Functionally Fitted Explicit Two-Step Peer Methods

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This talk deals with the numerical solution of IVPs for ODE systems

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0 \in \mathbb{R}^N, \tag{1}$$

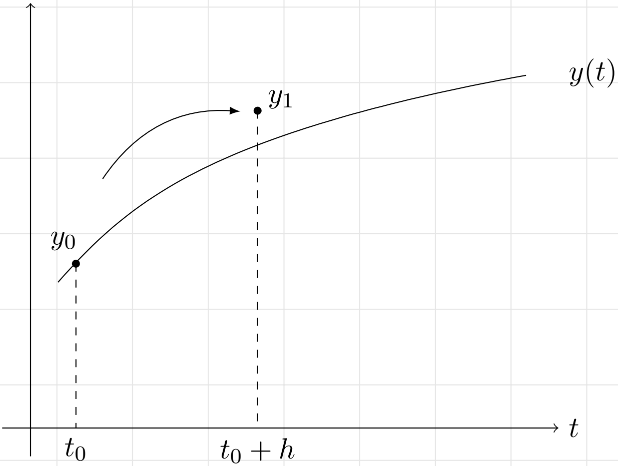
with oscillating or periodic solutions by means of explicit two-stage peer methods.

We present a class of numerical methods called "fitted two-step peer methods" for the numerical integration of periodic problems whose frequency is approximately known in advance.

- These methods combine the advantages of Runge-Kutta and multistep ones to obtain high stage order.
- Introduced by Weiner *et al* (2004, 2005, 2009), . . . for parallel computation and extended to sequential computation.

A step of a RK method

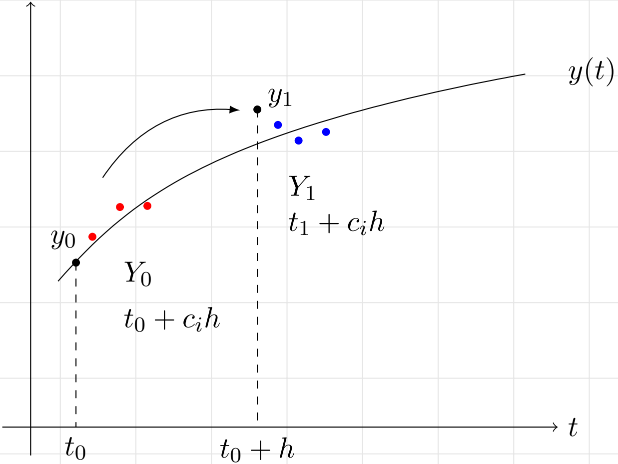
$$y_0 = y(t_0) \longrightarrow y_1 \simeq y(t_1)$$



A step of a Peer method

$$Y_1 = AY_0 + hBF(Y_0) + hRF(Y_1)$$

$$Y_0 \simeq (y(t_0), y(t_0 + c_1 h), \ldots) \longrightarrow Y_1 \simeq (y(t_1), y(t_1 + c_1 h), \ldots)$$



Functionally Fitted Peer

- **Standard methods** for IVPs are fitted to a polynomial approximation to the local solution. The fitting space is $\mathcal{F} = \{1, t, t^2, \ldots\}$.
- **Exponential Fitted methods** The fitting space is $\mathcal{F} = \{1, \mathrm{e}^{\pm iwt}, \mathrm{e}^{\pm i2wt}, \ldots\}$.
- **Functional Fitted methods** The fitting space is $\mathcal{F} = \{1, \varphi_1(t), \varphi_2(t), \ldots\}$.

Ref: Bettis (1979), Paternoster (1998), Simos (1998), Vanden Berghe *et al* (1999), Coleman *et al* (2000), Franco (2002), Ixaru *et al* (2004), ...

Background

In Functionally Fitted s-stages RK methods the solution of the IVP (1) advances to $(t_n, y_n) \to (t_{n+1} = t_n + h, y_{n+1})$ by means of the formulas

$$y_{n+1} = \gamma_0 \, y_n + h \sum_{j=1}^{s} b_j \; f(t_n + c_j h, Y_{n,j}), \qquad (2)$$

$$Y_{n,j} = \gamma_j \, y_n + h \sum_{k=1}^{s} a_{jk} \; f(t_n + c_k h, Y_{n,k}), \quad j = 1, \ldots, s, \qquad (3)$$

where

$$\begin{array}{c|c|c} c = (c_j)_{j=1}^{s} & \gamma = (\gamma_j)_{j=1}^{s} & A = (a_{jk}) \in \mathbb{R}^{s \times s} \\ \hline & \gamma_0 & b^T = (b_j)_{j=1}^{s} \end{array} \qquad (4)$$

are the real coefficients that define the method.

In standard RK methods all $\gamma_j = 1$ and the remaining coefficients are fixed numbers. In fitted methods they depend in general on the time step h, the starting time t_n and the space $\mathcal{F}$ of fitting functions.

We will assume $(q+1)$-dim. fitting spaces

$$\mathcal{F} = \mathcal{F}_q = \langle \varphi_0(t), \varphi_1(t), \ldots, \varphi_q(t) \rangle,$$

of smooth linearly independent real functions in $[t_0, t_0+T]$ in the sense that the Wronskian matrix

$$W\left(\varphi_0, \varphi_1, \ldots, \varphi_q\right)(t) = \begin{pmatrix} \varphi_0(t) & \varphi_1(t) & \ldots & \varphi_q(t) \\ \dot{\varphi}_0(t) & \dot{\varphi}_1(t) & \ldots & \dot{\varphi}_q(t) \\ \vdots & \vdots & & \vdots \\ \varphi_0^{(q)}(t) & \varphi_1^{(q)}(t) & \ldots & \varphi_q^{(q)}(t) \end{pmatrix}$$

is non singular for all $t \in [t_0, t_0+T]$.

To have a RK method fitted to $\mathcal{F}_q$, the available coefficients a_{ij}, c_i, γ_i, b_i are selected so that they satisfy the fitting conditions

$$\varphi(t_n + h) = \gamma_0 \, \varphi(t_n) + h \sum_{j=1}^{s} b_j \, \dot{\varphi}(t_n + c_j h), \quad (5)$$

$$\varphi(t_n + c_j h) = \gamma_j \, \varphi(t_n) + h \sum_{k=1}^{s} a_{jk} \, \dot{\varphi}(t_n + c_k h), \quad j = 1, \ldots, s \quad (6)$$

for all $\varphi \in \mathcal{F}_q$.

The above conditions imply by linearity that the corresponding RK method integrates exactly any local solution $y(t; t_n, y_n)$ of $y' = f(t, y)$ such that $y(t; t_n, y_n) \in \mathcal{F}_q$.

Drawback

For explicit RK fitted methods $c_1 = 0, \gamma_1 = 1$ and then the second one

$$\varphi(t_n + c_2 h) = \gamma_2 \varphi(t_n) + h a_{21} \dot{\varphi}(t_n),$$

for a fixed node c_2, has only the two free parameters $(\gamma_2,\, a_{21})$ and $q \leq 1$ and this implies serious restrictions in the dimensionality of the fitting space.

One remedy

We consider the so called explicit two-step peer methods, recently introduced by R. Weiner, B. A. Schmitt *et al* as an alternative to classical Runge–Kutta (RK) and multistep methods attempting to combine the advantages of these two classes of methods.

Two-Step Peer Methods

Given a set of admissible fixed nodes $\{c_j\}_{j=1}^{s}$ in the sense that

$$c_1, c_2, \ldots, c_s, 1 + c_1, 1 + c_2, \ldots, 1 + c_s$$

is a non confluent set of nodes, and starting from known approximations $Y_{0,j}$ to $y(t_0 + c_j h)$, $j = 1, \ldots, s$ we obtain the new approximations

$$Y_{1,j} \simeq y(t_1 + c_j h) \quad \text{where} \quad t_1 = t_0 + h,$$

by means of

$$\begin{aligned} Y_{1,j} = & \sum_{k=1}^{s} a_{jk} Y_{0,k} + h \sum_{k=1}^{s} b_{jk} f(t_0 + c_k h, Y_{0,k}) \\ & + h \sum_{k=1}^{j-1} r_{jk} f(t_1 + c_k h, Y_{1,k}), \qquad (j = 1, \ldots, s). \end{aligned} \tag{7}$$

A, $B \in \mathbb{R}^{s \times s}$ full matrices and $R \in \mathbb{R}^{s \times s}$ strictly lower triangular are the free parameters that define the method with $A\mathbf{e} = \mathbf{e}$ to ensure the preconsistency condition.

Extending the definition of fitted RK methods to PEER methods, we will say that the explicit two-step peer method is fitted to $\mathcal{F}_q$ if

$$\varphi(t_1 + c_j h) = \sum_{k=1}^{s} a_{jk}\, \varphi(t_0 + c_k h) + h \sum_{k=1}^{s} b_{jk}\, \dot{\varphi}(t_0 + c_k h) + h \sum_{k=1}^{j-1} r_{jk}\, \dot{\varphi}(t_1 + c_k h), \quad j = 1, \ldots, s \tag{8}$$

holds for all $\varphi \in \mathcal{F}_q$.

- At each stage we have at least $2s - 1$ free parameters
- It is possible to obtain explicit methods that attain high stage order
- Are good candidates to obtain explicit methods fitted to spaces $\mathcal{F}_q$ with q large.
- The authors have derived in (2010), s stage methods with $q = 2s - 1$ taking into account some stability and accuracy requirements

In our study of the order of a Fitted Peer Methods it will be sufficient to consider a scalar (non-linear) equation ($m = 1$), and they can be written in the vector form

$$\mathbf{Y}_1 = A\,\mathbf{Y}_0 + h\,B\,\mathbf{F}_0 + h\,R\,\mathbf{F}_1, \tag{9}$$

where

$$\begin{aligned} \mathbf{Y}_k &= (Y_{k,1}, \dots, Y_{k,s})^T \in \mathbb{R}^s, \\ \mathbf{e} &= (1, \dots, 1)^T \in \mathbb{R}^s, \\ \mathbf{c} &= (c_1, \dots, c_s)^T \in \mathbb{R}^s, \\ \mathbf{F}_k &= f(t_k\mathbf{e} + h\mathbf{c}, \mathbf{Y}_k) = (f(t_k + hc_j, Y_{k,j}))_{j=1}^s \in \mathbb{R}^s. \end{aligned}$$

0-Stability

For the zero stability we only consider Peer Methods with the stronger requirement

$$\lambda_1(A) = 1, \qquad \lambda_j(A) = 0, \quad j = 2, \ldots, s, \tag{10}$$

and take A with the form

$$A = P^{-1}\,\widehat{A}\,P, \tag{11}$$

with P and $\widehat{A}$ of type

$$P = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ p_{21} & 1 & \ldots & 0 \\ p_{31} & p_{32} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{s1} & p_{s2} & \ldots & 1 \end{pmatrix}, \quad \widehat{A} = \begin{pmatrix} 1 & \widehat{a}_{12} & \ldots & \widehat{a}_{1s} \\ & 0 & \widehat{a}_{23} & \widehat{a}_{2s} \\ & & \ddots & \vdots \\ & & & \widehat{a}_{s-1,s} \\ & & & 0 \end{pmatrix},$$

that satisfy (10). Note that the pre-consistency condition $A\mathbf{e} = \mathbf{e}$ implies that $P\mathbf{e} = \mathbf{e}_1$.

We associate to $\mathbf{Y}_1 = A\mathbf{Y}_0 + hB\mathbf{F}_0 + hR\mathbf{F}_1$ the linear s-dim vector valued operator $\mathcal{L}[\varphi; h]$ defined by

$$\begin{aligned} \mathcal{L}[\varphi; h](t) \equiv \varphi((t+h)\mathbf{e} + h\mathbf{c}) - A\,\varphi(t\mathbf{e} + h\mathbf{c}) \\ - h\,B\,\dot{\varphi}(t\mathbf{e} + h\mathbf{c}) - h\,R\,\dot{\varphi}((t+h)\mathbf{e} + h\mathbf{c}). \end{aligned} \tag{12}$$

Definition

For a given set of admissible nodes and a fitting space $\mathcal{F}_q = \langle \varphi_0(t), \varphi_1(t), \dots, \varphi_q(t) \rangle$ the Peer Method is fitted to the linear space $\mathcal{F}_q$ with step size h at t_0 if

$$\mathcal{L}[\varphi; h](t_0) = 0, \qquad \forall \varphi \in \mathcal{F}_q. \tag{13}$$

Remarks:

- If the starting values $(Y_{0,j})_{j=1}^{s}$ belong to a solution of the differential equation contained in the fitting space $\mathcal{F}_q$, then the Peer method gives the exact values of the solution.

- In the polynomial case, $\mathcal{F}_q = \Pi_q$, q is the stage order and

$$\mathcal{L}[y;h](t) = \mathcal{O}(h^{q+1}) \quad \text{for all} \quad y \in \mathcal{C}^\infty,$$

  and this condition turns out to be independent of t.

- If $\mathbf{Z}(t) = P\,\varphi(t\mathbf{e} + h\mathbf{c})$, we have $\mathcal{L}[\varphi;h](t_0) = P^{-1}\,\widehat{\mathcal{L}}[\varphi;h](t_0)$, with

$$\widehat{\mathcal{L}}[\varphi;h](t_0) = \mathbf{Z}(t+h) - \widehat{A}\,\mathbf{Z}(t) - h\widehat{B}\,\dot{\mathbf{Z}}(t) - h\widehat{R}\,\dot{\mathbf{Z}}(t+h)$$

  Then, the Peer method is fitted to $\mathcal{F}_q$ iff $\widehat{\mathcal{L}}[\varphi;h](t_0) = 0, \forall \varphi \in \mathcal{F}_q$.

When the coefficients are independent of t_n?

We give sufficient conditions on the functions of $\mathcal{F}_q$ that ensure that $\mathcal{L}[\varphi; h](t_0)$ is independent of t_0 and therefore the coefficients of the fitted method can be chosen independent of t_0.

Theorem

Let $\mathcal{F}_q$ be the $(q+1)$-dim space of solutions of an homogeneous linear differential equation with constant coefficients with order $(q+1)$. If the linear operator $\mathcal{L}$ with A, B and R independent of t satisfies

$$\mathcal{L}[\varphi; h](t_0) = 0, \quad \forall \varphi \in \mathcal{F}_q$$

then

$$\mathcal{L}[\varphi; h](t) = 0, \quad \forall \varphi \in \mathcal{F}_q, \quad \forall\, t \in [t_0, t_0 + T].$$

Remarks

- For fitting spaces of solutions of linear homogeneous solutions with constant coefficients if the available coefficients A, B, R (that may depend of the nodes and the step size h) are fitted for some particular t_0 then they are fitted for all t.
- For fitting spaces that satisfy the assumptions of the above Theorem, if we take as basis point $t_0 = -hc_1$ then $\widehat{\mathcal{L}}[\varphi; h](-hc_1)$ depends on the nodes in the form of differences $(c_2 - c_1), \ldots, (c_s - c_1)$.

We assume that $F_q = \langle \varphi_0(t) = 1, \varphi_1(t), \ldots, \varphi_q(t) \rangle$ is a $(q+1)$-dim basis of solutions of the linear equation with constant coefficients

$$Q(D)u(t) \equiv u^{(q+1)}(t) + a_q u^{(q)}(t) + \ldots + a_1 u^{(1)}(t) = 0,$$

whose characteristic polynomial is

$$Q(z) = z^{q+1} + a_q z^q + \ldots + a_1 z = z^{\beta_0} (z - w_1)^{\beta_1} \ldots (z - w_r)^{\beta_r},$$

with $\beta_0 + \beta_1 + \ldots + \beta_r = q + 1$.

In the polynomial case it has been shown that:

- **i)** Given a set of admissible nodes $c_1, \ldots, c_s, 1 + c_1, \ldots, 1 + c_s$.
- **ii)** Given a lower triangular matrix $P = (p_{ij})$ with $p_{ii} = 1$ and $P\mathbf{e} = \mathbf{e}_1$.

The parameters in $\widehat{A}$, $\widehat{B}$ and $\widehat{R}$ can be obtained, under usual hypothesis, as solutions of s independent sets of linear equations in the unknowns

$$\begin{array}{r|cccc|cccc|cccc}
\text{Eq1}: & \widehat{a}_{12} & \widehat{a}_{13} & \ldots & \widehat{a}_{1s} & \widehat{b}_{11} & \widehat{b}_{12} & \ldots & \widehat{b}_{1s} & 0 & & & \\
\text{Eq2}: & & \widehat{a}_{23} & \ldots & \widehat{a}_{2s} & \widehat{b}_{21} & \widehat{b}_{22} & \ldots & \widehat{b}_{2s} & \widehat{r}_{21} & & & \\
\vdots & & & & & \vdots & & & \vdots & \vdots & & & \\
\text{Eq}s: & & & & 0 & \widehat{b}_{s1} & \widehat{b}_{s2} & \ldots & \widehat{b}_{ss} & \widehat{r}_{s1} & \ldots & \widehat{r}_{s,s-1} &
\end{array} \qquad (14)$$

It has been proved that if the free parameters are selected by attempting its exactness for the polynomials $\varphi(t) = t^k$, $k = 1, \ldots, 2s - 1$ then the corresponding method would have (stage) order $p = 2s - 1$.

We extend this situation to a more general case of spaces $\mathcal{F}_q$.

Theorem

Suppose that for a given set of admissible fixed nodes and constant matrix P the polynomially fitted two-step peer method with s stages has a unique solution with stage order $2s-1$, then:

1. *For any linear space $\mathcal{F}_{2s-1} = \langle 1, \varphi_1(t), \ldots, \varphi_{2s-1}(t) \rangle$ there exist a unique s-stage two step peer method fitted to this space for h sufficiently small. This peer method has the same nodes and P-matrix as the polynomially fitted method to Π_{2s-1} and the coefficients*

$$\widehat{A}_{\mathcal{F}} = \widehat{A}(t_0, h), \quad \widehat{B}_{\mathcal{F}} = \widehat{B}(t_0, h), \quad \widehat{R}_{\mathcal{F}} = \widehat{R}(t_0, h),$$

   may depend (apart of the fitting space) on t_0 and h.

2. *If $\mathcal{F}_{2s-1}$ is a basis of solutions of a linear equation with constant coefficients, $\widehat{A}_{\mathcal{F}}$, $\widehat{B}_{\mathcal{F}}$, $\widehat{R}_{\mathcal{F}}$, are independent of t_0 and depend only on the roots of $Q(D)$.*

3. *Further when all the roots of the polynomial $Q(D)$ tend to zero the coefficients $\widehat{A}_{\mathcal{F}}$, $\widehat{B}_{\mathcal{F}}$, $\widehat{R}_{\mathcal{F}}$ tend to those of the polynomial case.*

Two-stage Peer Methods

With $s = 2$ the pre consistency condition $P\mathbf{e} = \mathbf{e}_1$ implies that

$$P = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \tag{15}$$

On the other hand, the matrices $\widehat{A}, \widehat{B}, \widehat{R}$ will have the form

$$\widehat{A} = \begin{pmatrix} 1 & \widehat{a}_{12} \\ 0 & 0 \end{pmatrix}, \quad \widehat{B} = \begin{pmatrix} \widehat{b}_{11} & \widehat{b}_{12} \\ \widehat{b}_{21} & \widehat{b}_{22} \end{pmatrix}, \quad \widehat{R} = \begin{pmatrix} 0 & 0 \\ \widehat{r}_{21} & 0 \end{pmatrix}, \tag{16}$$

and

$$\mathbf{Z}(t) = P\,\varphi(t\mathbf{e} + h\mathbf{c}) = \begin{pmatrix} \varphi(t + c_1 h) \\ -\varphi(t + c_1 h) + \varphi(t + c_2 h) \end{pmatrix}. \tag{17}$$

Since the linear operator $\widehat{\mathcal{L}}$ is

$$\widehat{\mathcal{L}}[\varphi; h](t) = \mathbf{Z}(t+h) - \widehat{A}\,\mathbf{Z}(t) - h\widehat{B}\,\dot{\mathbf{Z}}(t) - h\widehat{R}\,\dot{\mathbf{Z}}(t+h)$$

we have two order conditions and in each condition there are three free parameters. The first equation with the parameters $\widehat{a}_{12}, \widehat{b}_{11}, \widehat{b}_{12}$ can be written in the form

$$\begin{aligned}&[\varphi(t+c_2h) - \varphi(t+c_1h)]\widehat{a}_{12} + h\dot{\varphi}(t+c_1h)\widehat{b}_{11}\\ &+ h[\dot{\varphi}(t+c_2h) - \dot{\varphi}(t+c_1h)]\widehat{b}_{12} = \varphi(t+h+c_1h) - \varphi(t+c_1h),\end{aligned}$$

and the second one with the parameters $\widehat{b}_{21}, \widehat{b}_{22}, \widehat{r}_{21}$ is

$$\begin{aligned}&h\dot{\varphi}(t+c_1h)\widehat{b}_{21} + h[\dot{\varphi}(t+c_2h) - \dot{\varphi}(t+c_1h)]\widehat{b}_{22}\\ &\qquad h\dot{\varphi}(t+h+c_1h)\widehat{r}_{21} = \varphi(t+h+c_2h) - \varphi(t+h+c_1h).\end{aligned}$$

These parameters will be determined by imposing that the above equations hold for the functions $\varphi_j(t), j = 1, 2, 3$ of the fitting space

$$\mathcal{F}_3 = \langle 1, t, \cos\omega t, \sin\omega t \rangle.$$

Solving the equations for $\widehat{r}_{21}$, $\widehat{a}_{12}$, $\widehat{b}_{11}, \widehat{b}_{12}$, $\widehat{b}_{21}, \widehat{b}_{22}$ we obtain

$$\begin{aligned}
\widehat{a}_{12} &= \frac{-1+\cos\nu+\cos(d\nu)-\cos\nu\cos(d\nu)+\nu\sin(d\nu)-\sin\nu\sin(d\nu)}{\Delta_1}, \\
\widehat{b}_{11} &= \frac{(-2+d-d\cos\nu)(1-\cos(d\nu)+d\sin\nu\sin(d\nu)}{\Delta_1}, \\
\widehat{b}_{12} &= -\frac{\nu-d\nu+d\nu\cos\nu-\nu\cos(d\nu)-\sin\nu+\sin(d\nu)\sin(\nu-d\nu)}{\nu\Delta_1},
\end{aligned}$$

with $d=c_2-c_1$, $\nu=hw$ and $\Delta_1=-2+2\cos(d\nu)+d\nu\sin(d\nu)$.
In a similar way, we get

$$\begin{aligned}
\widehat{r}_{21} &= \frac{\sin(d\nu/2)\,(d\nu\cos(d\nu/2)-2\cos\nu\sin(d\nu/2))}{\Delta_2}, \\
\widehat{b}_{21} &= -\frac{\sin(d\nu/2)\,(d\nu\cos(\nu-d\nu/2)-2\cos\nu\sin(d\nu/2))}{\Delta_2}, \\
\widehat{b}_{22} &= -\frac{\sin(\nu/2)\,(d\nu\cos(\nu/2)+\sin(\nu/2)-\sin(d\nu+\nu/2))}{\Delta_2},
\end{aligned}$$

with $\Delta_2=2\nu\sin(d\nu/2)\sin(\nu/2)\sin((\nu-d\nu)/2)$.

Three-stages Peer Methods

In the case of three stages Peer Methods, we have several possibilities,

- $\mathcal{F}_q = \{1, t, \cos wt, \sin wt, \cos 2wt, \sin 2wt\}$
- $\mathcal{F}_q = \{1, t, t^2, t^3, \cos wt, \sin wt\}$
- $\mathcal{F}_q = \{1, t, \cos w_1 t, \sin w_1 t, \cos w_2 t, \sin w_2 t\}$
- $\mathcal{F}_q = \{1, t, \cos wt, \sin wt, t\cos wt, t\sin wt\}$

For the sake of simplicity, we derive the fitted method associated to the 3-stage method developed by the authors (2010) and given by the coefficients:

$$[A|B] = \left[\begin{array}{ccc|ccc} 0.000855 & 0.692006 & 0.307138 & 0.000172 & 0.041579 & -0.01777 \\ 5.04047 & 6.19552 & -10.236 & 1.11675 & 41.799 & 21.9221 \\ 2.63153 & 3.56484 & -5.1963 & 0.593029 & 20.477 & 10.6664 \end{array}\right]$$

$$[c|R] = \left[\begin{array}{c|ccc} 0 & 0.000172 & 0 & 0 \\ 0.904 & -56.8554 & 0 & 0 \\ 1.141 & -27.3595 & 0.470412 & 0 \end{array}\right]$$

Test Problems

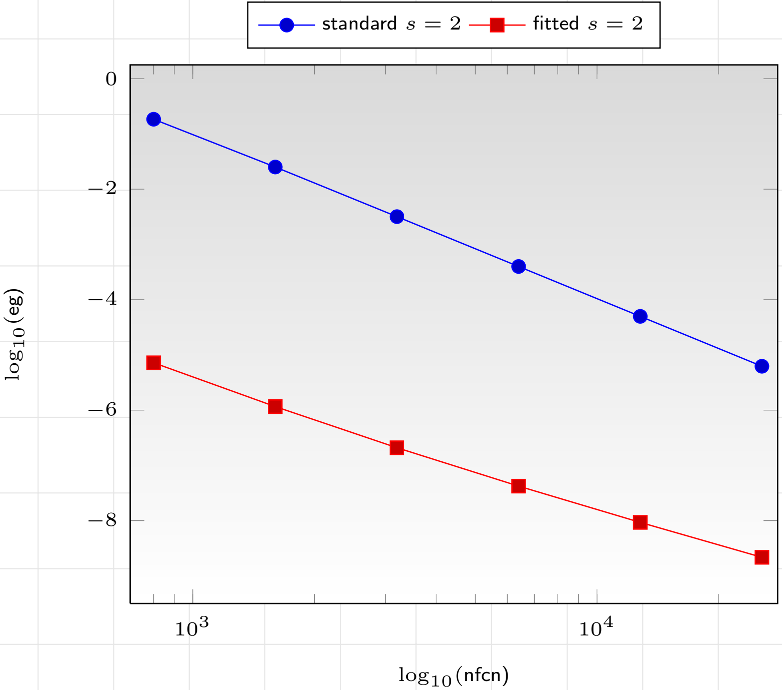
Duffing's equation

$$y'' + (\lambda^2 + k^2)y = 2k^2y^3, \quad t \in [0, 20]$$
$$y(0) = 0, \quad y'(0) = \lambda,$$

with $k = 0.035$ and $\lambda = 5$. The analytic solution is given by:

$$y(t) = \mathrm{sn}\left(\lambda t, (\tfrac{k}{\lambda})^2\right).$$

where sn represents the elliptic Jacobi function. We choose $\omega = 5i$, and the numerical results have been computed with the integration steps $\Delta t = \dfrac{1}{5 \times 2^m}$, $m = 1, \ldots, 6$.



standard $s = 2$
fitted $s = 2$
0
-2
-4
-6
-8
$\log_{10}$(eg)
10^3
10^4
$\log_{10}$(nfcn)

The Euler equations

$$y' = f(y) = \left((\alpha-\beta)y_2y_3,\ (1-\alpha)y_3y_1,\ (\beta-1)y_1y_2\right)^T,$$

with the initial values $y(0) = (0,1,1)^T$.

- It possesses two quadratic invariants: $G_1 = y_1^2 + y_2^2 + y_3^2$ and $G_2 = y_1^2 + \beta y_2^2 + \alpha y_3^2$
- Parameter values $\alpha = 1 + \frac{1}{\sqrt{1.51}}$ and $\beta = 1 - \frac{0.51}{\sqrt{1.51}}$. $\omega = 2\pi/T$, with $T = 7.45056320933095$. The exact solution of this IVP is given by

$$y(t) = \left(\sqrt{1.51}\,\mathrm{sn}(t,0.51),\ \mathrm{cn}(t,0.51),\ \mathrm{dn}(t,0.51)\right)^T,$$

  where sn, cn, dn are the elliptic Jacobi functions.
- The integration is carried out on the interval $[0,40]$ with step sizes $h = 1/(5\times 2^{j-2})$, $j = 1,\ldots,5$ and $w = 2\pi/Ti$.

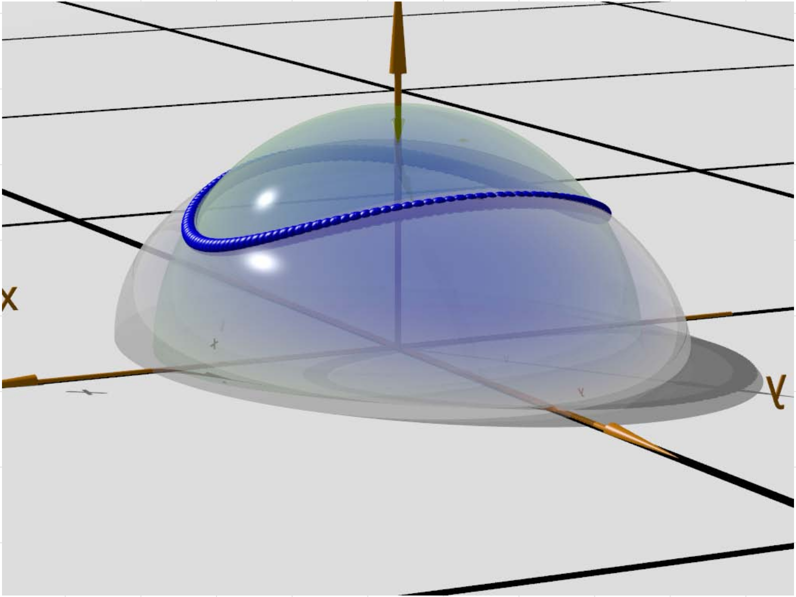
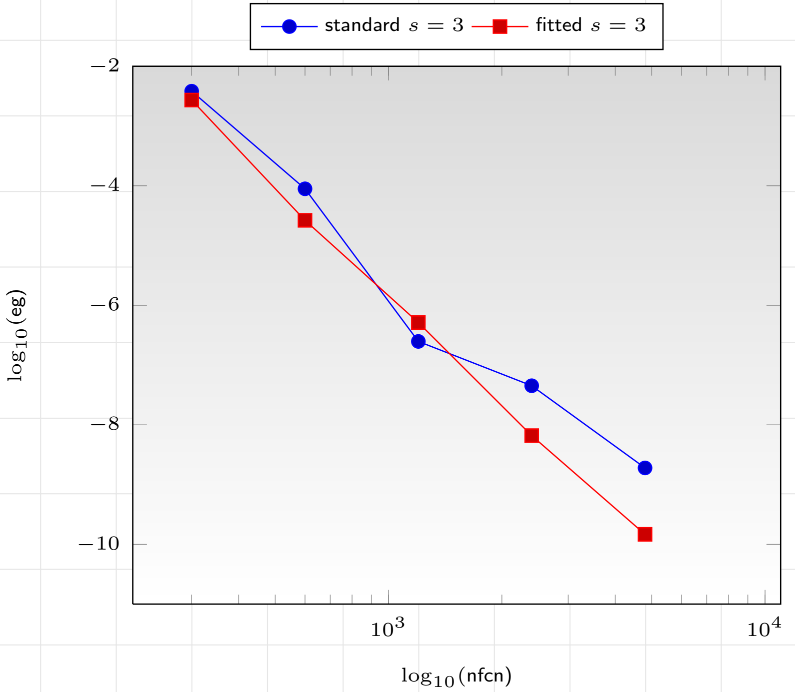


Figure: Intersection of the quadratic invariants G_1 and G_2



standard $s = 3$
fitted $s = 3$
-2
-4
-6
-8
-10
$\log_{10}(\text{eg})$
10^3
10^4
$\log_{10}(\text{nfcn})$

A perturbed Kepler's problem

The Hamiltonian function is

$$H(p,q) = \frac{1}{2}\left(p_1^2 + p_2^2\right) - \left(q_1^2 + q_2^2\right)^{-1/2} - (2\varepsilon + \varepsilon^2)/3\left(q_1^2 + q_2^2\right)^{-3/2},$$

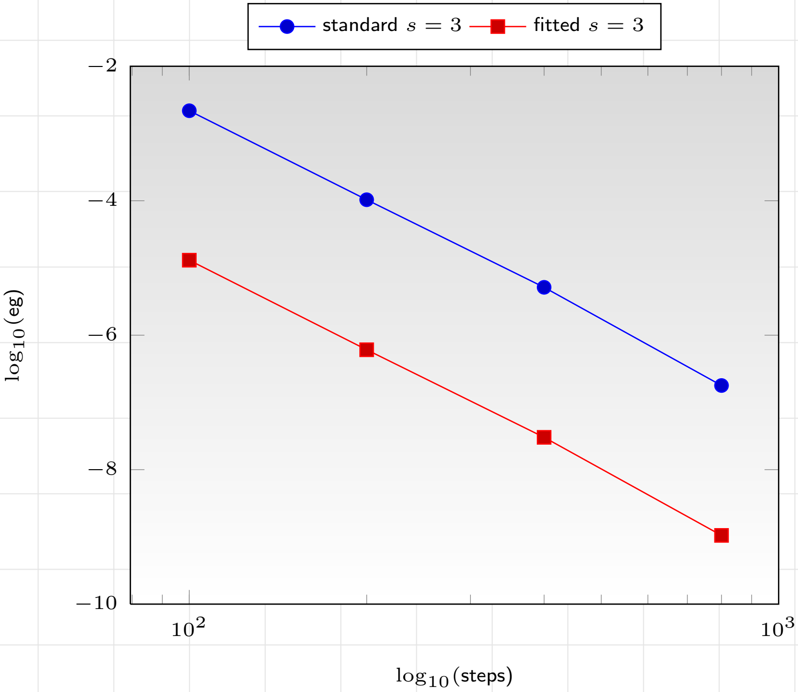
- Initial conditions:

$$q_1(0) = 1,\ q_2(0) = 0,\ p_1(0) = 0,\ p_2(0) = 1 + \varepsilon, \quad 0 < \varepsilon << 1$$

- The exact solution of this IVP is given by

$$q_1(t) = \cos(t + \varepsilon t), \quad q_2(t) = \sin(t + \varepsilon t), \quad p_i(t) = q_i'(t),\ i = 1, 2.$$

- The numerical results have been computed with the integration steps $\Delta t = \dfrac{\pi}{10 \times 2^m}$, $m = 0, \ldots, 3$. We take the parameter values $\varepsilon = 10^{-3}$, $\lambda = i$ and the problem is integrated up to $t_{end} = 10\pi$.


standard $s = 3$
fitted $s = 3$
-2
-4
-6
-8
-10
10^2
10^3
$\log_{10}$(eg)
$\log_{10}$(steps)

Conclusions

- In our numerical experiments, we have considered fitted methods for systems of equations with all components fitted to the same given frequency ω.
- It appear that an accurate estimation of the frequency is essential for the integrators based on fitted methods. This fact was already recognised by Vanden Berghe *et al* (2001)
- The accuracy of the fitted methods is in general superior to the non fitted ones of the same order.