Discontinuous Galerkin Methods for Bifurcation Phenomena in the Flow through Open Systems

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Acknowledgements

- Andrew Cliffe and Paul Houston (University of Nottingham)
- Tom Mullin and James Seddon (University of Manchester)
- Eric Phipps and Andy Salinger (Sandia National Laboratories)

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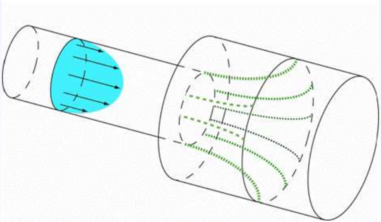
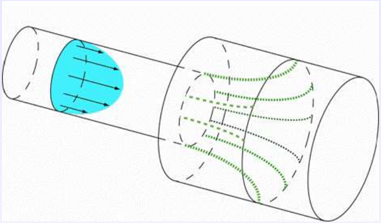
- This work is supported by the EPSRC under grant EP/E013724/1: Bifurcation phenomena in the flow through a sudden expansion in a pipe.



Outline

1. Background
2. Bifurcation in the presence of $O(2)$ symmetry
3. DG discretisation and *a posteriori* error estimation
4. Numerical experiments
5. Summary and Outlook



- Incompressible, viscous fluid satisfying Navier-Stokes equations

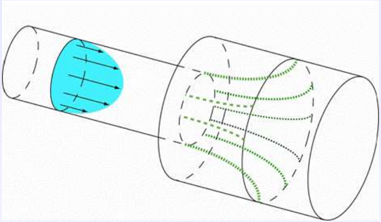
$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u} + \nabla\cdot(\mathbf{u}\otimes\mathbf{u}) + \nabla p &= 0, \\ \nabla\cdot\mathbf{u} &= 0. \end{aligned}$$

- Re is the non-dimensional Reynolds number

$$\mathrm{Re} = \frac{vL}{\nu},$$

where v is a typical velocity, L a typical length scale and ν the fluid viscosity.

Bifurcation in an Expanding Pipe



Problems:

- 3 spatial dimensions.
- Length of recirculation region varies linearly with Re.
- Bifurcations occur at high Re, therefore very long pipe is required.

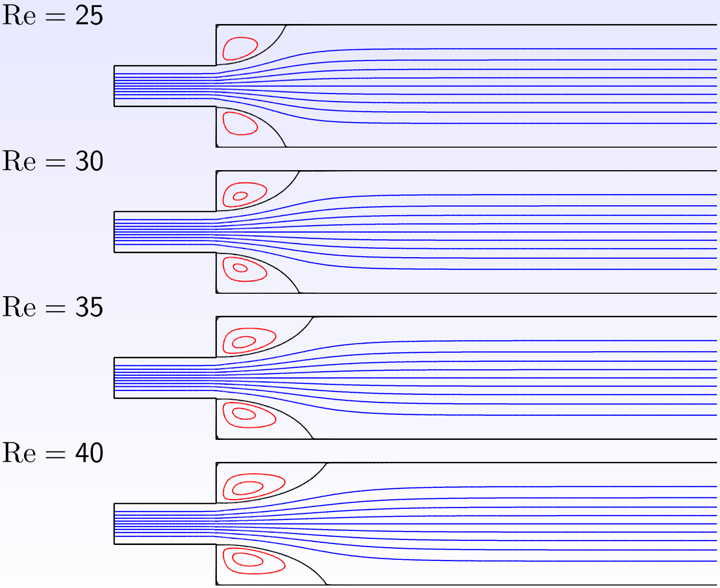
Solutions:

- Utilize symmetry of the pipe.
- Use mesh adaptivity.

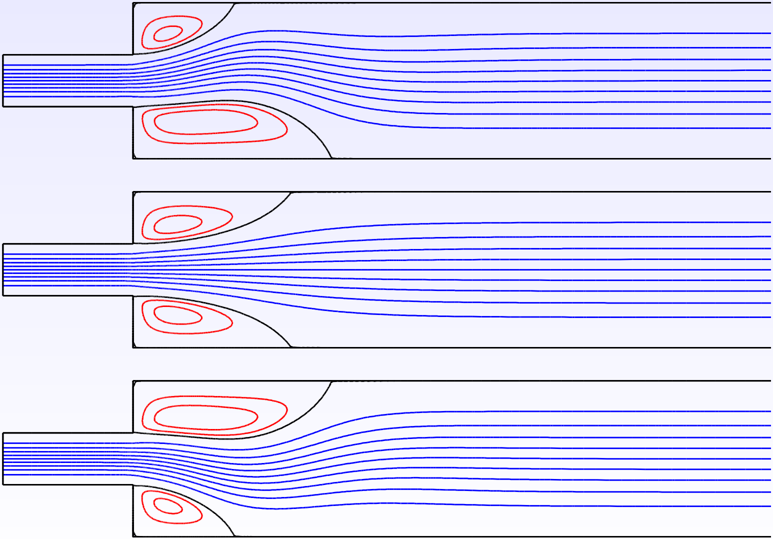
Bifurcation Phenomena in Open Systems

- Flow through a sudden expansion in a channel.
  Fearn, Mullin and Cliffe 1990.
  - Steady, Z_2 symmetry-breaking bifurcation.
    ($Re^c \approx 40$ for a $1:3$ expansion ratio)
- Flow past a cylinder in a channel.
  Jackson 1987; Cliffe and Tavener 2004.
  - Z_2 symmetry-breaking Hopf bifurcation.
    ($Re^c \approx 123$ for a $1:2$ blockage ratio)
- Flow past a sphere in a pipe.
  Tavener 1994; Cliffe, Spence and Tavener 2000.
  - Steady, $O(2)$ symmetry-breaking bifurcation.
    ($Re^c \approx 359$ for a $1:2$ blockage ratio)
- Flow in a pipe with a stenotic region.
  Sherwin & Blackburn 2005, 2007, Sherwin, Blackburn & Barkley 2008.
  - Steady, $O(2)$ symmetry-breaking bifurcation.
    ($Re^c \approx 721$ for a 75% occlusion)

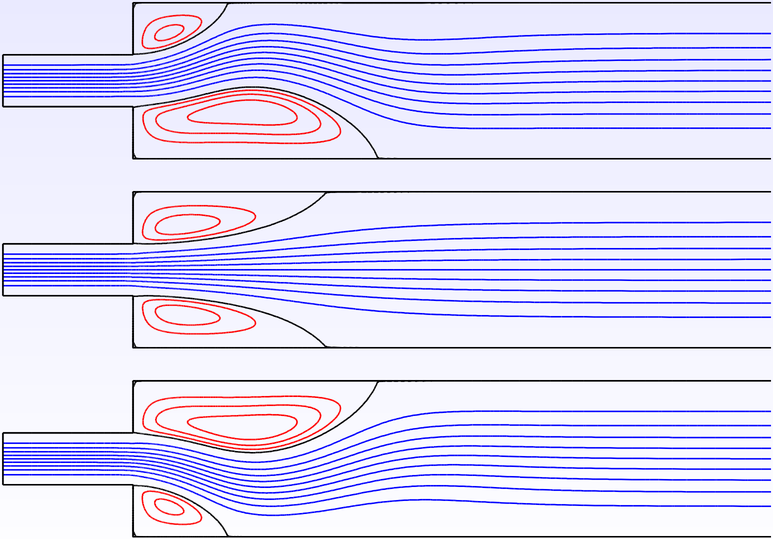
Channel with a Sudden Expansion



Channel with a Sudden Expansion - $\mathrm{Re} = 45$



Channel with a Sudden Expansion - $\mathrm{Re} = 55$



Nonlinear Problem

Consider the solution of the following nonlinear problem:

$$\frac{\partial u}{\partial t} + F(u, \lambda) = 0,$$

where

- u is the state variable(s);
- λ is a parameter (or set of parameters) of physical interest.
- F is a differential operator.

Bifurcation Problems

Nonlinear Problem

Consider the solution of the following nonlinear problem:

$$\frac{\partial u}{\partial t} + F(u, \lambda) = 0,$$

where

- u is the state variable(s);
- λ is a parameter (or set of parameters) of physical interest.
- F is a differential operator.

Fundamental questions include:

- How many solutions exist as λ is varied?
- Are the steady state solutions linearly stable?
- At what critical parameter value does a bifurcation occur?

Bifurcation Problems

- A (steady) bifurcation occurs at λ^* when the Jacobian

$$F_u(u^*, \lambda^*; \cdot)$$

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- The eigenvalues of $F_u(u, \lambda; \cdot)$ tell us whether a solution is stable or unstable.
  - If all eigenvalues have positive real part then the solution is linearly stable.
  - If any eigenvalue has negative real part, the solution is linearly unstable.

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- The eigenvalues of $F_u(u, \lambda; \cdot)$ tell us whether a solution is stable or unstable.
  - If all eigenvalues have positive real part then the solution is linearly stable.
  - If any eigenvalue has negative real part, the solution is linearly unstable.
  - The eigenvalues with smallest real part are termed the most dangerous eigenvalues.

We have two options:

- For a particular λ solve the eigenvalue problem: find $\mathbf{u} := (u, \phi, \mu)$ such that

$$\mathbf{E}(\mathbf{u}) \equiv \begin{pmatrix} F(u,\lambda) \\ F_u(u,\lambda;\phi) - \mu\phi \\ \langle \phi, g \rangle - 1 \end{pmatrix} = \mathbf{0},$$

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  for some appropriate g.
- Locate the critical parameter by solving directly:
  - In the case of a steady bifurcation: find $\mathbf{u}^c := (u^c, \phi^c, \lambda^c)$ such that

$$\mathbf{G}(\mathbf{u}^c) \equiv \begin{pmatrix} F(u^c,\lambda^c) \\ F_u(u^c,\lambda^c;\phi^c) \\ \langle \phi^c, g \rangle - 1 \end{pmatrix} = \mathbf{0},$$

    for some appropriate g.
  - Hopf bifurcation, similar but larger extended system.

Bifurcation in the presence of $O(2)$ Symmetry

- Exploit the underlying group structure within a physical system in order to rigorously justify the study of (equivalent) simplified problems.

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- Bifurcation with $O(2)$ symmetry.

  Vanderbanwhede 1982; Golubitsky & Schaeffer 1985; Golubitsky, Stewart & Schaeffer 1988; Healey & Treacy 1991; Aston 1991; Cliffe, Spence & Tavener 2000.

$O(2)$ Group

$O(2)$ is a group generated by

- Rotations r_α, $\alpha \in \mathbb{R}$;
- A Reflection s.

For any $\alpha, \beta \in \mathbb{R}$, the group actions satisfy

$$r_{\alpha+2\pi} = r_\alpha, \quad r_{\alpha+\beta} = r_\alpha r_\beta = r_\beta r_\alpha, \quad s^2 = r_0 = r_{2\pi} = I, \quad s r_\alpha = r_{-\alpha} s,$$

where I is the group identity.

Bifurcation in the Presence of $O(2)$ Symmetry

- Assume that the problem has $O(2)$ symmetry.
- F is $O(2)$ equivariant, i.e.,

$$\rho_\gamma F(u,\lambda) = F(\rho_\gamma(u),\lambda) \quad \forall \gamma \in O(2),$$

where ρ_γ is the representation of γ on $\mathbb{H}$.

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- Moreover, taking the Fréchet derivative, we note that for $u \in \mathbb{H}^{O(2)}$

$$\rho_\gamma F_u(u,\lambda)\phi = F_u(u,\lambda)\rho_\gamma(\phi) \quad \forall\gamma \in O(2) \;\; \forall\phi \in \mathbb{H},$$

where $\mathbb{H}^{O(2)} = \{v \in \mathbb{H} : v = \rho_\gamma(v)\ \forall\gamma \in O(2)\}$.

- Standard decomposition

$$\mathbb{H} = \sum_{m=0}^{\infty} \oplus \mathbb{V}_m, \quad \mathbb{V}_m \perp \mathbb{V}_l, \quad m \neq l.$$

where the $\mathbb{V}_m$ are $O(2)$ invariant.

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Theorem (Cliffe, Spence & Tavener 2000)

Let A be any $O(2)$-equivariant linear operator on the Hilbert space $\mathbb{H}$, i.e. $\rho_\gamma A = A\rho_\gamma$ for all $\gamma \in O(2)$. Then,

$$A : \mathbb{V}_m \to \mathbb{V}_m, \quad m = 0, 1, 2, \ldots.$$

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- Eigenvalue problem

$$F_u(u, \lambda)\phi = \mu\phi, \quad \phi \in \mathbb{H},$$

decouples into the infinite set of simpler eigenvalue problems

$$F_u(u, \lambda)\phi = \mu\phi, \quad \phi \in V_m, \quad m = 0, 1, 2, \ldots$$

Navier-Stokes in cylindrical coordinates

Find $\mathbf{u} = (u_r(r,\theta,z), u_\theta(r,\theta,z), u_z(r,\theta,z), p(r,\theta,z))^\top \in \mathbb{H}$ such that

$$F(\mathbf{u}, \mathrm{Re}) \equiv \begin{pmatrix} -\frac{1}{\mathrm{Re}}\nabla^2 u_z + \nabla\cdot(u_z\mathbf{u}) + \frac{\partial p}{\partial z} \\ -\frac{1}{\mathrm{Re}}\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi}\right) + \nabla\cdot(u_r\mathbf{u}) - \frac{u_\phi^2}{r} + \frac{\partial p}{\partial r} \\ -\frac{1}{\mathrm{Re}}\left(\nabla^2 u_\phi - \frac{u_\phi}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \phi}\right) + \nabla\cdot(u_\phi\mathbf{u}) + \frac{u_r u_\phi}{r} + \frac{1}{r}\frac{\partial p}{\partial \phi} \\ -\nabla\cdot\mathbf{u} \end{pmatrix} = \mathbf{0},$$

where $\mathbb{H} = H^1(\Omega)^3 \times L^2(\Omega)$.

Bifurcation in the Presence of $O(2)$ Symmetry

Navier-Stokes in cylindrical coordinates

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where $\mathbb{H} = H^1(\Omega)^3 \times L^2(\Omega)$.

- Can show $O(2)$ equivariance of Navier-Stokes equations in cylindrical coordinates.

$O(2)$ Invariant Subspaces of $\mathbb{H}$

$$\mathbb{V}_m = \text{Span}\left\{ \begin{pmatrix} u_r^m(r,z)\cos(m\theta) \\ u_\theta^m(r,z)\sin(m\theta) \\ u_z^m(r,z)\cos(m\theta) \\ p^m(r,z)\cos(m\theta) \end{pmatrix}, \begin{pmatrix} u_r^m(r,z)\sin(m\theta) \\ u_\theta^m(r,z)\cos(m\theta) \\ u_z^m(r,z)\sin(m\theta) \\ p^m(r,z)\sin(m\theta) \end{pmatrix} \right\}.$$

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- Hence, the eigenvalue problems

$$F_u(u, \mathrm{Re})\phi = \mu\phi, \quad \phi \in \mathbb{V}_m, \quad m = 0, 1, 2, \ldots$$

  only have to be discretised in (r, z).
- Can study stability to three dimensional disturbances using a sequence of two dimensional problems.

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Extended System

Seek $\hat{\mathbf{u}}^c \in \mathbb{H}^{O(2)} \times \mathbb{V}_m \times \mathbb{R}$, $m = 1, 2, \ldots$ such that

$$\mathbf{G}(\hat{\mathbf{u}}^c) = \mathbf{0}.$$

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$O(2)$ Invariant Subspaces of $\mathbb{H}$

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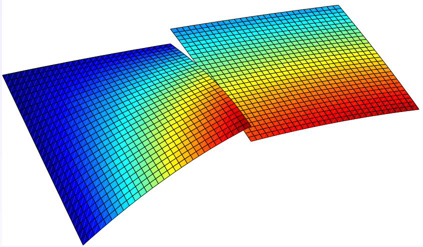
Weak formulation:

Seek $\hat{\mathbf{u}}^c \in \mathbb{H}^{O(2)} \times \mathbb{V}_m \times \mathbb{R}, m = 1, 2, \ldots$ such that

$$\mathcal{N}(\hat{\mathbf{u}}^c, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbb{H}^{O(2)} \times V_m \times \mathbb{R}.$$

- Method Construction
  - Employ local spaces of discontinuous piecewise polynomials;
  - Inter-element continuity weakly enforced.



$\Rightarrow$ Hybrid FE/FV Method

- Robustness/stability.
- Locally conservative.
- Ease of implementation.
- Highly parallelisable.
- Flexible mesh design (hybrid grids, non-matching grids, non-uniform/anisotropic polynomial degrees).
- Wider choice of stable FE spaces for mixed problems.
- Computational overhead/efficiency (increase in DoFs).

Interior Penalty DG Method

- $\mathcal{T}_h = \{\kappa\}$ is a non-degenerate mesh;
- For $\mathbf{p} = \{p_\kappa\}$, $p_\kappa \geq 1$, define the finite element space

$$S_{h,p} = \{v \in L_2(\Omega) : v|_\kappa \in \mathcal{R}_{p_\kappa} \quad \forall \kappa \in \mathcal{T}_h\},$$

where $\mathcal{R}_p$ is either $\mathcal{P}_p$ or $\mathcal{Q}_p$.

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DG Discretization

$$\mathcal{S}_{h,\mathbf{p}} = [S_{h,\mathbf{p}}]^2 \times S_{h,\mathbf{p}-1} \times [S_{h,\mathbf{p}}]^3 \times S_{h,\mathbf{p}-1} \times \mathbb{R}.$$

DGFEM: Find $\hat{\mathbf{u}}_h^c \in \mathcal{S}_{h,\mathbf{p}}$, $m = 1, 2, \ldots$, such that

$$\mathcal{N}_h(\hat{\mathbf{u}}_h^c, \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathcal{S}_{h,\mathbf{p}}.$$

Schötzau, Schwab & Toselli 2003, 2004, Cockburn, Kanschat & Schötzau 2005

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- The numerical solution $\hat{\mathbf{u}}_h^c$ is computed using Newton's method, together with a block elimination technique. Werner & Spence 1984

DWR A Posteriori Error Estimation

- Measurement Problem: Given a functional $J(\cdot)$ and a user–defined tolerance $\mathtt{TOL} > 0$, can we efficiently design $\mathcal{S}_{h,p}$ such that

$$|J(u) - J(u_h)| \leq \mathtt{TOL}.$$

Fluid dynamics: drag and lift coefficients.
Electromagnetics: far field pattern.
Other examples: Eigenvalues, point value, flux, mean value, etc.

Becker & Rannacher 1996, 2001, Larson & Barth 2000, Heuveline & Rannacher 2001 Houston & Süli 2001, 2002 Bangerth & Rannacher 2003, Hartmann & Houston 2002, 2006, Cliffe, H., Houston 2010.

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- Eigenvalue problem:

$$J(\hat{\mathbf{u}}) = \mu.$$

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- Bifurcation problem:

$$J(\hat{\mathbf{u}}^c) = \mathrm{Re}^c.$$

Dual problem

Find $\mathbf{z}$ such that

$$\mathcal{M}(\mathbf{u}, \mathbf{u}_h; \mathbf{w}, \mathbf{z}) = J(\mathbf{w}) \quad \forall \mathbf{w}.$$

- Gâteaux derivative of $\mathcal{N}_h(\cdot, \cdot)$:

$$\mathcal{N}'_{h,\mathbf{u}}[\mathbf{w}](\mathbf{v}, \cdot) = \lim_{\epsilon \longrightarrow 0} \frac{\mathcal{N}_h(\mathbf{w} + \epsilon \mathbf{v}, \cdot) - \mathcal{N}_h(\mathbf{w}, \cdot)}{\epsilon}.$$

- Linearization of $\mathcal{N}_h(\cdot, \cdot)$:

$$\begin{aligned} \mathcal{M}(\mathbf{u}, \mathbf{u}_h; \mathbf{u} - \mathbf{u}_h, \mathbf{v}) &= \mathcal{N}_h(\mathbf{u}, \mathbf{v}) - \mathcal{N}_h(\mathbf{u}_h, \mathbf{v}) \\ &= \int_0^1 \mathcal{N}'_{h,\mathbf{u}}[\theta \mathbf{u} + (1 - \theta)\mathbf{u}_h](\mathbf{u} - \mathbf{u}_h, \mathbf{v}) \, \mathrm{d}\theta \\ &\approx \mathcal{N}'_{h,\mathbf{u}}[\mathbf{u}_h](\mathbf{u} - \mathbf{u}_h, \mathbf{v}) \end{aligned}$$

Dual problem

Find $\mathbf{z}$ such that

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Proposition (Error Representation Formula)

Assuming the dual problem is well–posed, the following result holds:

$$\mathrm{Re}^c - \mathrm{Re}_h^c = -\mathcal{N}_h(\hat{\mathbf{u}}_h^c, \mathbf{z} - \mathbf{z}_h) \equiv \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa,$$

for all $\mathbf{z}_h \in \mathcal{S}_{h,p}$.

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for all $\mathbf{z}_h \in \mathcal{S}_{h,p}$.

- Linearize about $\hat{\mathbf{u}}_h^c$.
- Approximate $\mathbf{z}$ with DG method.

Error Estimation/Mesh Adaptivity

- Adaptivity is carried out based on $|\eta_\kappa|$. We use a fixed fraction strategy - 25%-refinement, 10%-derefinement.

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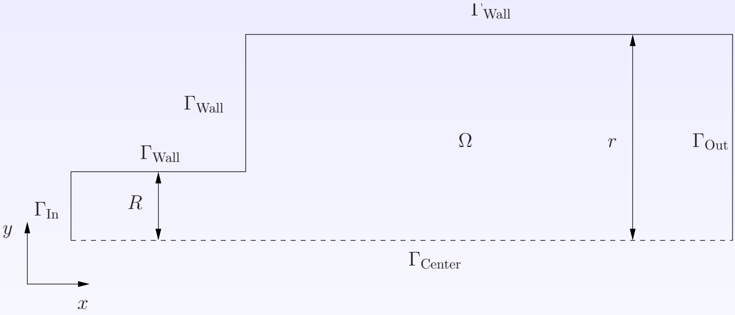
- We can use a purely *h*-refinement strategy, or an *hp*-refinement strategy.
- The choice of *h*- or *p*-refinement is based on the smoothness of both the primal and dual solutions.
- Smoothness determined via decay rate of Legendre coefficients.

  Houston, Senior, Süli, 2003.

Sudden Expansion in a Channel: Problem Setup



Sudden Expansion in a Channel: Error Effectivities

- $r : R = 3 : 1$
- $Re = 35$
- Eigenvalue = 0.00613553131999

Mesh No	No. Eles	Eig. Dof	Error	Effectivity
1	760	16720	6.027E-05	1.92
2	1387	30514	1.540E-05	2.47
3	2479	54538	9.795E-06	1.98
4	4387	96514	6.327E-06	1.58
5	7645	168190	3.845E-06	1.33
6	13243	291346	2.231E-06	1.16
7	22585	496870	1.281E-06	1.00

- Effectivity $= |Error|/|\sum_{\kappa \in \mathcal{T}_h} \eta_\kappa|$.

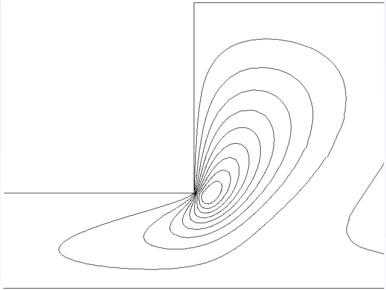
Sudden Expansion in a Channel: Mesh under Refinement

Mesh after 5 refinement steps

Contour plot of $\mathbf{z}_x^m$

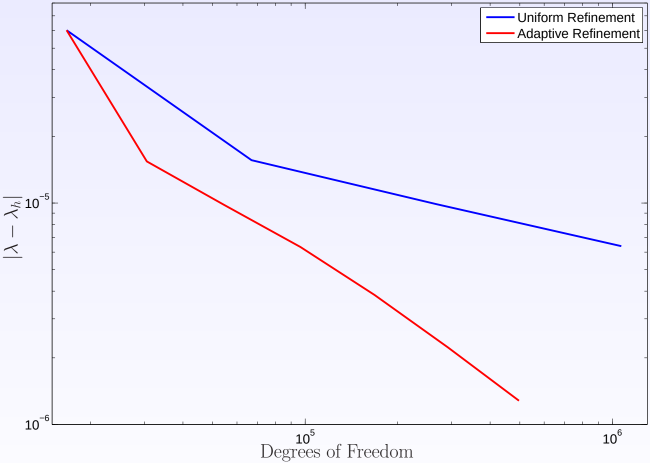
Sudden Expansion in a Channel: Mesh Detail under Refinement

Mesh detail near expansion

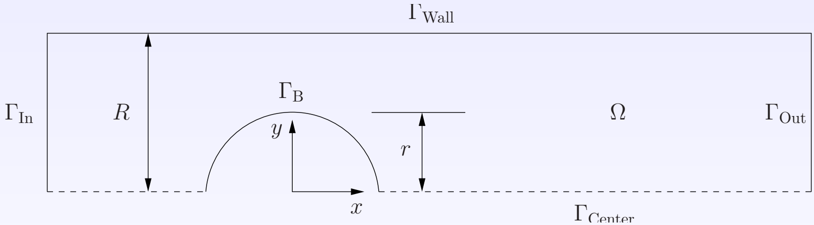


Contour plot of $\mathbf{z}_y^0$ near expansion

Sudden Expansion in a Channel: Error Convergence



Cylindrical Blockage in a Channel: Problem Setup



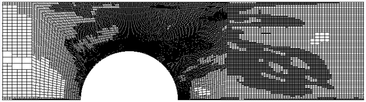
Cylindrical Blockage in a Channel: Error Effectivities

- $r : R = 1 : 2$
- $Re = 100$
- Eigenvalue $= 0.114789963956350 + 2.116719676204527i$

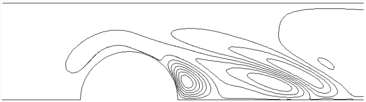
Mesh No	No. Eles	Eig. Dof	Error	Effecticity
1	816	17952	8.966E-02	1.08
2	1443	31746	2.229E-03	1.54
3	2577	56694	1.455E-04	1.31
4	4590	100980	4.089E-05	0.980
5	8190	180180	1.033E-05	1.01
6	14400	316800	3.870E-06	0.946
7	24843	546546	1.060E-06	1.00

Cylindrical Blockage in a Channel: Mesh under Refinement

Full Mesh

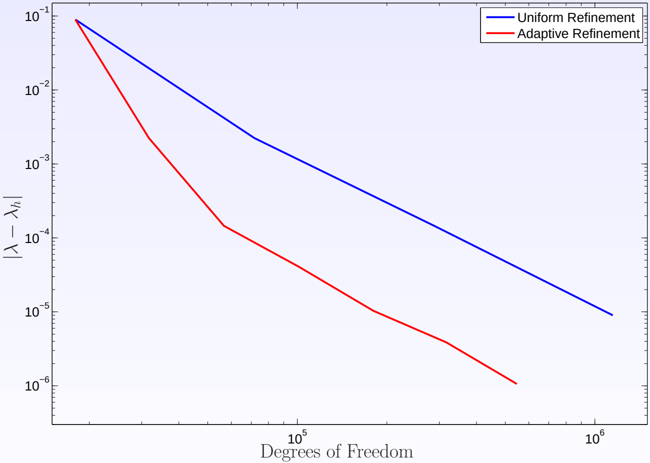


Mesh Detail near Blockage



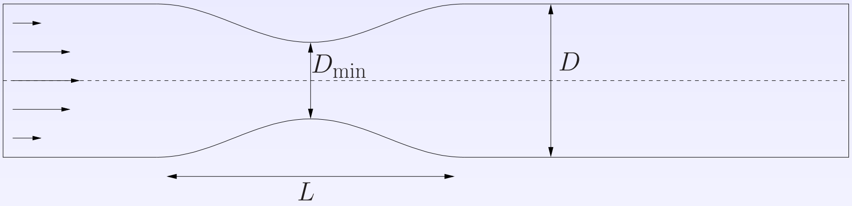
Contour plot of $\mathbf{z}_y^0$ near blockage

Cylindrical Blockage in a Channel: Error Convergence



Stenosis Problem

- Critical $\mathrm{Re} \approx 721$.



- Lengths in ratio $D_{\min} : D : L = 1 : 2 : 4$

Stenosis: Error Estimation

- *h*-adaptivity

Base DOF	Null DOF	Re_h^c	Error Estimate
84480	119040	688.07858	27.337
148962	209901	717.87440	3.629
258588	364374	720.31797	7.739E-01
445830	628215	720.82707	2.280E-01
771408	1086984	720.93597	1.168E-01
1334916	1881018	720.97594	7.677E-02

Stenosis: Error Estimation

- *h*-adaptivity

Base DOF	Null DOF	Re_h^c	Error Estimate
84480	119040	688.07858	27.337
148962	209901	717.87440	3.629
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445830	628215	720.82707	2.280E-01
771408	1086984	720.93597	1.168E-01
1334916	1881018	720.97594	7.677E-02

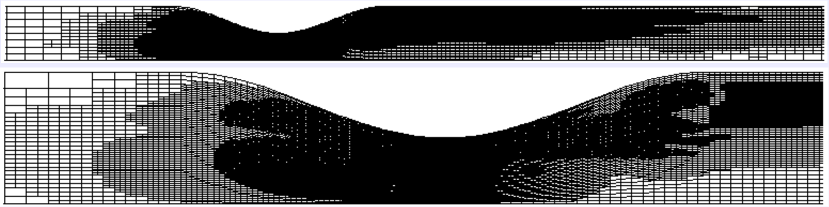
- *hp*-adaptivity

Base DOF	Null DOF	Re_h^c	Error Estimate
84480	119040	688.07858	27.337
146362	206090	708.96275	11.296
193518	271480	716.36055	4.680
259439	363362	721.0237123	3.575E-02
327537	456501	721.0519498	8.054E-04
398569	553522	721.0524660	5.477E-05
499025	691978	721.0524361	4.326E-05

- Sherwin & Blackburn 2005: $Re_h^c \approx 722$.

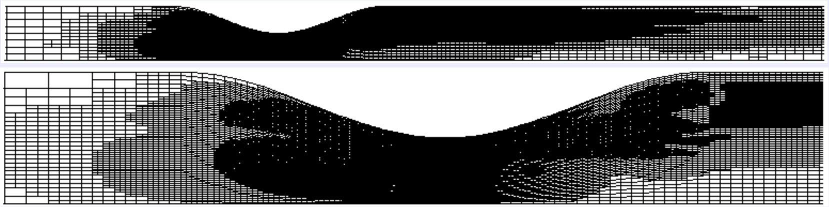
Stenosis Grid

Mesh distribution after 5 h-adaptive refinements

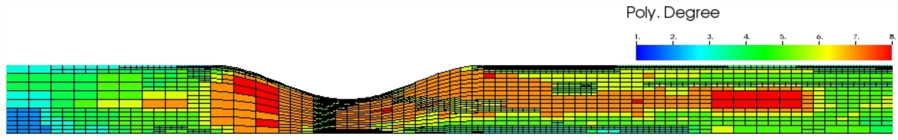


Stenosis Grid

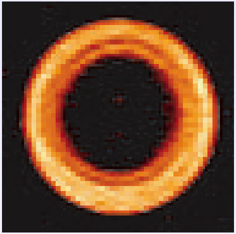
Mesh distribution after 5 *h*-adaptive refinements



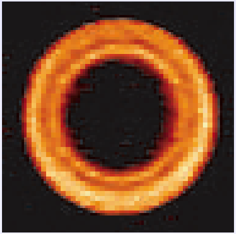
Mesh distribution after 6 *hp*-adaptive refinements



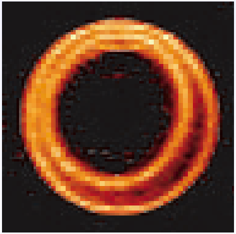
Flow Through 1:2 Pipe Expansion



$Re=372$



$Re=649$



$Re=1522$

$Re=1567$

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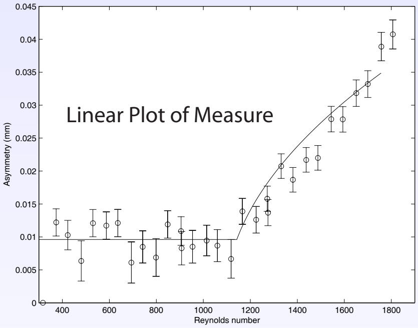
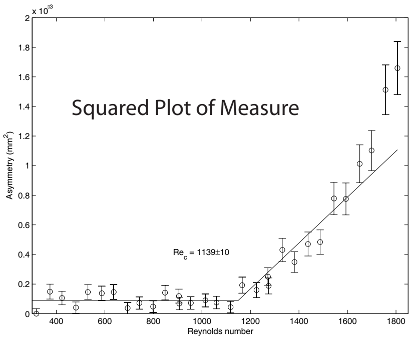
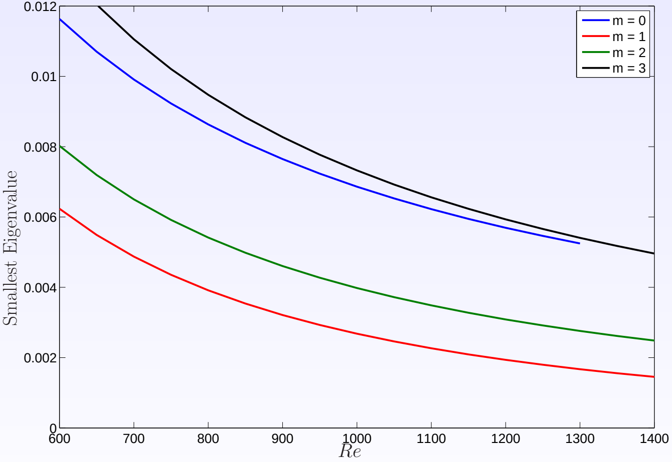




- Steady bifurcation occurs at $\mathrm{Re} = 1139 \pm 10$.
- Onset of time dependence at $\mathrm{Re} \approx 1500$.

T. Mullin, J.R.T. Seddon, M.D. Mantle, and A.J. Sederman Phys. Fluids 21, 014110 (2009)

1:2 Pipe Expansion: Eigenvalues with Re

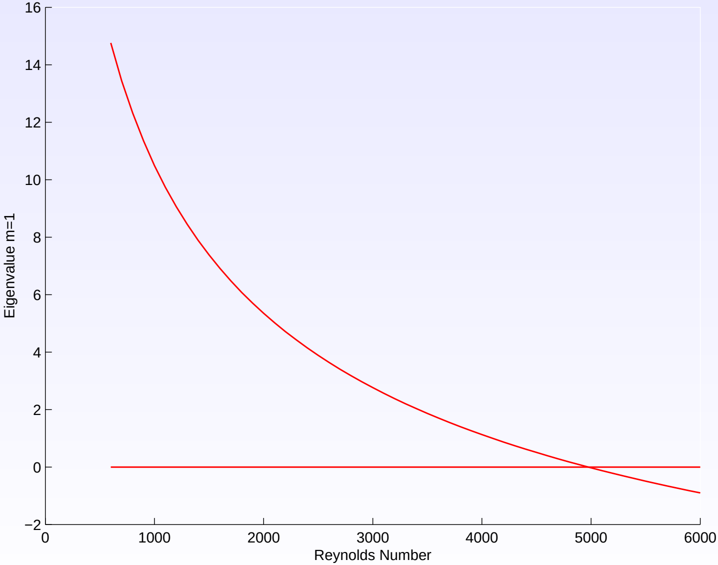


1:2 Pipe Expansion: Eigenvalue Errors

- $\mathrm{Re} = 1300,\ m = 1.$

Mesh No.	No. Eles	Eig. Dofs	Eigenvalue	$\sum_{\kappa \in \mathcal{T}_h} \eta_\kappa$
1	20000	420000	0.167241E-02	1.741E-06
2	34565	725865	0.167194E-02	1.914E-06
3	65909	1384089	0.167218E-02	9.771E-07
4	111956	2351076	0.167243E-02	5.765E-07

1:2 Pipe Expansion: Eigenvalues for $m = 1$



1:2 Pipe Expansion: Bifurcation Location

- *hp*-adaptive algorithm.

Mesh No.	Base Dofs	Null Dofs	Re_h^c	Error Indicator
1	232755	325857	4925.5119	211.880
2	311300	434264	4708.2944	342.372
3	448495	624696	4996.9118	85.547
4	844468	1158267	5084.7897	1.663
5	995544	1363010	5084.9472	7.869E-02

Summary and Outlook

- Successfully applied DG and goal-oriented *a posteriori* error estimation to bifurcation and stability analysis of incompressible Navier-Stokes equations.
- First mesh converged results for this problem.
- There is a steady, supercritical, $O(2)$-symmetry-breaking bifurcation at Reynolds number approximately 5080 ± 5.
- This is the same phenomenon as witnessed in the numerical experiments, but at a very different Reynolds number.
- Is $O(2)$ symmetry the wrong model for this problem?
- Investigate effect of perturbations that destroy the $O(2)$ symmetry.