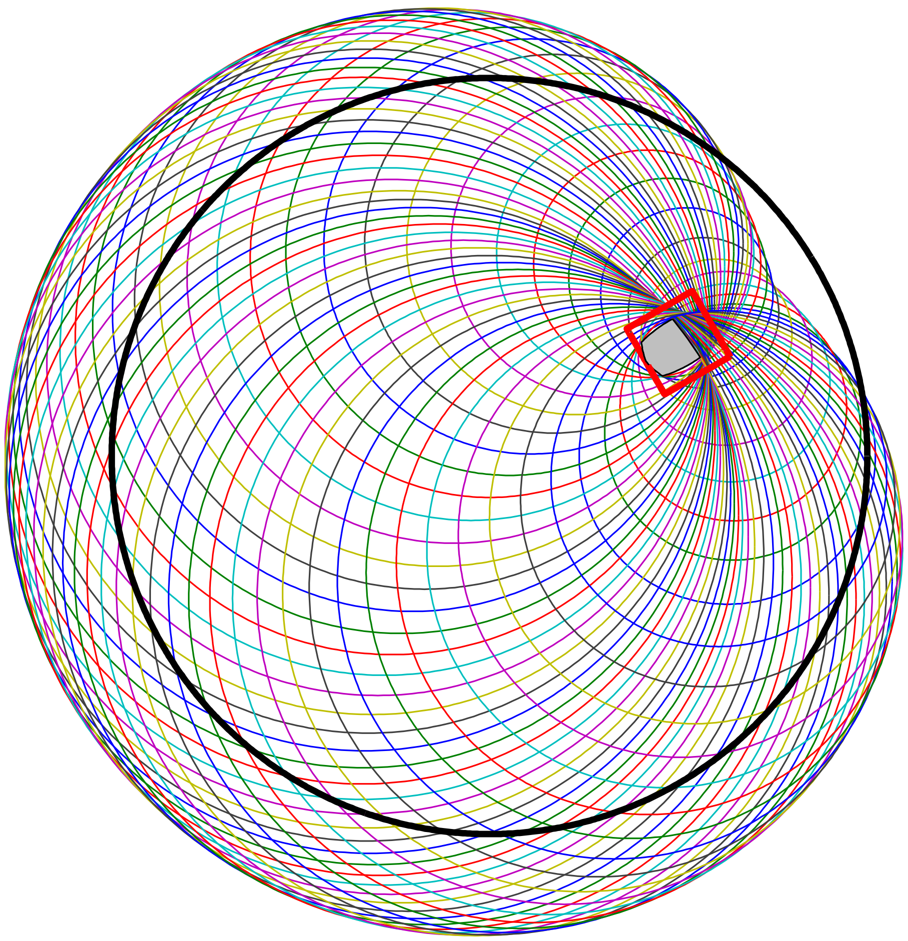
A reconstruction algorithm

Due to the above theorem, the localization of the inhomogeneity Ω from sweep data can be recast as an inverse source problem for the Poisson equation.

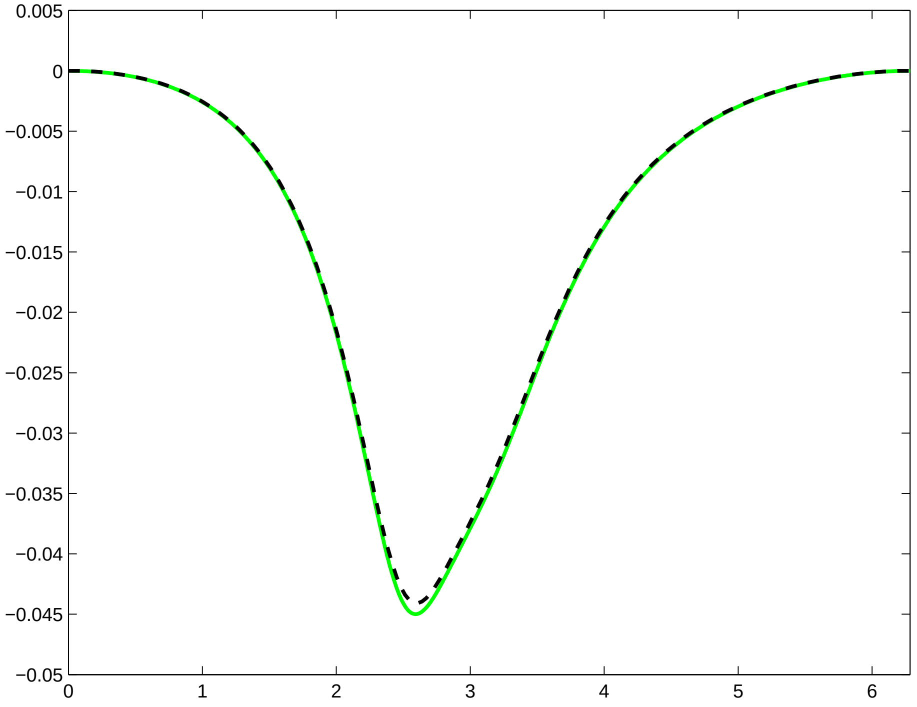
The following reconstructions have been computed using the so-called convex source support algorithm (Kusiak and Sylvester, 2003; Hanke, H, Reusswig, 2008). To put it very short, the leading idea is to use suitable Möbius transformations and Fourier series representations to test whether the Cauchy data $(\varsigma, 0)$ can be continued harmonically up to the boundary of a given closed disk $B \subset \mathbb{R}^2$. The intersection of the disks having this property is then dubbed the reconstruction.

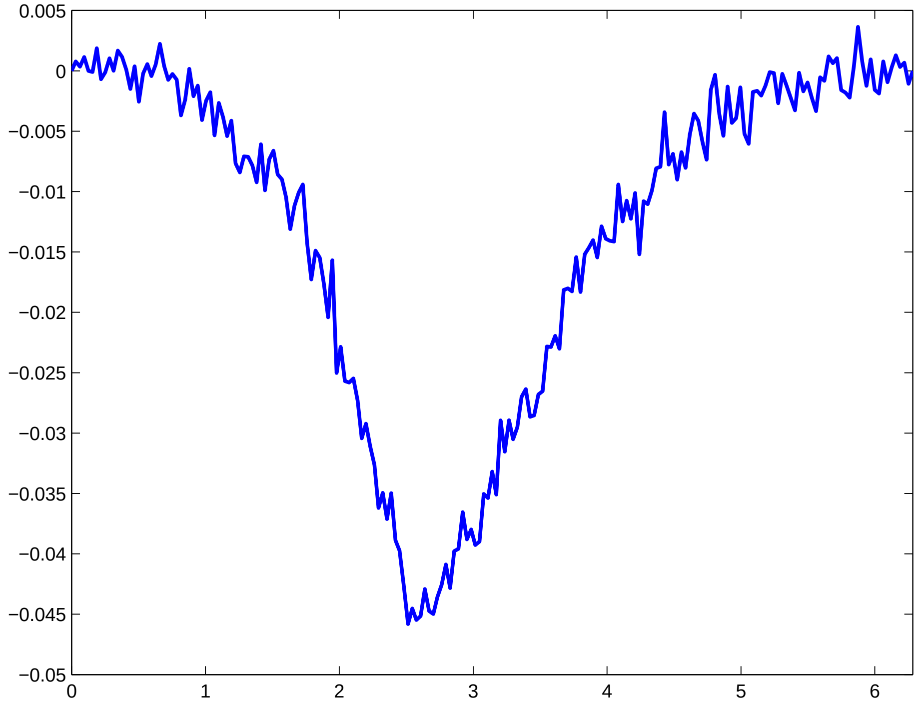
Reconstructions from exact data



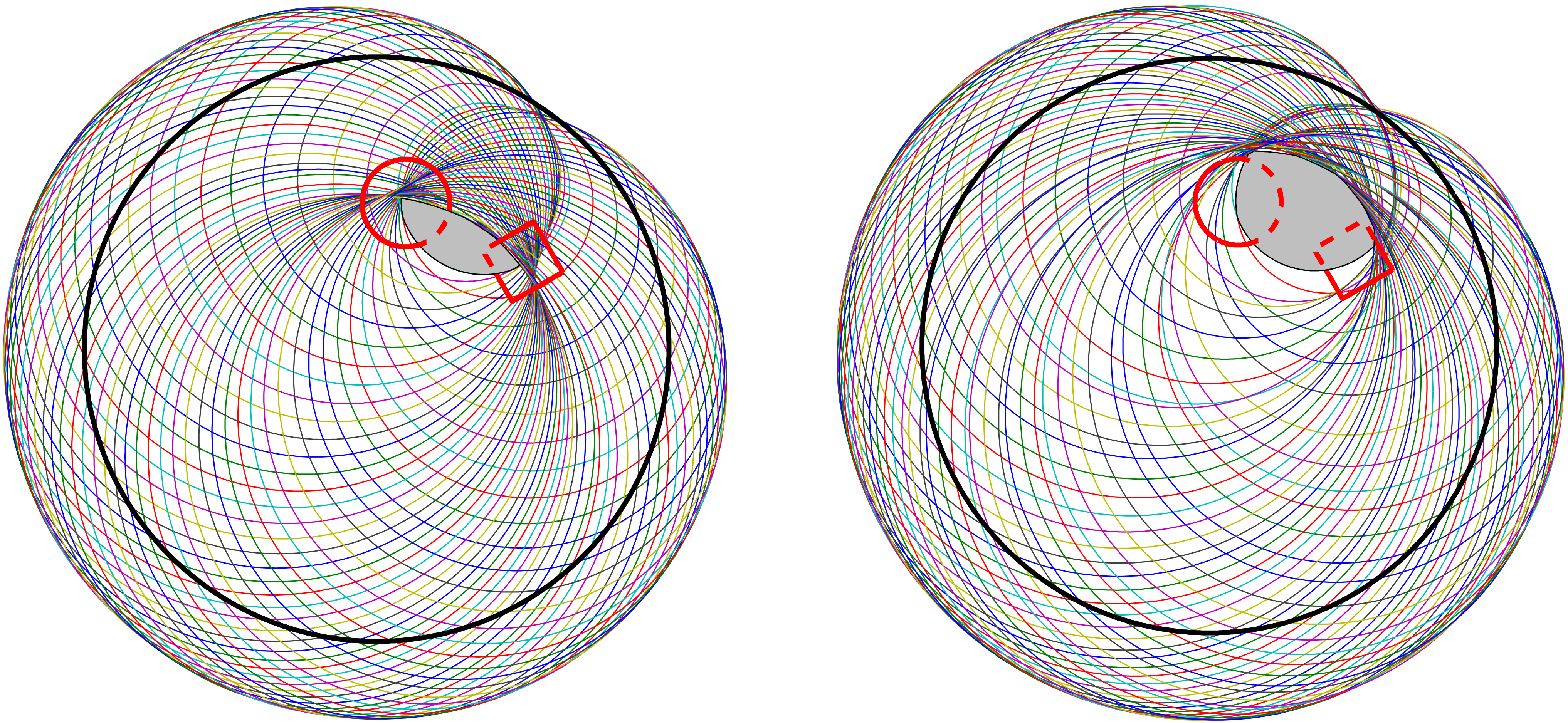


Comparison of exact and CEM data for $h \approx 0.2$





Reconstructions from simulated CEM data



Relevant publications

H. Hakula, L. Harhanen, and N. Hyvönen, *Sweep data of electrical impedance tomography*, submitted.

M. Hanke, N. Hyvönen, and S. Reusswig, *Convex source support and its application to electric impedance tomography*, SIAM Journal on Imaging Sciences, **1**, 364–378 (2008).

M. Hanke, N. Hyvönen, and S. Reusswig, *An inverse backscatter problem for electric impedance tomography*, SIAM Journal on Mathematical Analysis, **41**, 1948-1966 (2009).

M. Hanke, N. Hyvönen, and S. Reusswig, *Convex backscattering support in electric impedance tomography*, Numerische Mathematik, **117**, 373–396 (2011).

M. Hanke, B. Harrach, and N. Hyvönen, *Justification of point electrode models in electrical impedance tomography*, Mathematical Models and Methods in Applied Sciences, accepted.

S. Kusiak and J. Sylvester, *The scattering support*, Communications on Pure and Applied Mathematics, **56**, 1525–1548 (2003).