# Novel techniques for multiscale representations [1]

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[1] Supported by CSCAMM and IPAM.

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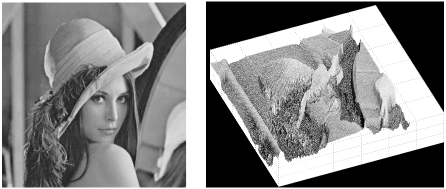
# Outline

## 1. Problems in image processing, a historical tour

# What is an image ?

- Digital images are sampled 2-D analogue signals
- Black and white images $\equiv f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$
- $f(x) \equiv$ intensity level at that point, which varies from zero to 255
- An image can be postulated as an $L^2(\Omega)$ object



(a) (b)

**Figure:** (a) Image of Lenna and (b) Image of Lenna as a graph of a function

# Problems in image processing...

- **Image denoising**: $f$ may have some noise $\eta$ in it.
- $f = u + \eta$, we need to get back the denoised image $u$.

(a) (b)

**Figure:** Can we go from a noisy image (a) to a restored image in (b) **?**

- $f$ may be blurry and noisy $f = Ku + \eta$
- **Image segmentation** $\equiv$ identifying 'components' in $f$ $\equiv$ edge detection

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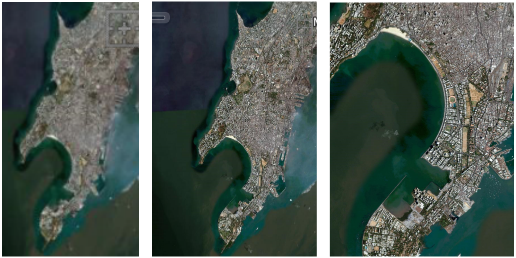
**Figure:** Can we identify components in (a) and get a segmented image as in (b) **?**

# Multiscale image representation

- **Multiscale image representation**: Finding different level of 'scales' in $f$



(a) (b) (c)

**Figure:** Multiscale images of the city of Mumbai.

- Multiscale representation: Family of images $\{u(t)\}$ for a scaling parameter $t$
- **Forward marching**: $u(0) = 0, u(t) \to u$
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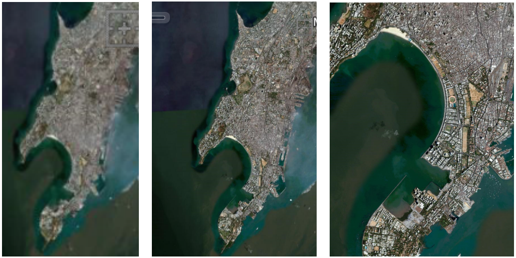

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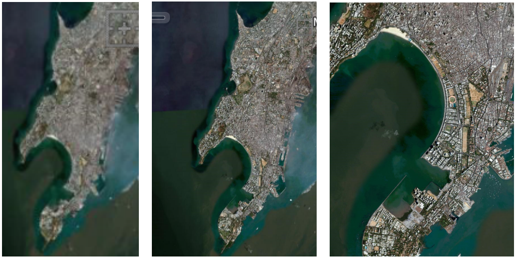

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There are two main approaches to solve above problems:

- **Variational approaches** - Tikhonov regularization, greedy algorithms, wavelets shrinkage etc.
- **PDE based approaches** - diffusion, Perona-Malik etc.

The approaches are related -

# Variational methods in image processing: Tikhonov regularization

- We need to solve the ill posed problem $f = Ku$ :

  Consider interpolation functional

$$\inf_{u \in X} \left( \|u\|_X + \lambda \|f - Ku\|_Y^2 \right)$$

  $X \subsetneq Y$, $\|u\|_X$ : regularizing term, $\|f - Ku\|_Y^2$ : fidelity term

- $(X, Y) \equiv (BV, L^2)$: Rudin-Osher-Fatemi (1992), Aubert-Vese (1997).

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- **Rudin-Osher-Fatemi (ROF) decomposition**
  $f = u_\lambda + v_\lambda$ for scale parameter $\lambda$.

$$[u_\lambda, v_\lambda] = \underset{\{f=u+v\}}{\text{arginf}} \left( \int_\Omega |\nabla u| + \lambda \int_\Omega |f-u|^2 \right)$$

- The BV seminorm $\int_\Omega |\nabla u|$ is a regularizing term
- $\int_\Omega |f-u|^2$: a fidelity term
- $\lambda$ : acts as an **inverse scale** of the $u_\lambda$ part ( smaller $\lambda \equiv$ larger scale )
- $u_\lambda :=$ smooth parts and edges in $f$
  $v_\lambda := f - u_\lambda$ texture, finer details, noise
- Many other variational methods ...

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# Other variational methods in image processing...

- **Mumford-Shah segmentation (1985)**

$$[u, v, \mathcal{C}] = \underset{\{f=u+v,\mathcal{C}\}}{\operatorname{arginf}} \left( \int_{\Omega-\mathcal{C}} |f-u|^2 + \lambda_1 \int_{\Omega-\mathcal{C}} |\nabla u|^2 + \lambda_2 \oint_{\mathcal{C}} d\sigma \right).$$

$u : \Omega \to \mathbb{R}$ : piecewise smooth image
$\mathcal{C} \in \Omega$ : the set of jump discontinuities

- **Ambrosio and Tortorelli approximation (1992)**
- **Kass-Witkin-Terzopoulos model (1988)**

$$\inf_{c \in \mathcal{C}} \left( \int_a^b |c'|^2 + \lambda_1 \int_a^b |c''|^2 + \lambda_2 \int_a^b g^2(|\nabla f(c)|) \right)$$

$\mathcal{C}$ : closed, piecewise regular, parametric curves (snakes)
$g$ : a decreasing function vanishing at infinity

- **Caselles, Kimmel, Sapiro: Geodesic active contours (1997)**
- **Osher, Sethian: Level set method (1988)**
- ... ...
- Now we look at some PDE methods ...

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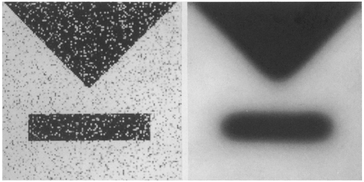
- Denoising with heat equation:



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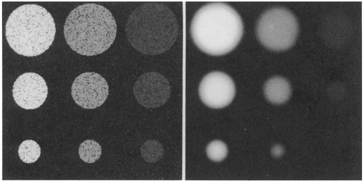

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**Figure:** Result of isotropic diffusion: reduction of noise at the expense of losing information at the edges

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- **Problem 2**: where to stop ?

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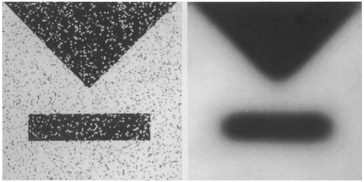

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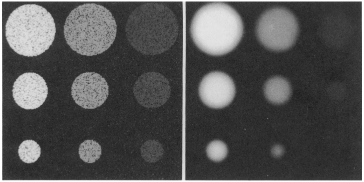

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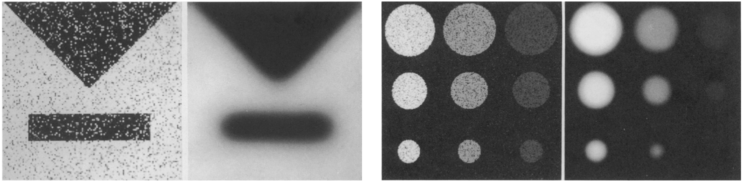

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# PDE methods in image processing: Perona-Malik model (1988)

- Heat equation $\equiv$ **isotropic diffusion** $\Rightarrow$ we lose information about edges
- Perona-Malik proposed an **anisotropic diffusion method**

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(g(|\nabla u|)\nabla u\right), \quad u(0) = f$$

- **The idea: preserve the edges**
  - Smooth regions $\equiv |\nabla u|$ is weak $\Rightarrow$ we need an isotropic smoothing
  - Near the edges $\equiv |\nabla u|$ is large $\Rightarrow$ we need to control the diffusion
  - Examples of suitable function $g(s)$ : $e^{-s}$, $\frac{1}{1+s^2}$, $\frac{1}{\sqrt{1+s}}$
- Perona-Malik is not well posed ! Catté et al. modification[3] :

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(g(|\nabla G_\sigma \star u|)\nabla u\right),$$

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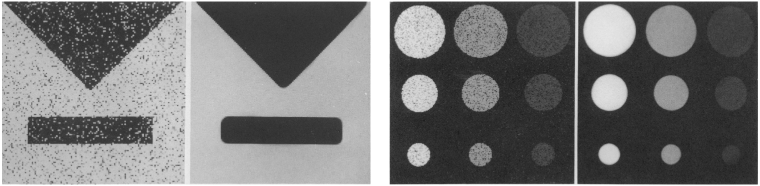

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**Figure:** Result of anisotropic diffusion: edges are preserved.

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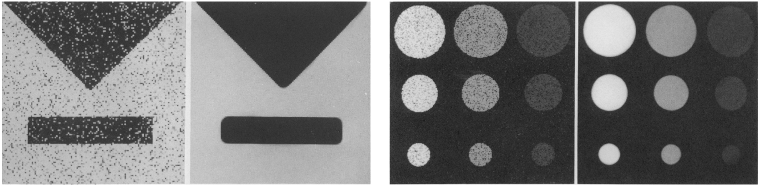

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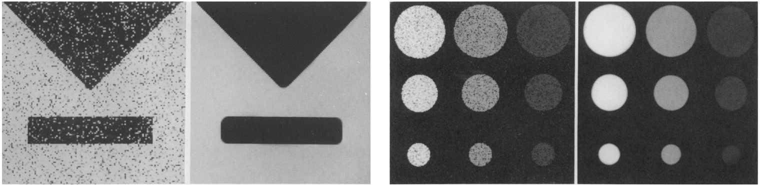

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**Figure:** Result of anisotropic diffusion: edges are preserved.

- **Problem**: As $t \to \infty$ the models discussed before diffuse completely.
  **... so where to stop ?**
- **Solution**: **Nordström** modified Perona-Malik model.

$$\frac{\partial u}{\partial t} = f - u + \operatorname{div}\left(g(|\nabla u|)\nabla u\right), \quad u(0) = 0.$$

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# PDE methods in image processing: Nordström's model

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The (time dependent) Euler-Lagrange equation:

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**Nordström's modification of Perona-Malik (1990)**

$$\frac{\partial u}{\partial t} = f - u + \operatorname{div}\left( g(|\nabla u|)\nabla u \right).$$

$g(s) = \frac{1}{2\lambda s} \Rightarrow$ steady-state of Nordström $\equiv$ Euler-Lagrange of ROF !

# PDE approach $\leftrightsquigarrow$ variational approach

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## 2. IDE based on $(BV, L^2)$ image decomposition

# A novel integro-differential model

- We propose a novel model.

$$\int_0^t u(x,s)ds = f(x) + \frac{1}{2\lambda(t)} \operatorname{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right).$$

- An **Integro**-differential equation (**IDE**).
- The scaling function $\lambda(t)$ : increasing function at our disposal.
- This model gives an inverse scale representation.
- ⋆ We do not need to associate with a variational problem anymore.⋆

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Where to **stop ?**

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- We propose a novel model.

$$\int_0^t u(x,s)ds = f(x) + \frac{1}{2\lambda(t)} \operatorname{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right).$$

- An **Integro**-differential equation (**IDE**).
- The scaling function $\lambda(t)$ : increasing function at our disposal.
- This model gives an inverse scale representation.
- $\star$ We do not need to associate with a variational problem anymore.$\star$

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What is the **motivation ?**

Where to **start ?**

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# Idea: Tadmor-Nezzar-Vese scheme with "intensity quanta"

- Let $\tau$ be the small intensity of quanta, with this the **ROF decomposition** becomes:

$$f = \tau u_{\lambda_0} + v_{\lambda_0}, \quad [u_{\lambda_0}, v_{\lambda_0}] = \underset{\{f=\tau u+v\}}{\operatorname{arginf}} \left( \int_\Omega |\nabla u| + \frac{\lambda_0}{\tau} \int_\Omega |f - \tau u|^2 \right).$$

- $v_{\lambda_0}$ can be decomposed with a scaling parameter $\lambda_1 > \lambda_0$.

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- **TNV multiscale decomposition**

$$v_{\lambda_{k-1}} = \tau u_{\lambda_k} + v_{\lambda_k}, \quad [u_{\lambda_k}, v_{\lambda_k}] = \underset{\{v_{\lambda_{k-1}}=\tau u+v\}}{\operatorname{arginf}} \left( \int_\Omega |\nabla u| + \frac{\lambda_k}{\tau} \int_\Omega |v_{\lambda_{k-1}} - \tau u|^2 \right).$$

- With this scheme after $N+1$ steps we get:

$$\begin{aligned} f &= \tau u_{\lambda_0} + v_{\lambda_0} \\ &= \tau u_{\lambda_0} + \tau u_{\lambda_1} + v_{\lambda_1} \\ &= \tau u_0 + \tau u_1 + \tau u_2 + v_2 \\ &= \ldots \\ &= \tau u_{\lambda_0} + \tau u_{\lambda_1} + \ldots + \tau u_{\lambda_N} + v_{\lambda_N}. \end{aligned}$$

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Telescopic sum of the above gives us:

$$\sum_{k=0}^{N} u_{\lambda_k} \tau + v_{\lambda_N} = f$$

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# Going from TNV to a novel integro-differential equation

New TNV formulation:

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This ‘motivates’ us to write the following model.

**The novel integro-differential model**

$$\int_0^t u(x,s)ds = f(x) + \frac{1}{2\lambda(t)}\operatorname{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right).$$

where $\lambda(t) > 0$ is an increasing scaling function at our disposal.

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# How to solve it numerically ?

- Let $\Delta t$ be the time interval step. Thus, after $N$ steps:

$$\mathcal{U}(t) := \int_0^t u(x,s)ds = \sum_{k=0}^{N-1} \int_{k\Delta t}^{(k+1)\Delta t} u(x,s)ds$$

- $\mathcal{U}^N := \int_0^{N\Delta t} u(x,s)ds$ and $u^{k+1} := u((k+1)\Delta t)$, with this we have

$$\mathcal{U}^N \approx \mathcal{U}^{N-1} + u^N \Delta t := \mathcal{U}^{N-1} + \omega^N.$$

- Thus, we have the following fixed point iteration.

$$\omega_{i,j}^n = \frac{2\lambda^N h^2 (f_{i,j} - \mathcal{U}_{i,j}^{N-1}) + c_E \omega_{i+1,j}^{n-1} + c_W \omega_{i-1,j}^{n-1} + c_S \omega_{i,j+1}^{n-1} + c_N \omega_{i,j-1}^{n-1}}{2\lambda^N h^2 + c_E + c_W + c_S + c_N}.$$

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# Proposed model $\lambda(t) = (0.002)2^t$, on Lenna.

Numerical result for $\int_0^t u(x,s)ds = f(x) + \frac{1}{2\lambda(t)} \operatorname{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right)$.

**Figure:** (a)–(d) As $\lambda(t) \to \infty$, the images $\int_0^t u(x,s)ds$ are shown above for $t = 1, 4, 6, 10$. Here, $\lambda(t) = 0.002 \times 2^t$.

# 3. A few theoretical results about $(BV, L^2)$-based IDE

# What does the scaling function, $\lambda(t)$, mean ?

Star-norm is the dual of the $BV$ norm w.r.t. the $L^2$ scalar product

$$\|w\|_* := \sup_{\varphi \neq 0} \frac{|(w, \varphi)_{L^2}|}{\int_\Omega |\nabla \varphi|}.$$

**Theorem (I)**

*For the IDE model*

$$\int_0^t u(x,s)\,ds = f(x) + \frac{1}{2\lambda(t)} \operatorname{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right),$$

*let* $\mathcal{U}(\cdot,t) := \int_0^t u(x,s)\,ds$ *and* $V(\cdot,t)$ *be the residual,*

$$V(\cdot,t) := f - \mathcal{U}(\cdot,t).$$

*Then size of the residual is dictated by the scaling function* $\lambda(t)$*,*

$$\|V(\cdot,t)\|_* = \frac{1}{2\lambda(t)}.$$

# What does the scaling function, $\lambda(t)$, mean ?

Star-norm is the dual of the $BV$ norm w.r.t. the $L^2$ scalar product

$$\|w\|_* := \sup_{\varphi \neq 0} \frac{|(w,\varphi)_{L^2}|}{\int_\Omega |\nabla \varphi|}.$$

**Theorem (I)**

*For the IDE model*

$$\int_0^t u(x,s)\,ds = f(x) + \frac{1}{2\lambda(t)} \operatorname{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right),$$

*let* $\mathcal{U}(\cdot,t) := \int_0^t u(x,s)\,ds$ *and* $V(\cdot,t)$ *be the residual,*

$$V(\cdot,t) := f - \mathcal{U}(\cdot,t).$$

*Then size of the residual is dictated by the scaling function* $\lambda(t)$*,*

$$\|V(\cdot,t)\|_* = \frac{1}{2\lambda(t)}.$$

# Energy decomposition

## Theorem (II)

*For the IDE model*

$$\int_0^t u(x,s)\,ds = f(x) + \frac{1}{2\lambda(t)} \operatorname{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right),$$

*associated with an $L^2$- image $f$, and let $V(\cdot,t)$ be the residual, $V(t) = f - \mathcal{U}(t)$. Then the following energy decomposition holds*

$$\int_{s=0}^t \frac{1}{\lambda(s)} |u(\cdot,s)|_{BV}\,ds + \|V(\cdot,t)\|_{L^2}^2 = \|f\|_{L^2}^2.$$

**Theorem (III)**

*Given an image $f \in BV$, we consider the IDE model*

$$\int_0^t u(x,s)\,ds = f(x) + \frac{1}{2\lambda(t)} \,\mathrm{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right),$$

*with rapidly increasing scaling function $\lambda(t)$ so that*

$$\frac{\lambda(t/2)}{\lambda(t)} \xrightarrow{t\to\infty} 0.$$

*Then, $f$ admits the multiscale representation (where equality is interpreted in $L^2$- sense)*

$$f(x) = \int_{s=0}^{\infty} u(x,s)\,ds,$$

*with energy decomposition*

$$\|f\|_{L^2}^2 = \int_{s=0}^{\infty} \frac{1}{\lambda(s)} |u(\cdot,s)|_{BV}\,ds.$$

We show that $\lim_{t\to\infty}\|V(\cdot,t)\|_{L^2} \to 0$. What happens for $f \in L^2$ ?

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# 4. Modifications to the $(BV, L^2)$-based IDE

# Filtered IDE model

- Recall heat equation :

$$\frac{\partial u}{\partial t} = \Delta u.$$

- Perona Malik model:

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(g(|G_\sigma \star \nabla u|)\nabla u\right).$$

**Filtered IDE model**

$$\int_0^t u(x,s)\,ds = f(x) + \frac{g(|G_\sigma \star \nabla u(x,t)|)}{2\lambda(t)} \operatorname{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right); \quad \left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0,$$

To compute this IDE we use a fixed point iteration as before with $g(|G_\sigma \star \nabla u(x,t)|)$.

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# Numerical results of the filtered IDE model

Numerical results of $\int_0^t u(x,s)ds = f(x) + \frac{g(|G_\sigma \star \nabla u(x,t)|)}{2\lambda(t)} \operatorname{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right)$.

**Figure:** (a)–(d) The above images depict $\int_0^t u(x,s)ds$ for $t = 1, 4, 6, 10$. Here, $\lambda(t) = 0.002 \times 2^t$. Here the function $g(s) = \frac{1}{1+(s/5)^2}$.

# The ORIGINAL IDE model applied to Lenna

Numerical result for $\int_0^t u(x,s)ds = f(x) + \frac{1}{2\lambda(t)} \operatorname{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right)$.

**Figure:** (a)–(d) As $\lambda(t) \to \infty$, the images $\int_0^t u(x,s)ds$ are shown above for $t = 1, 4, 6, 10$. Here, $\lambda(t) = 0.002 \times 2^t$.

# The ORIGINAL IDE model applied to MRI image

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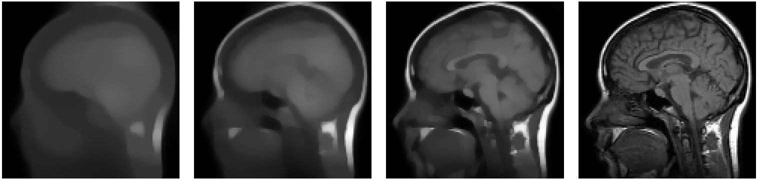

**Figure:** (a)–(d) The above images depict $\int_0^t u(x,s)ds$ for $t = 1, 4, 6, 10$ for the ORIGINAL IDE. Here, $\lambda(t) = 0.002 \times 2^t$.

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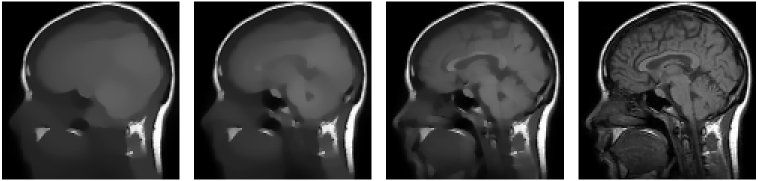

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# IDE with tangential smoothing modification

- The Heat equation

$$\frac{\partial u}{\partial t} = \Delta u.$$

- Note: $\Delta u = u_{TT} + u_{NN}$ and $u_{TT} := |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$.
  Alvarez et al. modification model:

$$\frac{\partial u}{\partial t} = g(|G_\sigma \star \nabla u|)|\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right),$$

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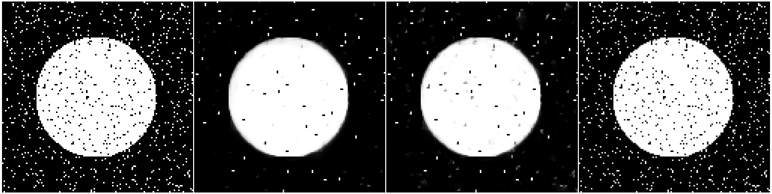

**Figure:** A given noisy image $f$ and the IDE images, $\int_0^t u(\cdot,s)\,ds$, at $t = 1, 4, 7$. Here, the scaling function is $\lambda(t) = 0.002 \times 2^t$. Most of the noise is present at scale $t = 7$.

Numerical results for

$$\int_0^t u(x,s)\,ds = f(x) + \frac{1}{2\lambda(t)} |\nabla u(x,t)| \operatorname{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right).$$

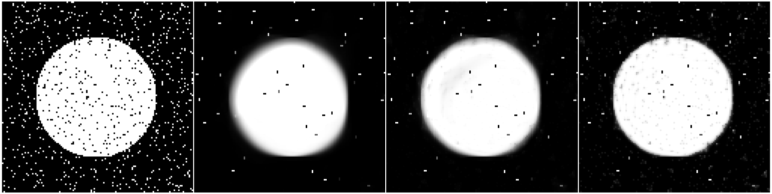

**Figure:** The same noisy image $f$ and the corresponding $\int_0^t u(\cdot,s)\,ds$, of the IDE with tangential smoothing at $t = 1, 4, 7$. The same scaling function as before, $\lambda(t) = 0.002 \times 2^t$. Large portion of the noise is suppressed at $t = 7$ but there is normal diffusion of edges.

# Numerical results for filtered IDE with tangential smoothing

Numerical results for

$$\int_0^t u(x,s)\,ds = f(x) + \frac{g(|G_\sigma \star \nabla u(x,t)|)}{2\lambda(t)} |\nabla u(x,t)| \operatorname{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right).$$

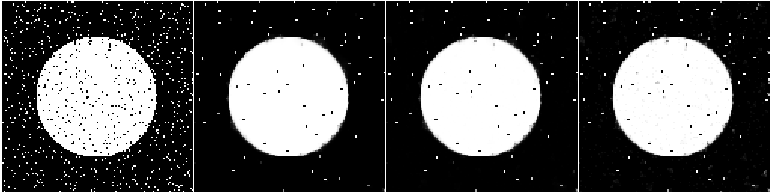

**Figure:** The same noisy image and the images, $\int_0^t u(\cdot,s)\,ds$, of IDE with tangential smoothing and filtering at $t = 1, 4, 7$. Here, $\lambda(t) = 0.002 \times 2^t$ and $g(s) = 1/(1+(s/5)^2)$. Noise is suppressed with minimal normal edge diffusion.

**Deblurring with IDE**

# TNV scheme with “intensity quanta” $\tau$ and blurring

- Let $\tau$ be the small intensity of quanta, with this the **ROF decomposition** becomes:

$$f = \tau K u_{\lambda_0} + v_{\lambda_0}, \quad [u_{\lambda_0}, v_{\lambda_0}] = \underset{\{f = \tau K u + v\}}{\text{arginf}} \left( \int_\Omega |\nabla u| + \frac{\lambda_0}{\tau} \int_\Omega |f - \tau K u|^2 \right).$$

- $v_{\lambda_0}$ can be decomposed with a scaling parameter $\lambda_1 > \lambda_0$.

$$v_{\lambda_0} = \tau K u_{\lambda_1} + v_{\lambda_1}, \quad [u_{\lambda_1}, v_{\lambda_1}] = \underset{\{v_{\lambda_0} = \tau K u + v\}}{\text{arginf}} \left( \int_\Omega |\nabla u| + \frac{\lambda_1}{\tau} \int_\Omega |v_{\lambda_0} - \tau K u|^2 \right).$$

- **TNV multiscale decomposition**

$$v_{\lambda_{k-1}} = \tau K u_{\lambda_k} + v_{\lambda_k}, \quad [u_{\lambda_k}, v_{\lambda_k}] = \underset{\{v_{\lambda_{k-1}} = \tau K u + v\}}{\text{arginf}} \left( \int_\Omega |\nabla u| + \frac{\lambda_k}{\tau} \int_\Omega |v_{\lambda_{k-1}} - \tau K u|^2 \right)$$

- With this scheme after $N + 1$ steps we get:

$$\begin{aligned} f &= \tau K u_{\lambda_0} + v_{\lambda_0} \\ &= \tau K u_{\lambda_0} + \tau K u_{\lambda_1} + v_{\lambda_1} \\ &= \tau K u_0 + \tau u_1 + \tau K u_2 + v_2 \\ &= \ldots \\ &= \tau K u_{\lambda_0} + \tau K u_{\lambda_1} + \ldots + \tau K u_{\lambda_N} + v_{\lambda_N}. \end{aligned}$$

i.e. a nonlinear multiscale decomposition: $f = \sum_{k=0}^{N} \tau K u_{\lambda_k} + v_{\lambda_N}$.

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# TNV scheme with $\tau$ and deblurring

- TNV scheme with deblurring reads:

$$\tau \sum_{k=0}^{N} K u_{\lambda_k} = f - v_{\lambda_N}.$$

$$\tau \sum_{k=0}^{N} K^* K u_{\lambda_k} = K^* f - K^* v_{\lambda_N}. \qquad (1)$$

- The Euler-Lagrange for the $N^{\mathrm{th}}$ step:

$$K^* v_{\lambda_{N-1}} = \tau K^* K u_{\lambda_N} \underbrace{- \frac{1}{2\lambda_N} \operatorname{div}\left(\frac{\nabla u_{\lambda_N}}{|\nabla u_{\lambda_N}|}\right)}_{K^* v_{\lambda_N}},$$

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$$\sum_{k=0}^{N} K^* K u_{\lambda_k} \tau = K^* f + \frac{1}{2\lambda_N} \operatorname{div}\left( \frac{\nabla u_{\lambda_N}}{|\nabla u_{\lambda_N}|} \right).$$

$$\int_0^t K^* K u(x,s)\, ds = K^* f(x) + \frac{1}{2\lambda(t)} \operatorname{div}\left( \frac{\nabla u(x,t)}{|\nabla u(x,t)|} \right).$$

# TNV scheme with $\tau$ and deblurring

- TNV scheme with deblurring reads:

$$\tau \sum_{k=0}^{N} K u_{\lambda_k} = f - v_{\lambda_N}.$$

$$\tau \sum_{k=0}^{N} K^* K u_{\lambda_k} = K^* f - K^* v_{\lambda_N}. \tag{1}$$

- The Euler-Lagrange for the $N^{\mathrm{th}}$ step:

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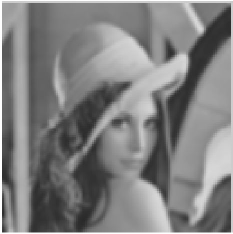

(a)

(b)

**Figure:** Image (a) shows a blurred image of Lenna blurred using a Gaussian kernel with $\sigma = 1$. Image (b) shows the result of the deblurring IDE model, as $t \to \infty$.

E. Tadmor, P. Athavale, *Multiscale image representation using novel integro-differential equations*, Inverse Problems in Imaging, **3** (2009), 693–710.

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E. Tadmor, P. Athavale, *Multiscale image representation using novel integro-differential equations*, Inverse Problems in Imaging, **3** (2009), 693–710.

## 5. IDE based on $(BV, L^1)$ image decomposition

# $(BV, L^1)$ image decomposition

- $(BV, L^1)$ model (Alliney, Nikolova, Chan-Esedoğlu, Allard, Aujol)

$$f = u_\lambda + v_\lambda, \quad [u_\lambda, v_\lambda] := \operatorname*{arginf}_{f=u+v} \left( \int_\Omega |\nabla u| + \lambda \int_\Omega |f - u| \right).$$

- This decomposition is contrast invariant and
- The scale-space generated is geometric in nature. (Chan-Esedoğlu, 2005)

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# $(BV, L^1)$ hierarchical scheme with $\tau$

- $N^{th}$ step in $(BV, L^1)$ scheme: $\tau u_{\lambda_k} + v_{\lambda_k} = v_{\lambda_{k-1}}$

$$[u_{\lambda_N}, v_{\lambda_N}] = \underset{\{v_{\lambda_{N-1}}=\tau u+v\}}{\operatorname{arginf}} \left( \int_\Omega |\nabla u| + \frac{\lambda_N}{\tau} \int_\Omega |v_{\lambda_{N-1}} - \tau u| \right)$$

$$\operatorname{sgn}(\tau u_{\lambda_N} - v_{\lambda_{N-1}}) = \frac{1}{\lambda_N} \operatorname{div} \left( \frac{\nabla u_{\lambda_N}}{|\nabla u_{\lambda_N}|} \right)$$

we have: $v_{\lambda_{N-1}} = f - \sum_{k=0}^{N-1} \tau u_{\lambda_k} \Rightarrow$ .

$$\operatorname{sgn}\left( \sum_{k=0}^{N} u_{\lambda_k} \tau - f \right) = \frac{1}{\lambda_N} \operatorname{div} \left( \frac{\nabla u_{\lambda_N}}{|\nabla u_{\lambda_N}|} \right).$$

This motivates the following IDE:

$$\operatorname{sgn}\left( \int_{s=0}^{t} u(x, s)\, dx - f(x) \right) = \frac{1}{\lambda(t)} \operatorname{div} \left( \frac{\nabla u(x, t)}{|\nabla u(x, t)|} \right)$$

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# Multiscale image representation using $(BV, L^1)$ IDE

$$\operatorname{sgn}\left(\int_{s=0}^{t} u(x,s)\,dx - f(x)\right) = \frac{1}{\lambda(t)}\operatorname{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right)$$

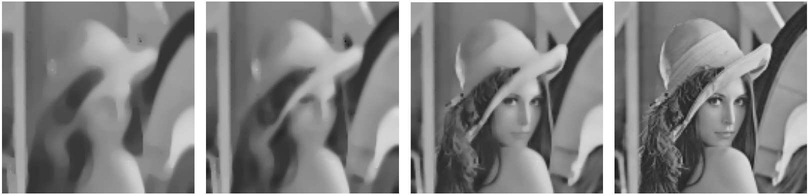

**Figure:** The above image show $\int_0^t u(\cdot,s)\,ds$ for the $(BV, L^1)$ IDE for $t = 1, 6, 9, 15$.

# Scale space generated by $(BV, L^1)$ IDE

$$\operatorname{sgn}\left(\int_{s=0}^{t} u(x,s)\,dx - f(x)\right) = \frac{1}{\lambda(t)} \operatorname{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right)$$

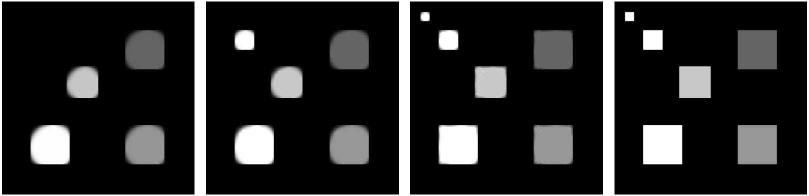

**Figure:** The above image show $\int_0^t u(\cdot, s)\,ds$ for the $(BV, L^1)$ IDE for $t = 1, 3, 5, 7$.

# Compare this with the scale space generated by $(BV, L^2)$ IDE

$$\int_{s=0}^{t} u(x,s)\,dx = f(x) + \frac{1}{2\lambda(t)}\,\mathrm{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right)$$

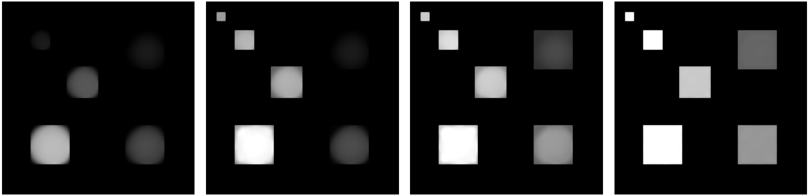

**Figure:** The above image show $\int_0^t u(\cdot, s)\,ds$ for the $(BV, L^2)$ IDE for $t = 1, 6, 7, 10$.

Athavale, Tadmor, *Integro-Differential Equations Based on* $(BV, L^1)$ *Image Decomposition*, SIAM J. Imaging Sci. 4, pp. 300-312.

## Proton therapy applications

# Denoising using $(BV, L^1)$ IDE

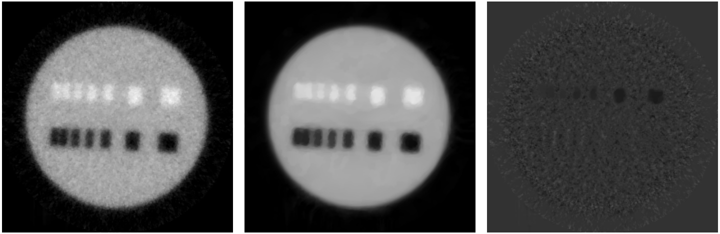

**Figure:** The above images show the original noisy image*, $\int_0^t u(\cdot, s)\, ds$ for the $(BV, L^1)$ IDE for $t = 7$ and the corresponding residual.

* *Noisy image provided by Dr. Reinhard, Loma Linda University.*

## 6. A few theoretical results for $(BV, L^1)$-IDE

# Some properties of this IDE

**Theorem (I)**

*For the IDE model*

$$\text{sgn}\left(\int_0^t u(x,s)\,ds - f(x)\right) = \frac{1}{\lambda(t)}\,\text{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right),$$

*let $V(\cdot,t)$ be the residual, and $\mathcal{U}(\cdot,t) := \int_0^t u(x,s)\,ds$*

$$V(\cdot,t) := f - \mathcal{U}(\cdot,t).$$

*Then size of the signum of residual is dictated by the scaling function $\lambda(t)$,*

$$\|\text{sgn}\,(V(\cdot,t))\|_* = \frac{1}{\lambda(t)}.$$

*Recall, for $(BV, L^2)$-based IDE we had*

$$\|V(\cdot,t)\|_* = \frac{1}{2\lambda(t)}.$$

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$$\|V(\cdot,t)\|_* = \frac{1}{2\lambda(t)}.$$

**Theorem (II)**

*Moreover, we have the following $L^1$-energy decomposition,*

$$\int_0^t \frac{1}{\lambda(s)} |u(\cdot, s)|_{BV} \, ds + \|V(\cdot, t)\|_{L^1} = \|f\|_{L^1}.$$

*Recall, for $(BV, L^2)$-based IDE we had the following $L^2$-energy decomposition:*

$$\int_0^t \frac{1}{\lambda(s)} |u(\cdot, s)|_{BV} \, ds + \|V(\cdot, t)\|_{L^2}^2 = \|f\|_{L^2}^2.$$

## Theorem (II)

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## 7. Modifications to the $(BV, L^1)$-IDE

# The $(BV, L^1)$ IDE with filtered diffusion.

Results for the $(BV, L^1)$ IDE with filtered diffusion:

$$\operatorname{sgn}\left(\int_{s=0}^{t} u(x,s)\,dx - f(x)\right) = \frac{g(|G_\sigma \star \nabla u(x,t)|)}{\lambda(t)} \operatorname{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right)$$

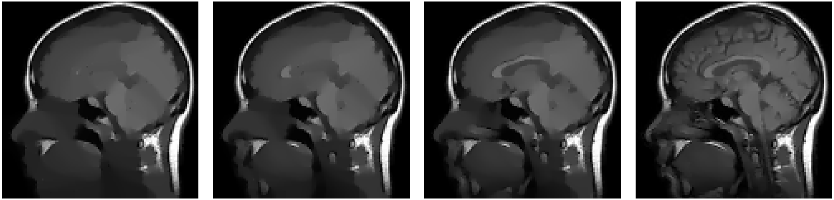

**Figure:** The above image show $\int_0^t u(\cdot, s)\,ds$ for the $(BV, L^1)$ IDE for $t = 1, 6, 7, 10$.

**Compare** these results for the original $(BV, L^1)$ IDE:

$$\operatorname{sgn}\left(\int_{s=0}^{t} u(x,s)\,dx - f(x)\right) = \frac{1}{\lambda(t)} \operatorname{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right)$$

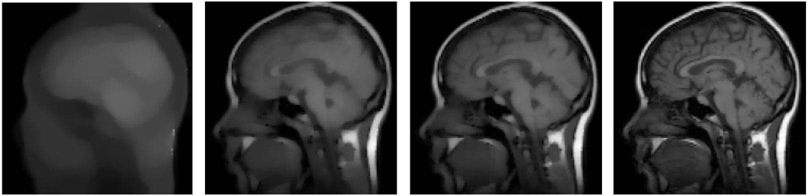

**Figure:** The above image show $\int_0^t u(\cdot, s)\,ds$ for the $(BV, L^1)$ IDE for $t = 1, 6, 7, 10$.

# Results for filtered $(BV, L^1)$ IDE with tangential smoothing

Numerical results for

$$\operatorname{sgn}\left(\int_0^t u(x,s)\,ds - f(x)\right) = \frac{g(|G_\sigma \star \nabla u(x,t)|)}{\lambda(t)} |\nabla u(x,t)| \operatorname{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right).$$

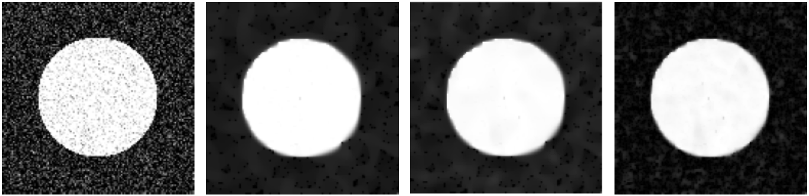

**Figure:** The same noisy image $f$ and the corresponding $\int_0^t u(\cdot, s)\,ds$, of the IDE with tangential smoothing at $t = 1, 4, 18$.

**Compare** these results with the numerical results for

$$\operatorname{sgn}\left(\int_0^t u(x,s)\,ds - f(x)\right) = \frac{1}{\lambda(t)}\operatorname{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right).$$

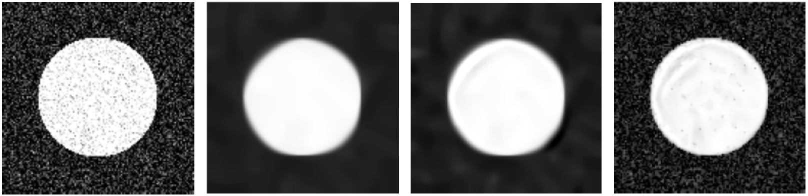

**Figure:** A given noisy image $f$ and the IDE images, $\int_0^t u(\cdot,s)\,ds$, at $t = 1, 4, 18$.

**Let's connect the dots!**

# Connecting the dots ...

**Heat equation**

$\Downarrow$

Perona-Malik

$\Downarrow$

Nordström

$\Downarrow$

Rudin Osher Fatemi

$\Downarrow$

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The novel integro-differential equations

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**The novel integro-differential equations**

*THANK YOU*

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