# Mathematical Methods for Breast Image Registration

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# Outline

1. PDE Approach to Joint Registration and Intensity Correction
   - Introduction
   - Mathematical Formulation
   - A Corresponding PDE
   - Discretization and Numerical Scheme
   - Results and Discussion
2. GN Approach to Joint Registration and Intensity Correction
   - Introduction
   - Mathematical Formulation
   - Discretization and Numerical Scheme
   - Results and Discussion

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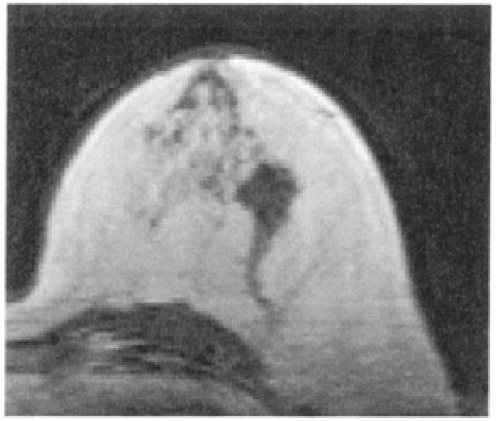
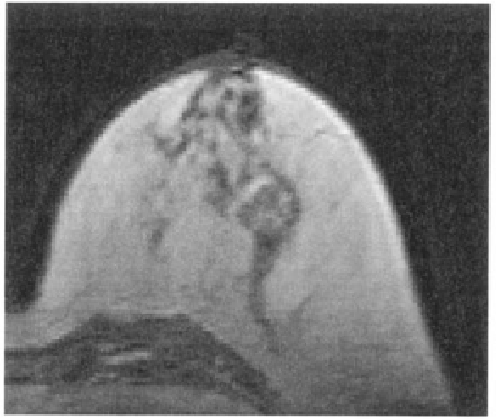
Introduction
Mathematical Formulation
A Corresponding PDE
Discretization and Numerical Scheme
Results and Discussion

# Motivation: Contrast Enhanced MR Breast Registration

*Pre-injection*

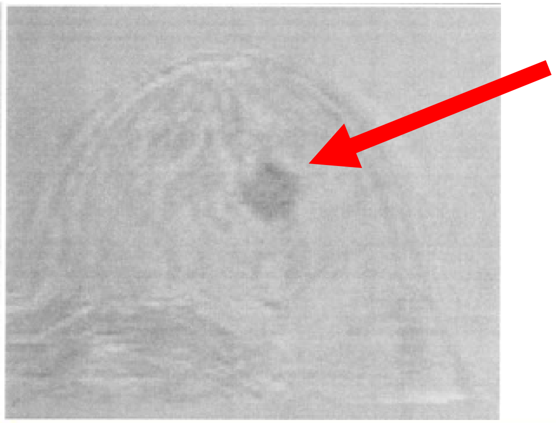


*Post-injection*



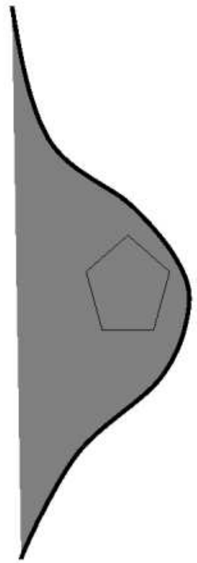
*Subtraction (No- registration)*

*Subtraction after registration*



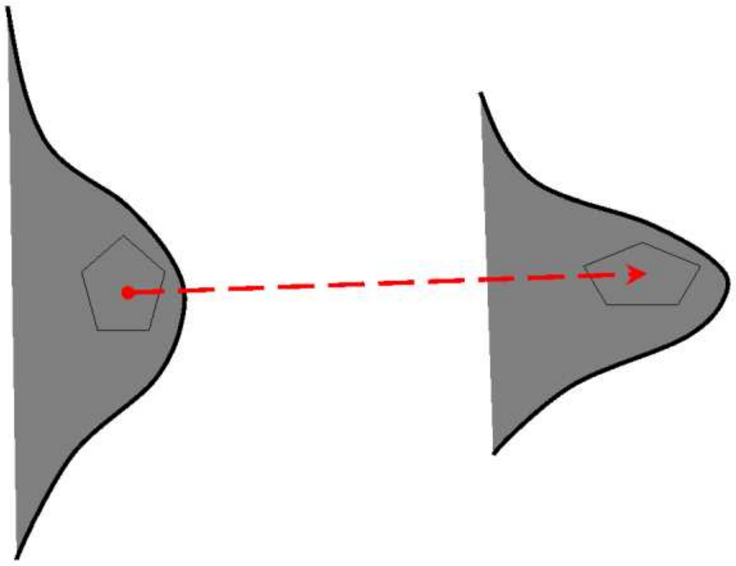
Rueckert et. al., 1999

# Schematic Example



An initial image is given.

## Schematic Example



The initial image is transformed to a second image.

Introduction
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A Corresponding PDE
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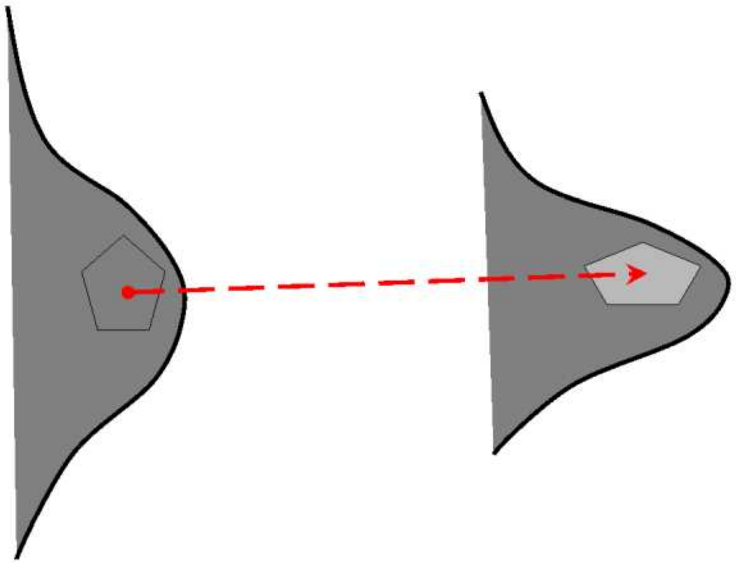
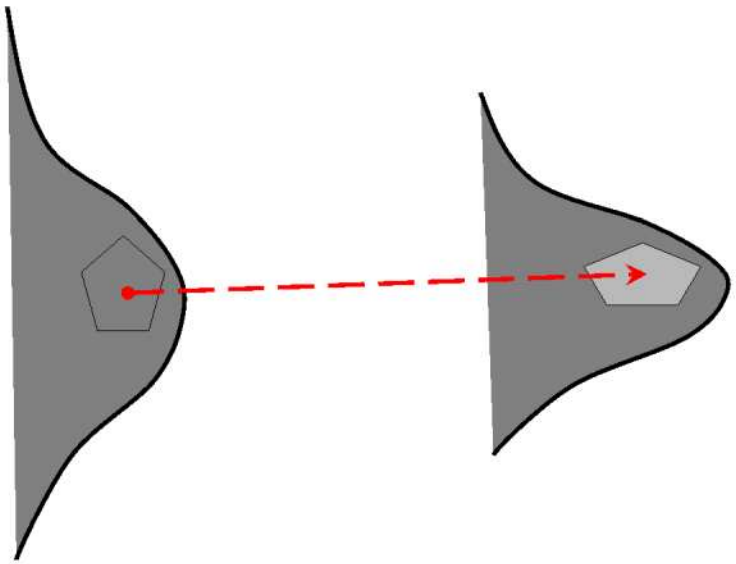
Results and Discussion

## Schematic Example



Parts of the second image have different **contrast** compared to the first.

## Schematic Example



The transformation and the contrast enhancement are the unknowns.

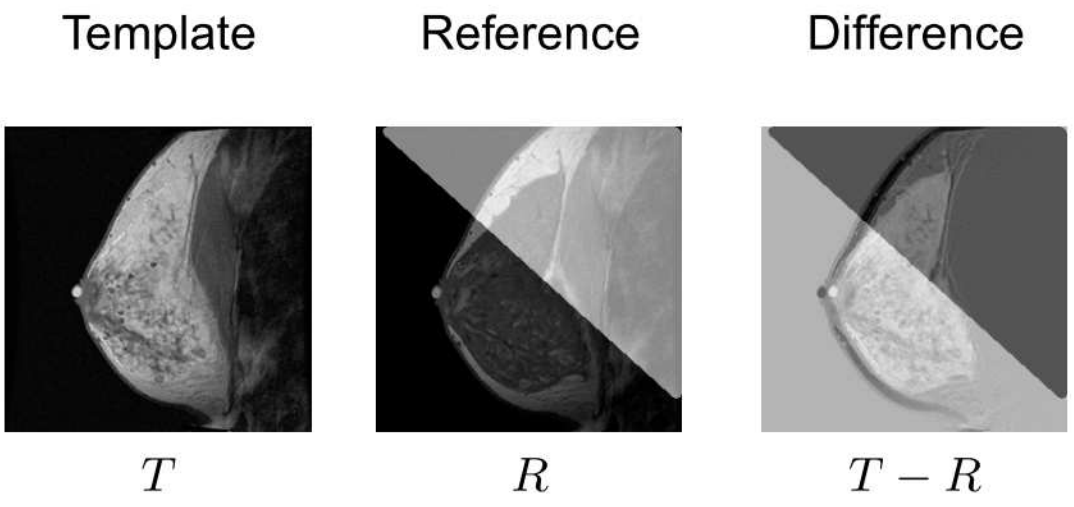
# Literature

Separate the contrast enhancement from the image and regularize it!

- M.A. Gennert, S. Negahdaripour (1987) (Following the seminal work of B.K.P. Horn, B.G. Schunck (1981))
- D.C. Barber, D.R. Hose (2005)
- Modersitzki, J., Papenberg, N. (2005)
- A.L. Martel, M.S. Froh, K. K. Brock, D. B. Plewes (2006-2007)

***The goal: Generalize and present efficient numerical schemes.***

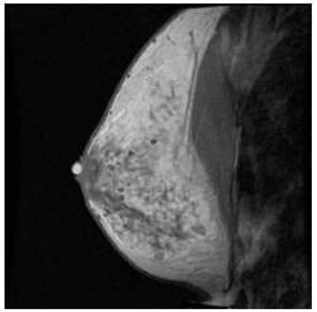
# Problem




Simulated data courtesy of
Dr. Kristy Brock (PMH)

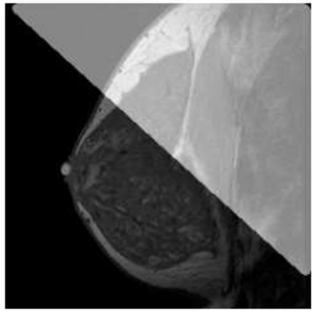
# Problem

Template



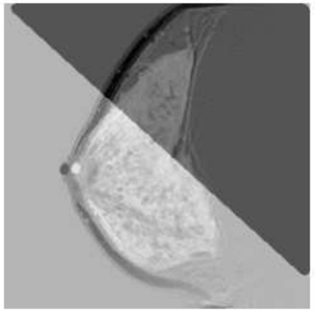
$T$

Reference



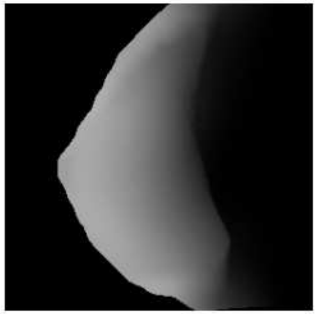
$R$

Difference

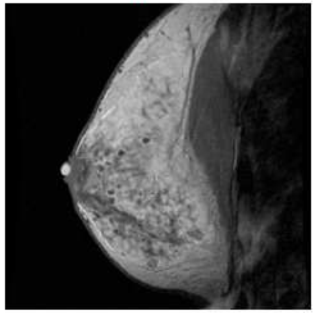


$T - R$

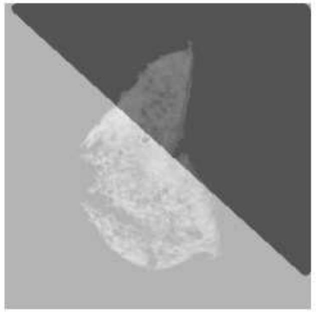
True Displacement ?



Deformed Template ?



True Contrast Enhancement ?



Simulated data courtesy of Dr. Kristy Brock (PMH)

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## Problem

*Given two images $R$ and $T$, find a displacement field $u$ and a contrast enhancement image $w$, that minimizes*

$$\mathcal{J}[u,w] := \mathcal{D}[R,T;u,w] + \mathcal{H}[u,w]$$

*in which $\mathcal{D}$ measures the dissimilarity of $T_u - w$ and $R$, and $\mathcal{H}$ is a regularization expression on $[u,w]$.*

## Problem

*Hence, the objective is to minimize*

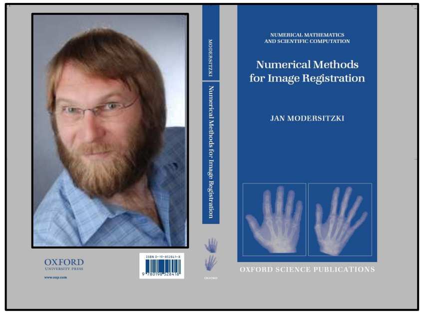
$$\mathcal{J}[u,w] := \frac{1}{2}\,||T_u - R - w||^2_{L_2(\Omega)} + \alpha\mathcal{S}[u] + \beta\mathcal{Q}[w].$$

**Assume diffusion regularization on both**

$$\mathcal{S}[u] := \frac{1}{2}\sum_{j=1}^{d}\int_\Omega \langle \nabla u_j, \nabla u_j\rangle\, dx,$$

$$\mathcal{Q}[w] := \frac{1}{2}\int_\Omega \langle \nabla w, \nabla w\rangle\, dx$$

**The PDE approach:**
**Optimize then Discretize**




*Numerical Methods for Image Registration, J. Modersitzki 2004*

Technical details of the joint PDE approach:
M. Ebrahimi and A. L. Martel, A General PDE-Framework for Registration of Contrast Enhanced Images, MICCAI 2009, *pp.811-819*

## Theorem

*The Euler-Lagrange equations corresponding to $\mathcal{J} = \mathcal{D} + \alpha\mathcal{S} + \beta\mathcal{Q}$, are*

$$\Phi(x, u(x), w(x)) + \alpha\mathcal{A}[u](x) + \beta\mathcal{B}[w](x) = 0, \qquad x \in \Omega,$$

*with Neumann boundary conditions.*
*These can also be written as*

$$[T_u(x) - R(x) - w(x)]\nabla T_u(x) + \alpha\Delta u(x) = 0_{\mathbb{R}^d} \qquad x \in \Omega,$$

$$[T_u(x) - R(x) - w(x)] + \beta\Delta w(x) = 0 \qquad x \in \Omega,$$

# Corresponding PDE

There exists various ways to solve the Euler-Lagrange equations. A possibility is to formulate its solution as the steady-state solution of a PDE. We propose

$$\partial_t (u(x,t), s\, w(x,t)) = \Phi(x, u(x,t), w(x,t)) + \alpha \mathcal{A}[u](x) + \beta \mathcal{B}[w](x)$$
$$x \in \Omega, \quad t \geq 0.$$

where $s$ is a nonzero real scale factor.

# Corresponding PDE

Assuming $\Phi = (f, g)$ this PDE can be written as

$$\partial_t u(x,t) = f(x, u(x,t), w(x,t)) + \alpha \Delta u(x,t), \quad x \in \Omega, \quad t \geq 0,$$
$$s\, \partial_t w(x,t) = g(x, u(x,t), w(x,t)) + \beta \Delta w(x,t), \quad x \in \Omega, \quad t \geq 0,$$

$$f(x,u,w) := [T_u(x) - R(x) - w(x)] \nabla T_u(x),$$
$$g(x,u,w) := [T_u(x) - R(x) - w(x)].$$

# Discretized Numerical Scheme

This yields

**Iterative Scheme**

$$U_j^{k+1} = \Big(I - \tau_1 \alpha A\Big)^{-1} \Big(U_j^k + \tau_1 \Big(T(X - U^k(X)) - R(X) - W^k(X)\Big) \partial_j T(X - U^k(X))\Big),$$

$$W^{k+1} = \Big((1+\tau_2) I - \tau_2 \beta A\Big)^{-1} \Big(W^k + \tau_2 \Big(T(X - U^{k+1}(X)) - R(X)\Big)\Big).$$

We use the initialization vectors

$$W^0 = U_j^0 = 0_{\mathbb{R}^{nd}}, \quad j = 1, \ldots, d.$$

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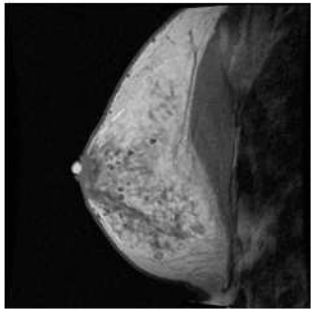
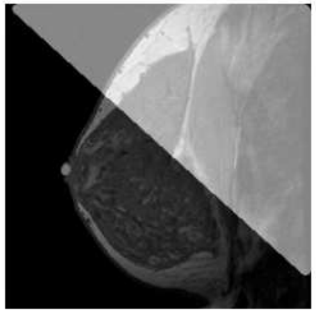
# A Simple Experiment

Template



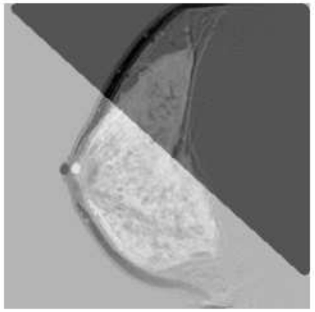
$T$

Reference



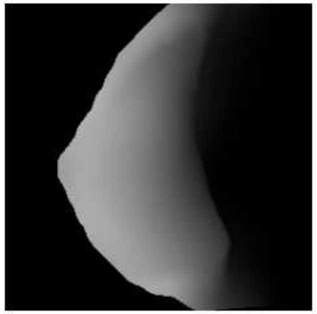
$R$

Difference

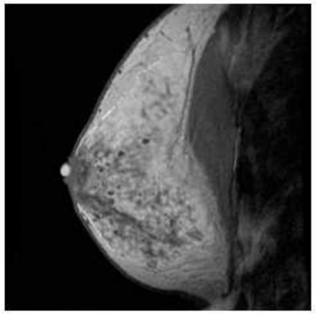


$T - R$

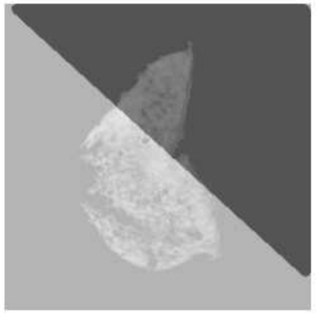
Magnitude of True Displ.



Deformed Template

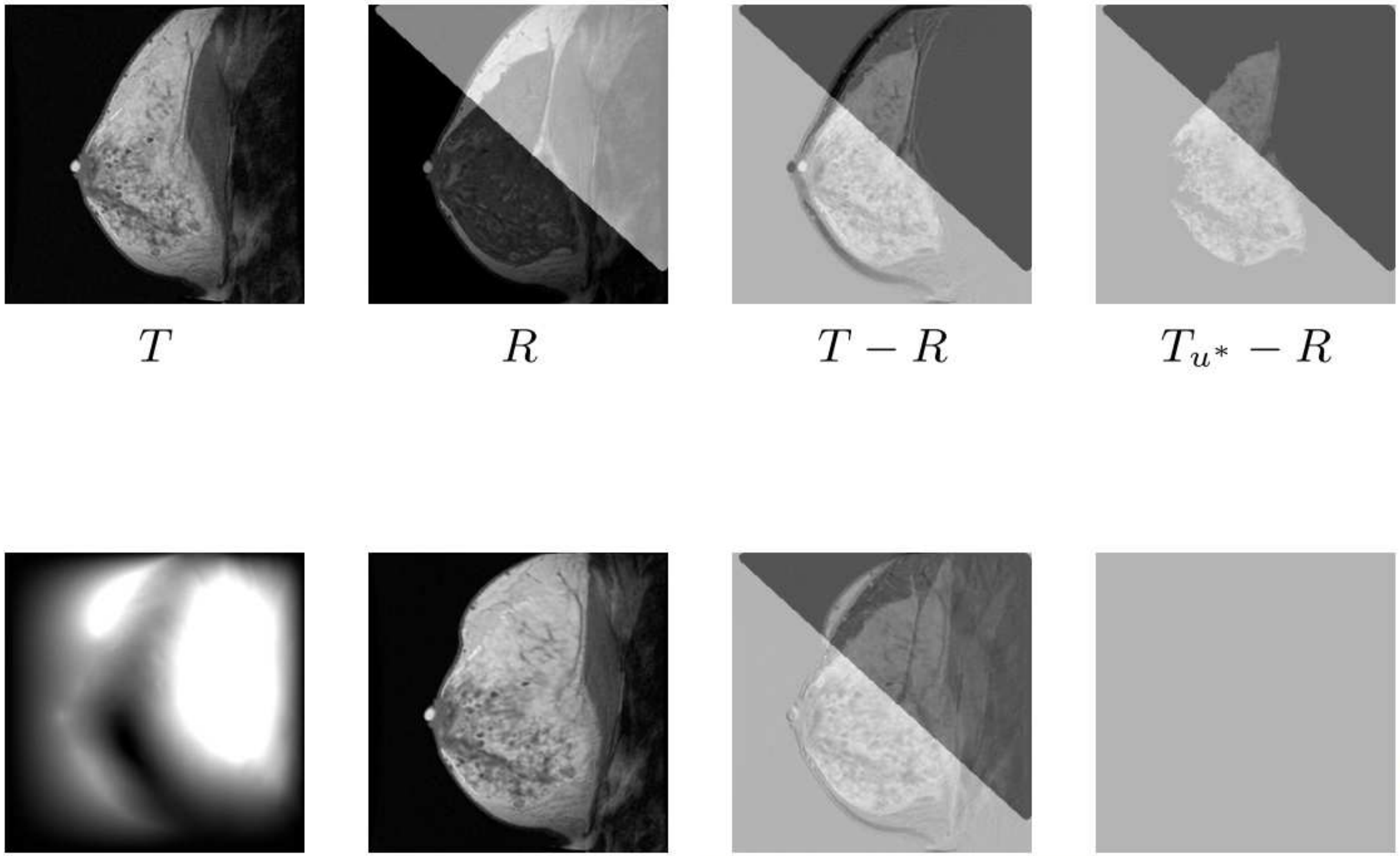


True Contrast Enhancement



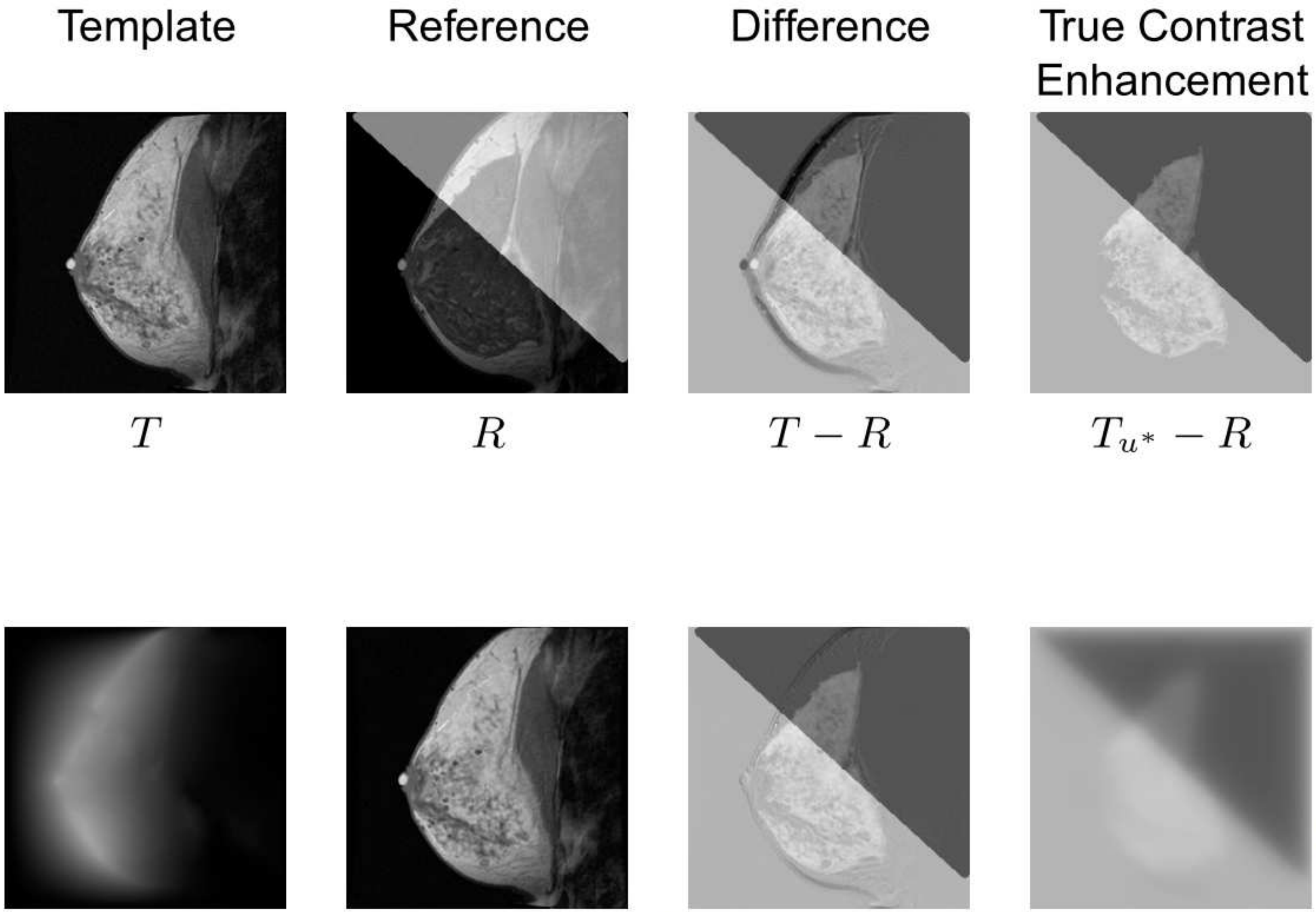
## $\beta = 10^{10}$

Template | Reference | Difference | True Contrast Enhancement

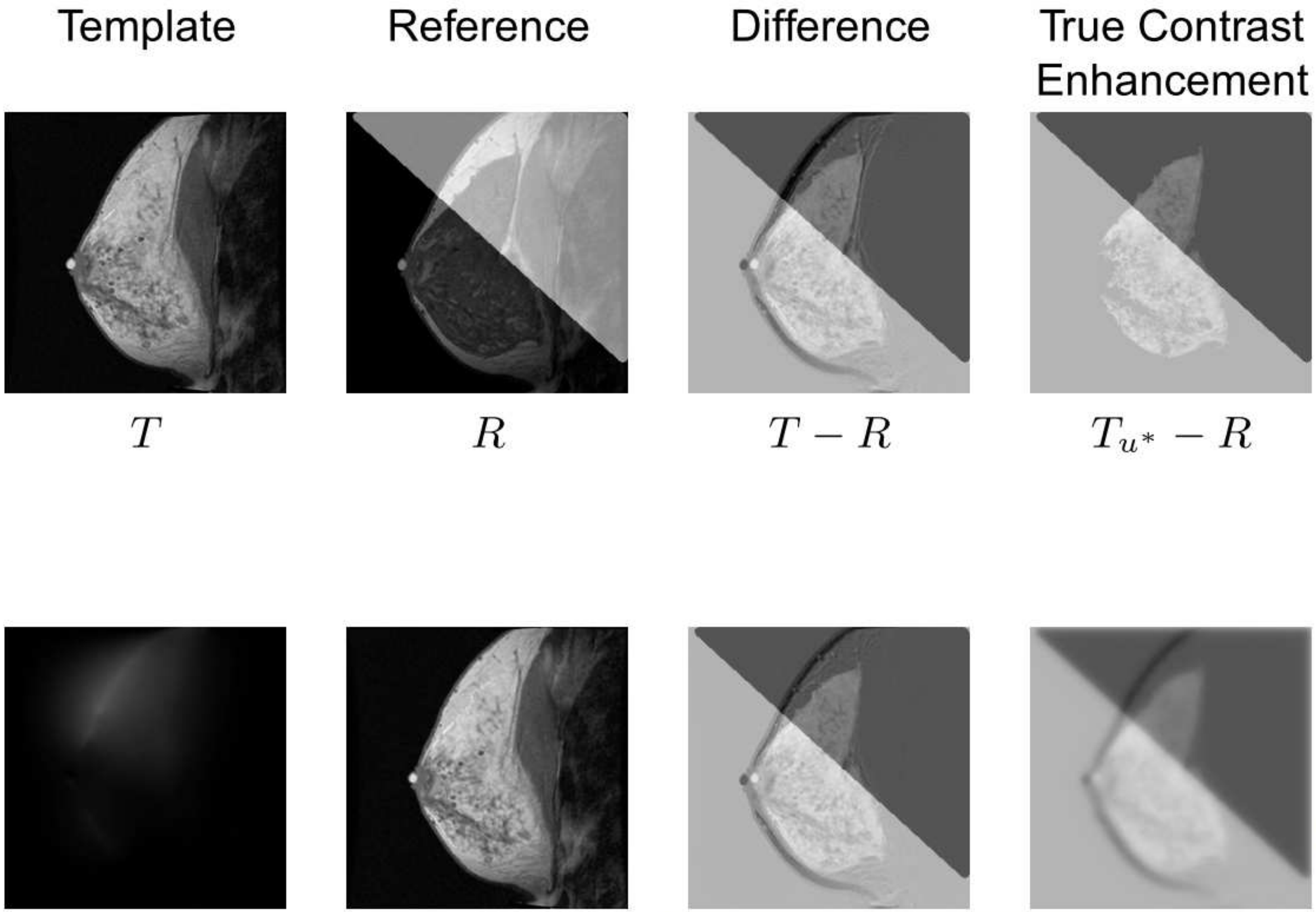


$T$ | $R$ | $T - R$ | $T_{u^*} - R$

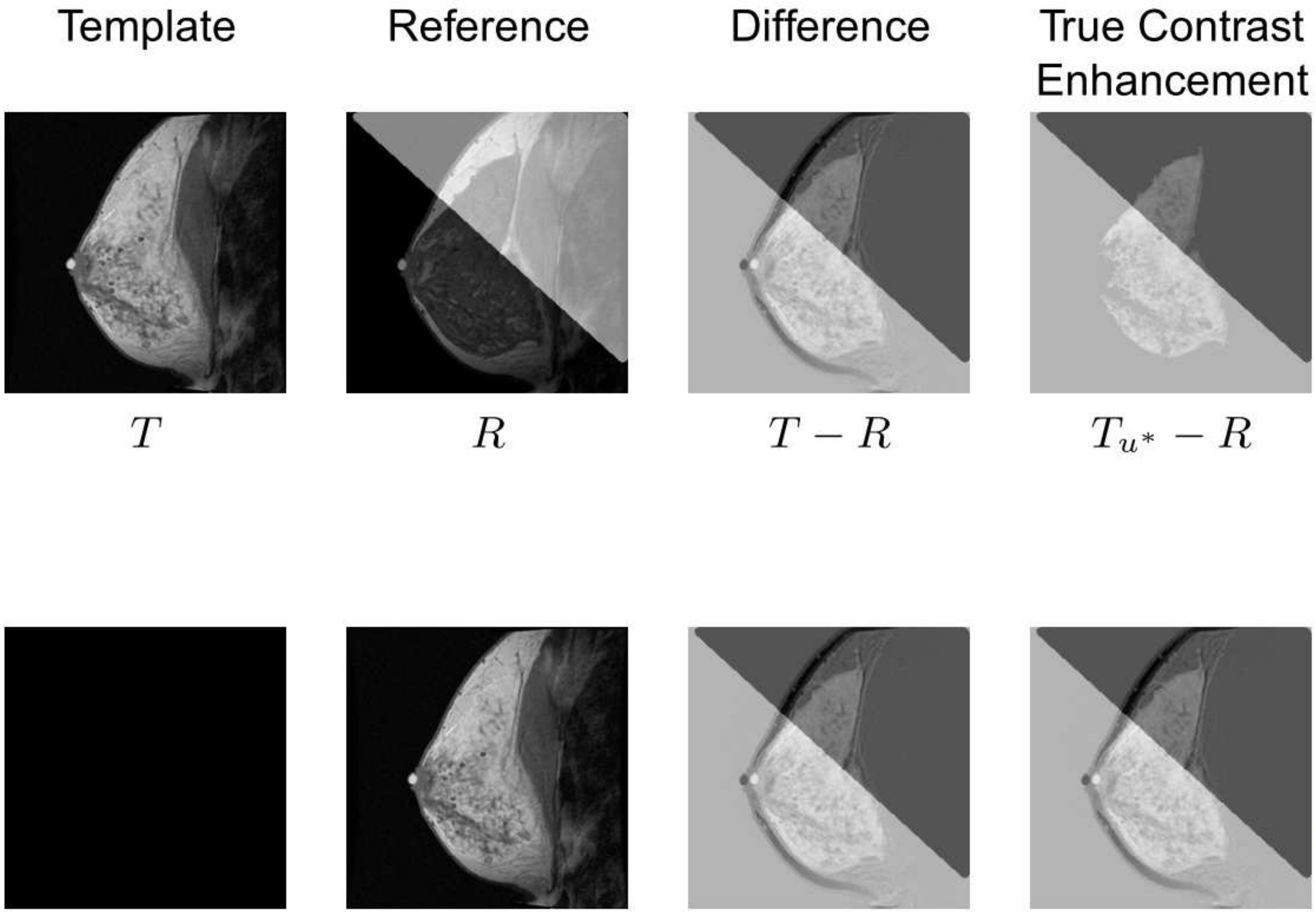
## $\beta = 100$


Template
Reference
Difference
True Contrast Enhancement
$T$
$R$
$T - R$
$T_{u^*} - R$

## $\beta = 10$


Template
Reference
Difference
True Contrast Enhancement
$T$
$R$
$T - R$
$T_{u^*} - R$

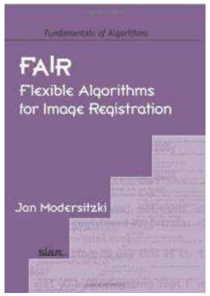
# $\beta = 0$


Template
Reference
Difference
True Contrast Enhancement
$T$
$R$
$T - R$
$T_{u^*} - R$

# Advantages to the coupled PDE approach

- Newton's method is proven to be far more efficient than steepest descent.

**The GN approach:**
**Discretize then optimize**




*Flexible Algorithms for Image Registration (FAIR), J. Modersitzki 2009*

## Mathematical Formulation

### Problem

*Given two images* $\mathcal{R}, \mathcal{T} : \Omega \subset \mathbb{R}^d \to \mathbb{R}$*, find a transformation* $y : \mathbb{R}^d \to \mathbb{R}^d$ *and an illumination image* $w : \Omega \subset \mathbb{R}^d \to \mathbb{R}$ *that minimize the joint objective functional*

$$\mathcal{J}[y; w] := \mathcal{D}[\mathcal{T}[y] + w, \mathcal{R}] + \alpha \mathcal{S}[y - y^{\text{ref}}] + \beta \mathcal{Q}[w].$$

*Here,* $\mathcal{D}$ *measures the dissimilarity of* $\mathcal{T}[y] + w$ *and* $\mathcal{R}$*, and* $\alpha \mathcal{S} + \beta \mathcal{Q}$ *is a regularization expression on* $[y; w]$*.*

# Mathematical Formulation

Here we assume

$$y^{\text{ref}}(x) = x,$$

$$\mathcal{D}[\mathcal{T},\mathcal{R}] = \mathcal{D}^{\text{SSD}}[\mathcal{T},\mathcal{R}] = \frac{1}{2}\int_\Omega (\mathcal{T}(x) - \mathcal{R}(x))^2 \, dx,$$

$$\mathcal{S}[y] = \frac{1}{2}\int_\Omega \mu\langle\nabla y, \nabla y\rangle + (\lambda+\mu)(\nabla \cdot y)^2 \, dx,$$

$$\begin{aligned} \mathcal{Q}[w] &= \mathcal{TV}_\epsilon[w] = \int_\Omega \sqrt{(\nabla w(x))^2 + \epsilon}dx \\ &\approx \int_\Omega |\nabla w(x)| \, dx. \end{aligned}$$

# Discretization

Let $\mathtt{x}$ denote a discretization of the $\Omega$, $\mathtt{y} \approx y(\mathtt{x})$, $\mathtt{w} \approx w(\mathtt{P} \cdot \mathtt{x})$, and $\mathtt{R} \approx R(\mathtt{P} \cdot \mathtt{x})$.

**Problem**

*Minimize the discretized functional*

$$J[\mathtt{y}; \mathtt{w}] := D[T(\mathtt{P} \cdot \mathtt{y})] + \mathtt{w}, R] + \alpha S(\mathtt{y} - \mathtt{yRef}) + \beta Q(\mathtt{w}).$$

# Gauss-Newton Approach

## Minimizing $J[\mathrm{y};\mathrm{w}]$ using Gauss-Newton Approach

- Initialize $\begin{bmatrix} \mathrm{y} \\ \mathrm{w} \end{bmatrix} \leftarrow \begin{bmatrix} \mathrm{y}_0 \\ \mathrm{w}_0 \end{bmatrix}$.
- Loop while not converged
  - Evaluate $H_J$ and $dJ$.
  - Solve the descent direction from the linear equation

$$H_J \begin{bmatrix} \delta\mathrm{y} \\ \delta\mathrm{w} \end{bmatrix} = -dJ^T.$$

  - Find a positive scalar step-size $\mathrm{s}$ using line-search.
  - Update $\begin{bmatrix} \mathrm{y} \\ \mathrm{w} \end{bmatrix} \leftarrow \begin{bmatrix} \mathrm{y} \\ \mathrm{w} \end{bmatrix} + \mathrm{s} \begin{bmatrix} \delta\mathrm{y} \\ \delta\mathrm{w} \end{bmatrix}$.
- End loop

# Jacobian of $J$

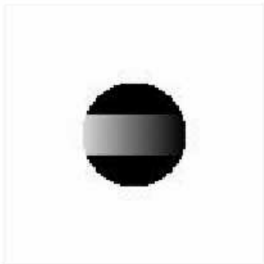
## Jacobian Computation

$$dJ = [\mathtt{r}^T dT\ \mathtt{P} + \alpha\ dS, \mathtt{r}^T + \beta\ dQ]$$

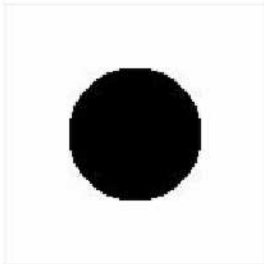
# Hessian of $J$

## Hessian Computation

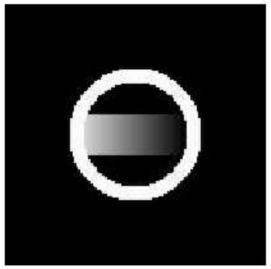
$$H_J = \left[\begin{array}{c|c} (dT\ \mathtt{P})^T dT\ \mathtt{P} + \alpha H_S & (dT\ \mathtt{P})^T \\ \hline dT\ \mathtt{P} & I_n + \beta\ H_Q \end{array}\right]$$



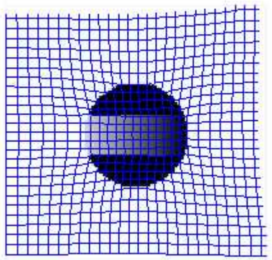
Template $T(\mathrm{x})$



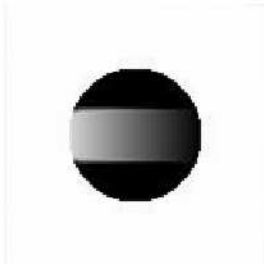
Reference $R(\mathrm{x})$



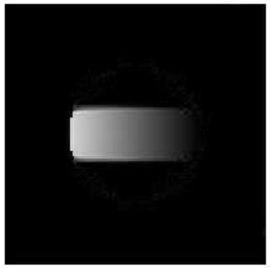
$|T(\mathrm{x}) - R(\mathrm{x})|$



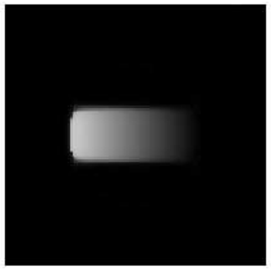
$T(\mathrm{x})$ and the grid $\mathrm{y}$



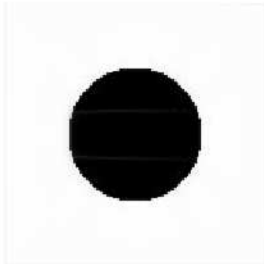
$T(\mathrm{y})$



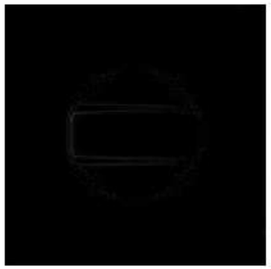
$|T(\mathrm{y}) - R(\mathrm{x})|$



Intensity correction $|\mathrm{w}(\mathrm{x})|$



$T(\mathrm{y}) + \mathrm{w}(\mathrm{x})$

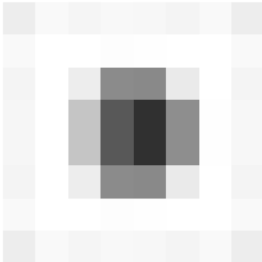


$|T(\mathrm{y}) + \mathrm{w}(\mathrm{x}) - R(\mathrm{x})|$

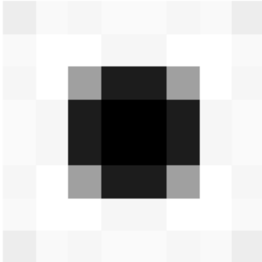
A

Multilevel

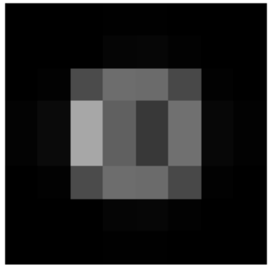
# Treatment



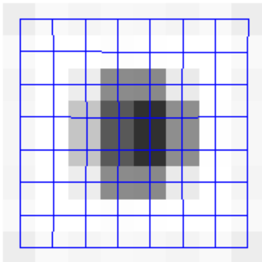
Template $T(\mathrm{x})$



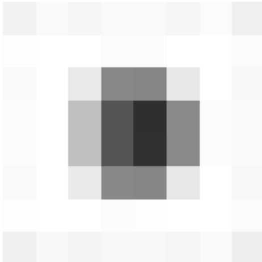
Reference $R(\mathrm{x})$



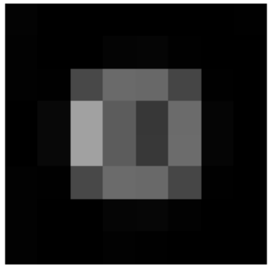
$|T(\mathrm{x}) - R(\mathrm{x})|$



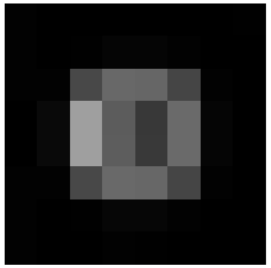
$T(\mathrm{x})$ and the grid $\mathrm{y}$



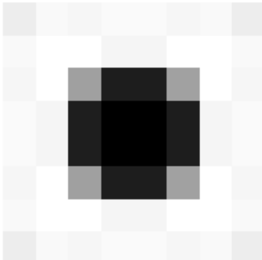
$T(\mathrm{y})$



$|T(\mathrm{y}) - R(\mathrm{x})|$



Intensity correction $|\mathrm{w}(\mathrm{x})|$



$T(\mathrm{y}) + \mathrm{w}(\mathrm{x})$

$|T(\mathrm{y}) + \mathrm{w}(\mathrm{x}) - R(\mathrm{x})|$

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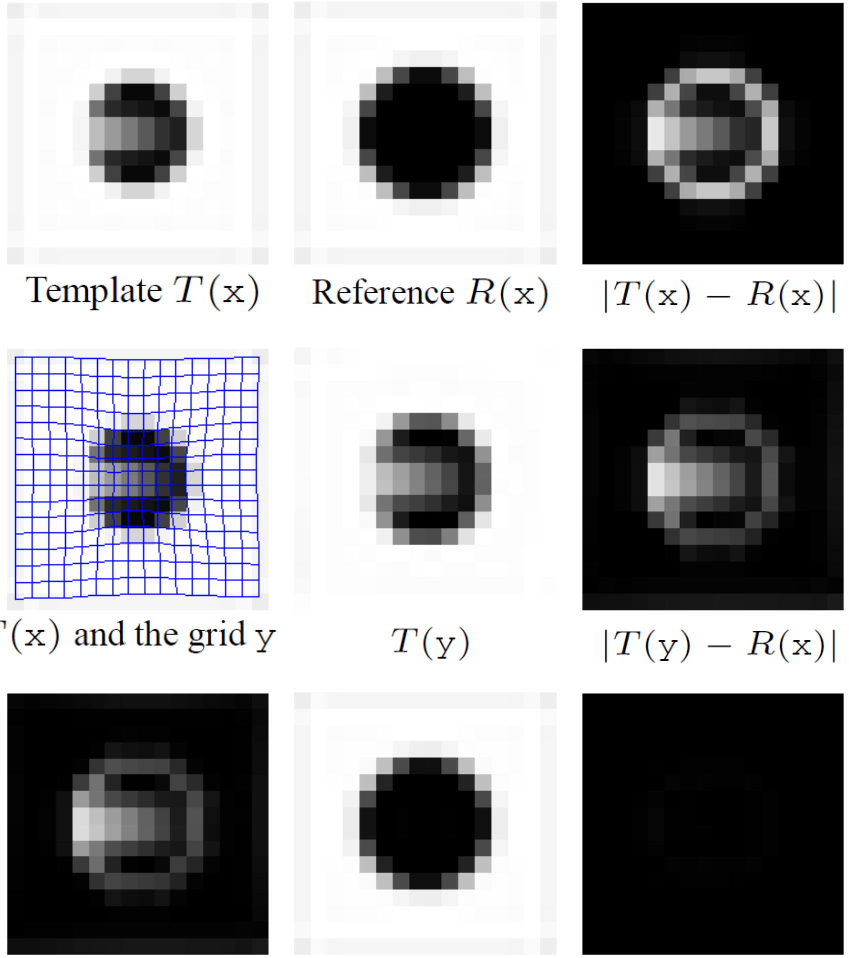
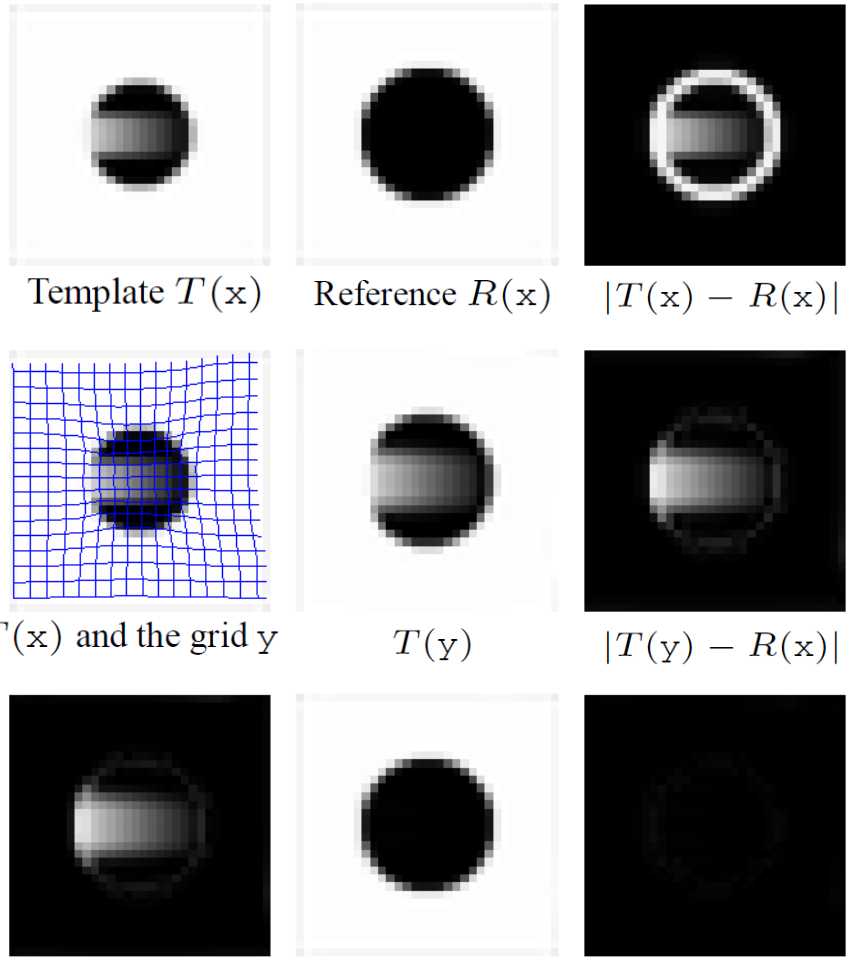


Template $T(\mathrm{x})$ | Reference $R(\mathrm{x})$ | $|T(\mathrm{x}) - R(\mathrm{x})|$

$T(\mathrm{x})$ and the grid $\mathrm{y}$ | $T(\mathrm{y})$ | $|T(\mathrm{y}) - R(\mathrm{x})|$

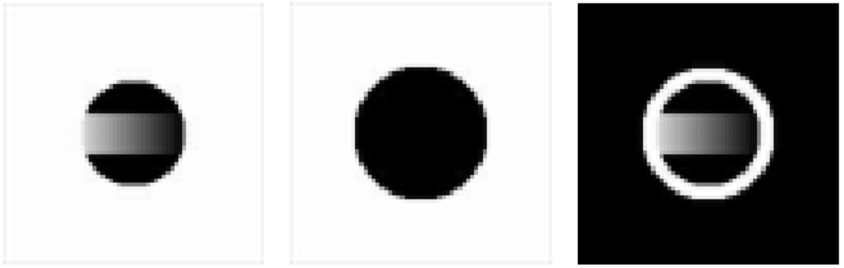
Intensity correction $|\mathrm{w}(\mathrm{x})|$ | $T(\mathrm{y}) + \mathrm{w}(\mathrm{x})$ | $|T(\mathrm{y}) + \mathrm{w}(\mathrm{x}) - R(\mathrm{x})|$



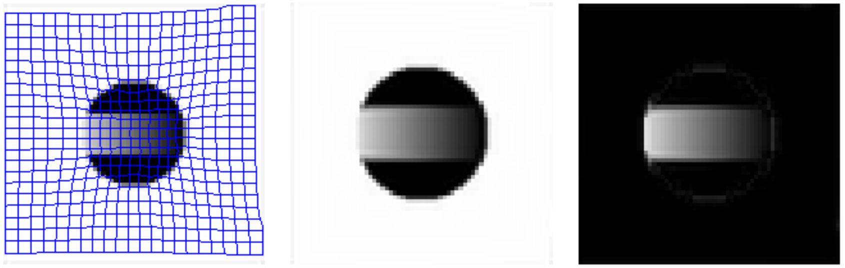
Template $T(x)$ | Reference $R(x)$ | $|T(x) - R(x)|$

$T(x)$ and the grid $y$ | $T(y)$ | $|T(y) - R(x)|$

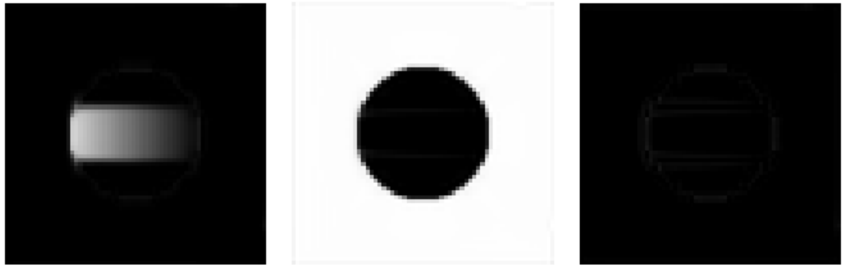
Intensity correction $|w(x)|$ | $T(y) + w(x)$ | $|T(y) + w(x) - R(x)|$



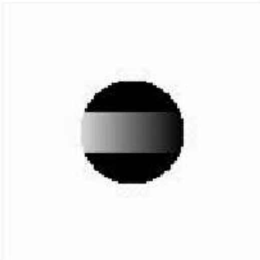
Template $T(\mathrm{x})$ | Reference $R(\mathrm{x})$ | $|T(\mathrm{x}) - R(\mathrm{x})|$

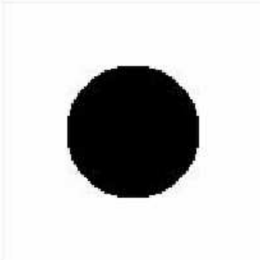


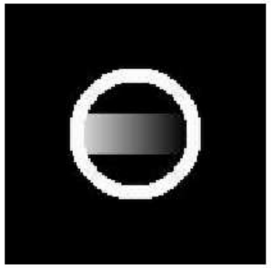
$T(\mathrm{x})$ and the grid $\mathrm{y}$ | $T(\mathrm{y})$ | $|T(\mathrm{y}) - R(\mathrm{x})|$

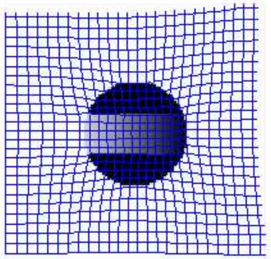


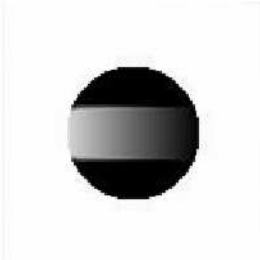
Intensity correction $|\mathrm{w}(\mathrm{x})|$ | $T(\mathrm{y}) + \mathrm{w}(\mathrm{x})$ | $|T(\mathrm{y}) + \mathrm{w}(\mathrm{x}) - R(\mathrm{x})|$

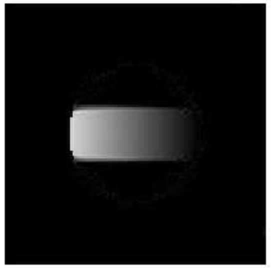

Template $T(\mathrm{x})$

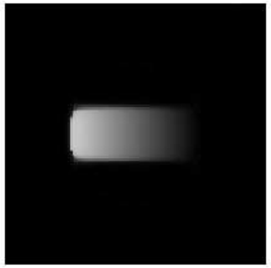

Reference $R(\mathrm{x})$

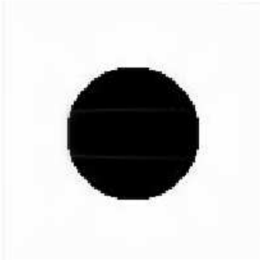

$|T(\mathrm{x}) - R(\mathrm{x})|$

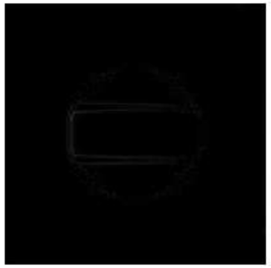

$T(\mathrm{x})$ and the grid $\mathrm{y}$

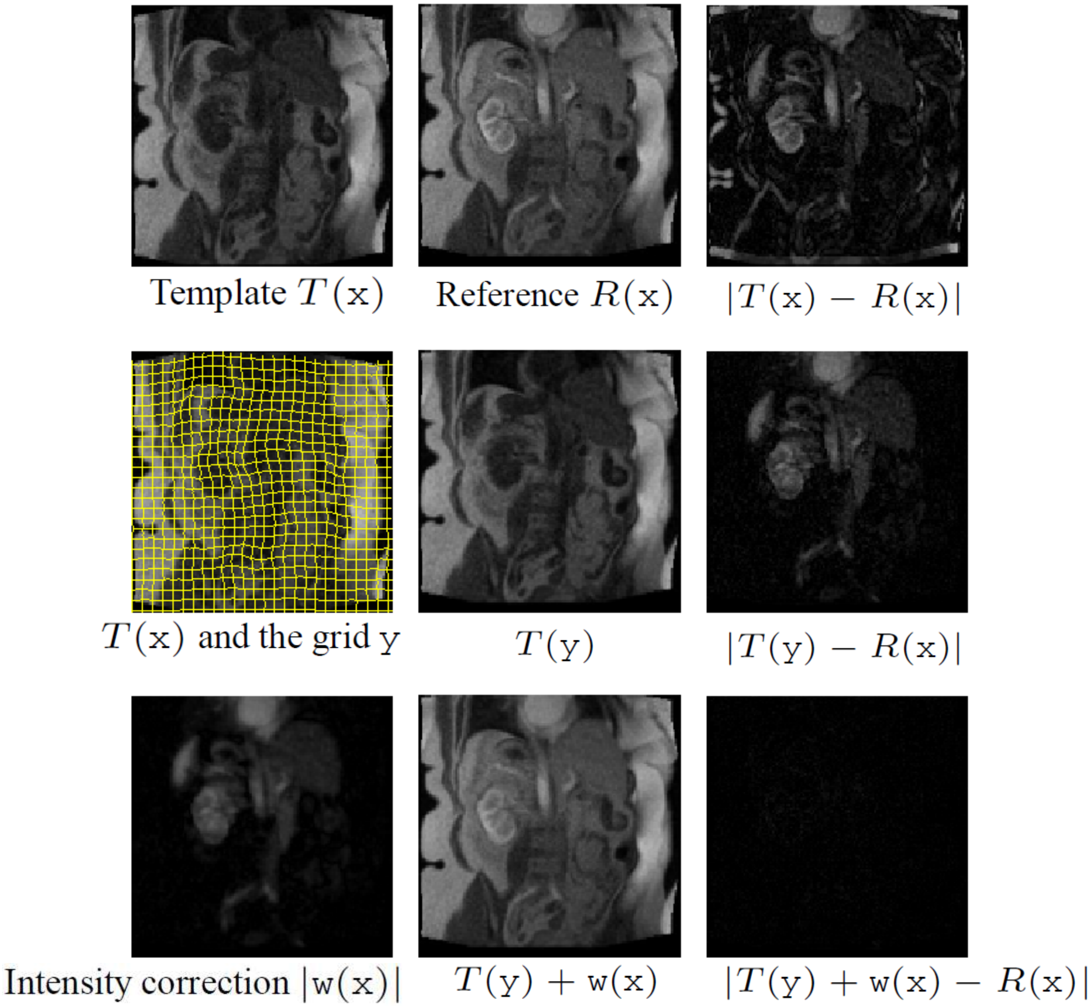

$T(\mathrm{y})$


$|T(\mathrm{y}) - R(\mathrm{x})|$

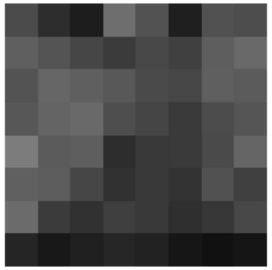

Intensity correction $|\mathrm{w}(\mathrm{x})|$


$T(\mathrm{y}) + \mathrm{w}(\mathrm{x})$


$|T(\mathrm{y}) + \mathrm{w}(\mathrm{x}) - R(\mathrm{x})|$

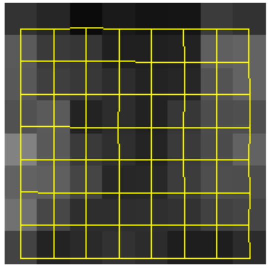

Template $T(x)$
Reference $R(x)$
$|T(x) - R(x)|$
$T(x)$ and the grid $y$
$T(y)$
$|T(y) - R(x)|$
Intensity correction $|w(x)|$
$T(y) + w(x)$
$|T(y) + w(x) - R(x)|$

Template $T(\mathrm{x})$



Reference $R(\mathrm{x})$

$|T(\mathrm{x}) - R(\mathrm{x})|$

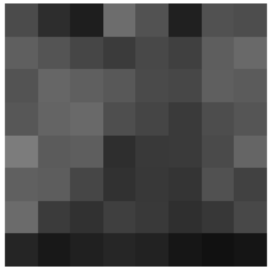


$T(\mathrm{x})$ and the grid $\mathrm{y}$

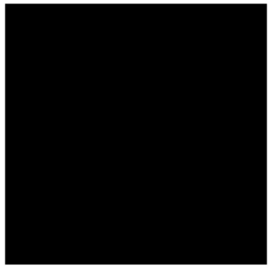
$T(\mathrm{y})$

$|T(\mathrm{y}) - R(\mathrm{x})|$

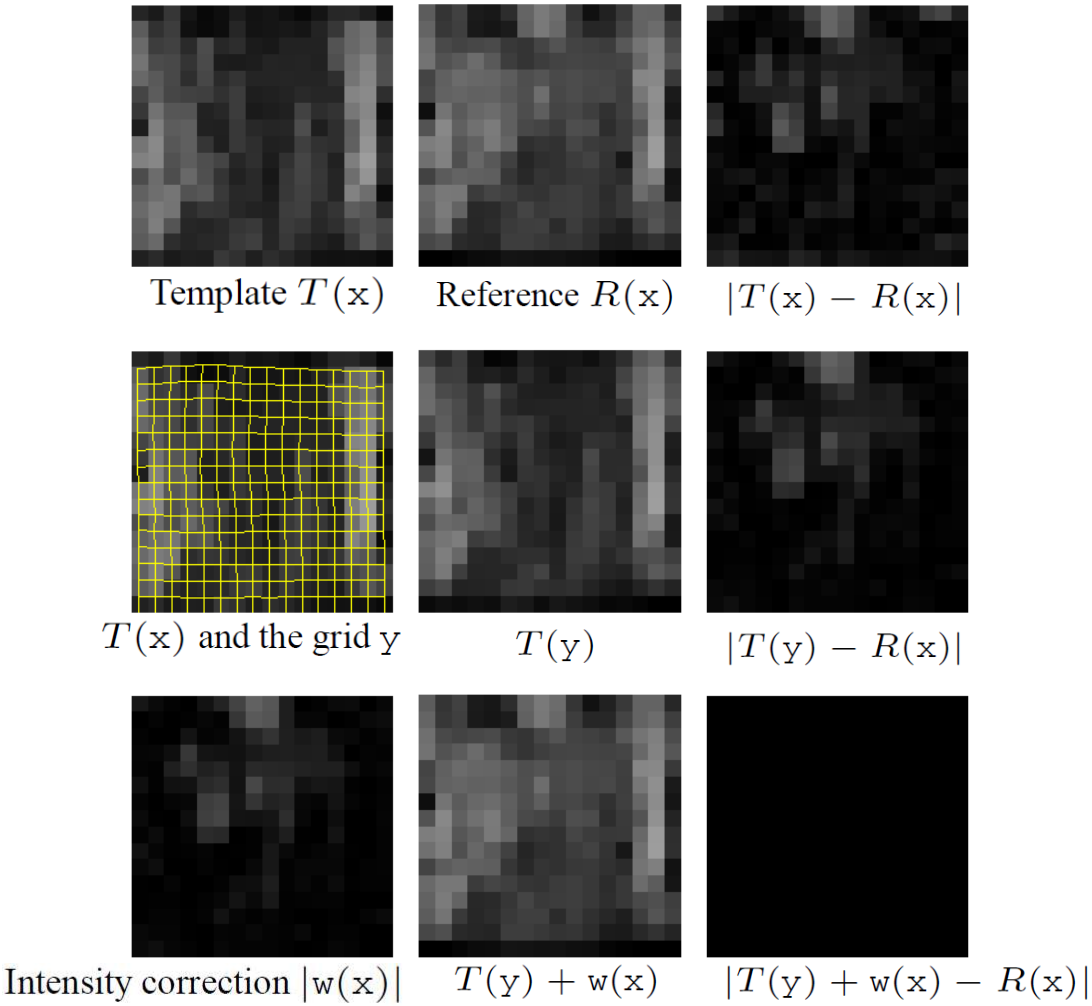
Intensity correction $|\mathrm{w}(\mathrm{x})|$

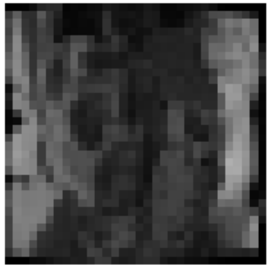


$T(\mathrm{y}) + \mathrm{w}(\mathrm{x})$



$|T(\mathrm{y}) + \mathrm{w}(\mathrm{x}) - R(\mathrm{x})|$


Template $T(x)$
Reference $R(x)$
$|T(x) - R(x)|$
$T(x)$ and the grid $y$
$T(y)$
$|T(y) - R(x)|$
Intensity correction $|w(x)|$
$T(y) + w(x)$
$|T(y) + w(x) - R(x)|$



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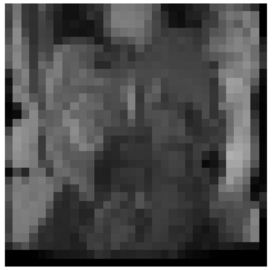
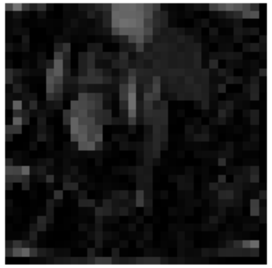
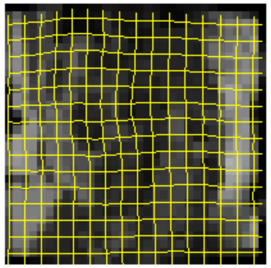


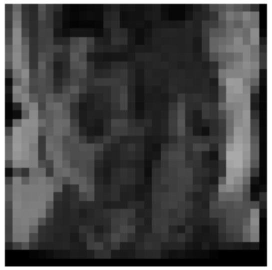
Reference $R(\mathrm{x})$



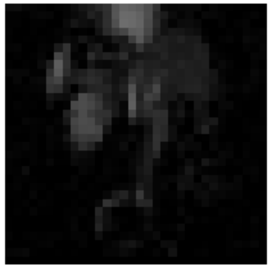
$|T(\mathrm{x}) - R(\mathrm{x})|$



$T(\mathrm{x})$ and the grid $\mathrm{y}$

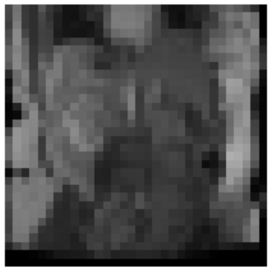


$T(\mathrm{y})$

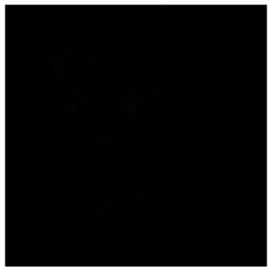


$|T(\mathrm{y}) - R(\mathrm{x})|$

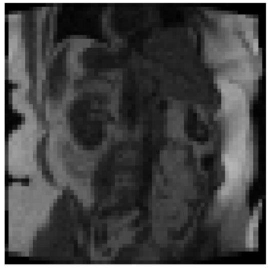
Intensity correction $|\mathrm{w}(\mathrm{x})|$



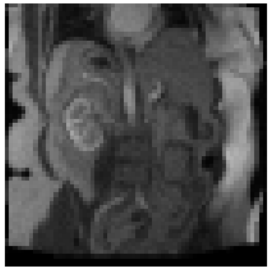
$T(\mathrm{y}) + \mathrm{w}(\mathrm{x})$



$|T(\mathrm{y}) + \mathrm{w}(\mathrm{x}) - R(\mathrm{x})|$

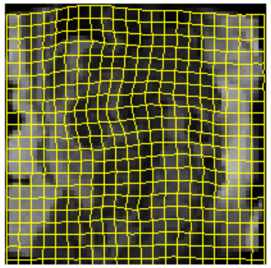


Template $T(\mathrm{x})$

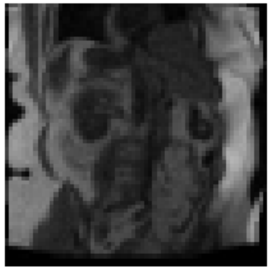


Reference $R(\mathrm{x})$

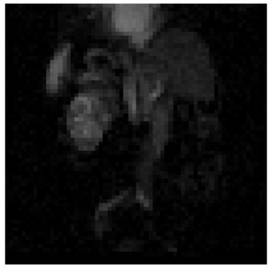
$|T(\mathrm{x}) - R(\mathrm{x})|$



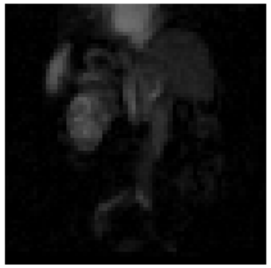
$T(\mathrm{x})$ and the grid $\mathrm{y}$



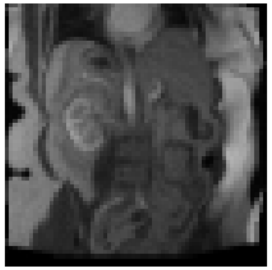
$T(\mathrm{y})$



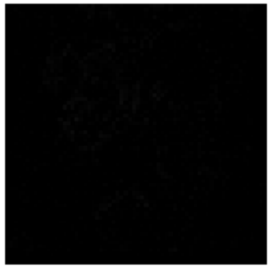
$|T(\mathrm{y}) - R(\mathrm{x})|$



Intensity correction $|\mathrm{w}(\mathrm{x})|$



$T(\mathrm{y}) + \mathrm{w}(\mathrm{x})$



$|T(\mathrm{y}) + \mathrm{w}(\mathrm{x}) - R(\mathrm{x})|$

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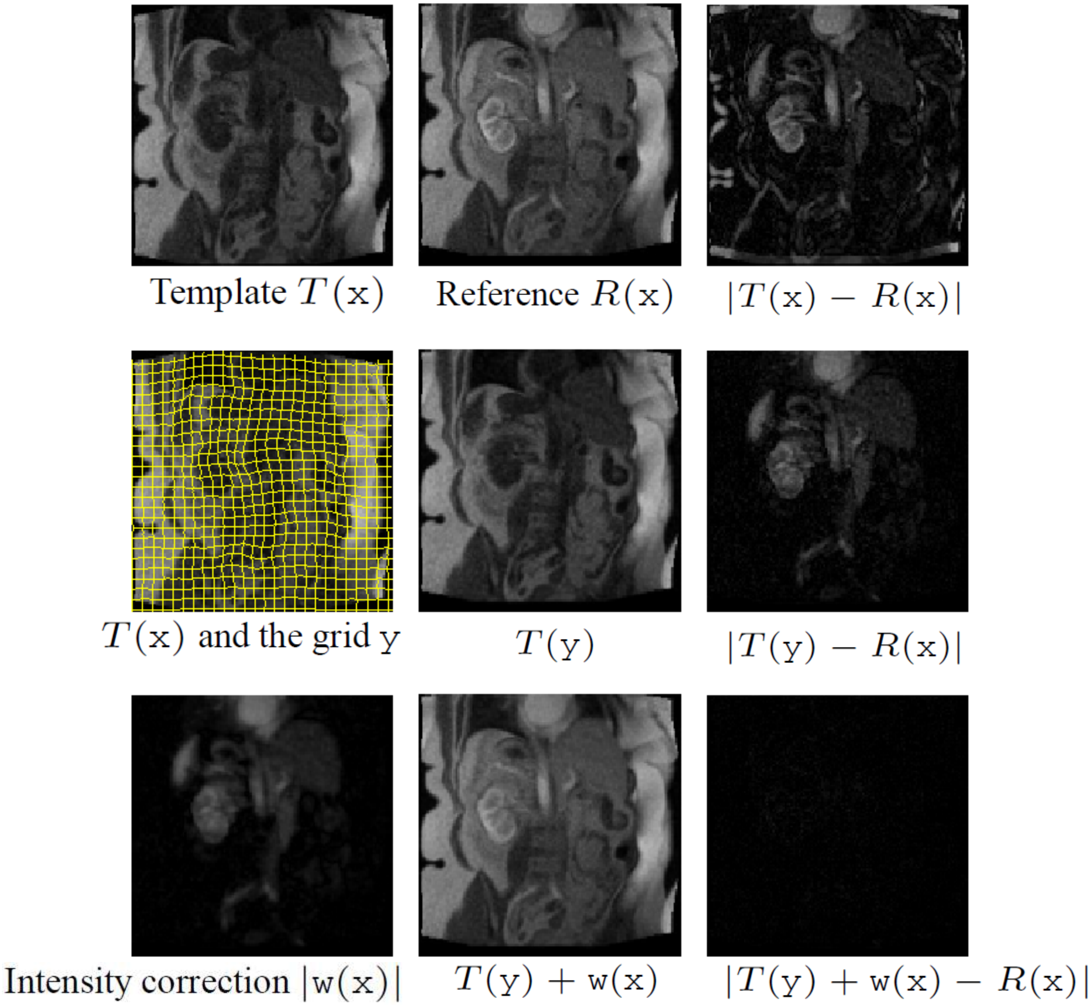
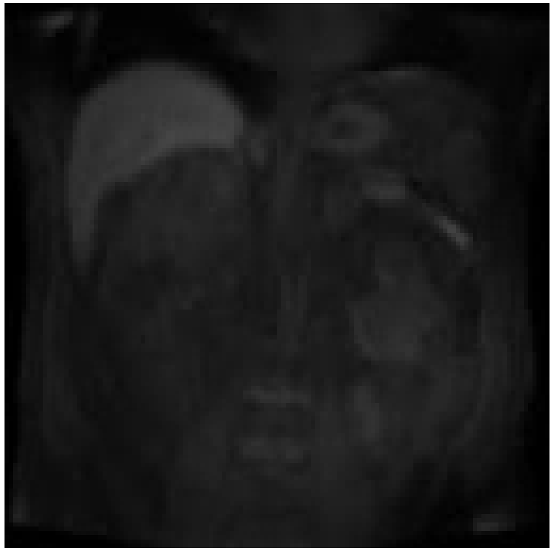
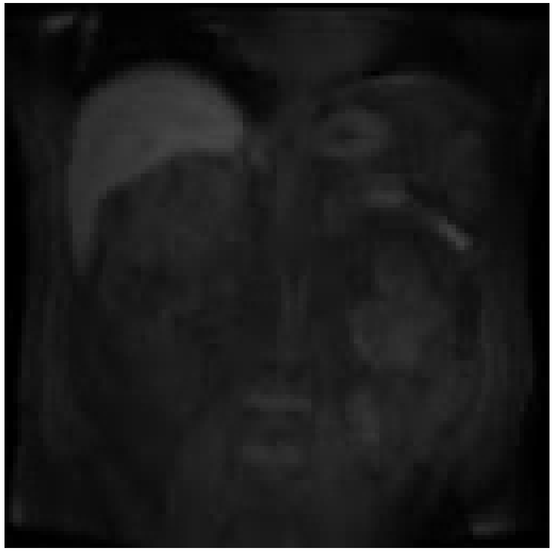

Template $T(\mathrm{x})$
Reference $R(\mathrm{x})$
$|T(\mathrm{x}) - R(\mathrm{x})|$
$T(\mathrm{x})$ and the grid $\mathrm{y}$
$T(\mathrm{y})$
$|T(\mathrm{y}) - R(\mathrm{x})|$
Intensity correction $|\mathrm{w}(\mathrm{x})|$
$T(\mathrm{y}) + \mathrm{w}(\mathrm{x})$
$|T(\mathrm{y}) + \mathrm{w}(\mathrm{x}) - R(\mathrm{x})|$



*Motionless*



*Simulated Motion*

*Simulated Data by Anthony Lausch (Sunnybrook Research Institute)*

*NGF*

*Proposed*

## Concluding Remarks

- We presented a general mathematical framework for registration and intensity correction.
- This is an explanation of the previous methods (Martel, Froh, Gennert, Negahdaripiour, Barber, etc.) that separate the contrast enhancement term in the regularization.
- Two numerical schemes were presented: GN approach is more efficient compared to the PDE one.
- Our approach is flexible: new regularizations may be used.

## Acknowledgments

*Anne Martel's group (Sunnybrook)*
*Funding : CBCF, NSERC*

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