Matching preclusion and conditional matching preclusion problems for regular graphs

Eddie Cheng

Oakland University
Rochester, MI
echeng@oakland.edu

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Joint work with László Lipták, Marc J. Lipman,
Philip Hu and Randy Jia
Earlier work: with other researchers including Jimmy Tan.

Outline

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Definitions (matching preclusion)

- Only interested in regular even graphs, say, r-regular with $r \geq 3$.
- The *matching preclusion number* of a graph G, denoted by $\mathrm{mp}(G)$, is the minimum number of edges whose deletion leaves the resulting graph without a perfect matching.
- It is bounded above by r as one can trivially delete edges to isolate a vertex.
- If $\mathrm{mp}(G) = r$, then G is *maximally matched*.
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- A trivial feasible solution is pick a path $u - v - w$ in the original graph and delete all the edges incident to either u or w but not v.
- $\nu_e(G)$ is the minimum size of such trivial solutions. (It is either $2r - 2$ or $2r - 3$.)
- If $mp_1(G) = \nu_e(G)$, then G is *conditionally maximally matched*.
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Cayley graph generated by (transposition) trees

- Slater 78, Tchuente 82, popularize as interconnection networks by Araki 06.
- T is a tree with labels $1, 2, \ldots, n$. It generates a graph G where the vertices are the $n!$ permutations on $\{1, 2, \ldots, n\}$. Two vertices are adjacent if one permutation (label of a vertex) can be obtained from another by switching the symbols in the ith and jth positions where (i, j) is an edge of T.
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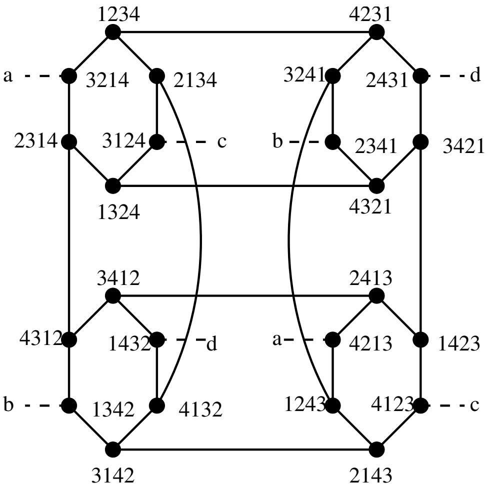
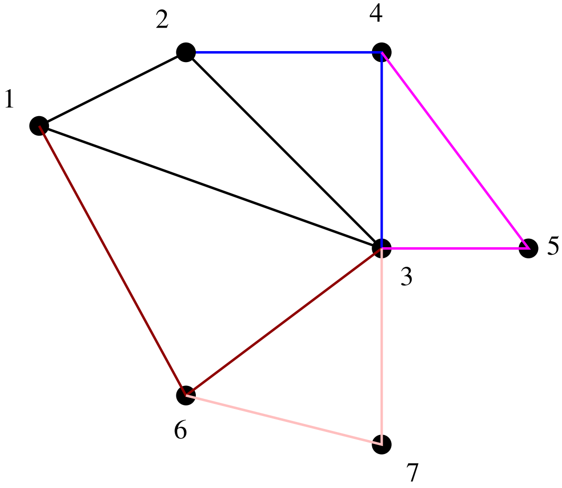
Ex: $n = 4$ generated by $K_{1,3}$ with 1 as center



A 2-tree



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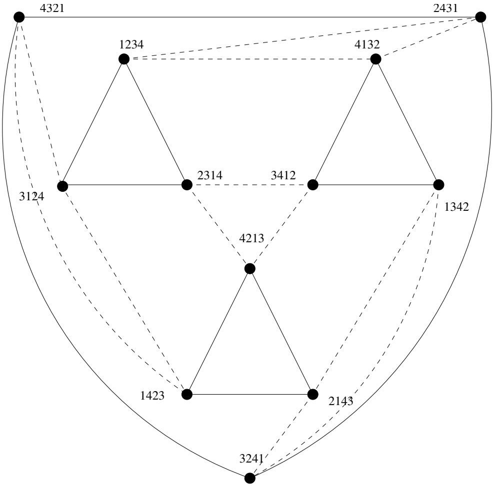
Ex: $n = 4$, only one graph



Hyperstars

- Introduced by Lee, Kim, Oh and Lim 02.
- An hyperstar HS(n,k) with $1 \leq k \leq n-1$ is defined as follows: its vertex-set is the set of $\{0,1\}$-strings of length n with exactly k 1's, and two vertices are adjacent if and only if one can be obtained from the other by exchanging the first symbol with a different symbol (1 with 0, or 0 with 1) in another position.
- Only interested in the regular subclass $HS(2k,k)$
- Isomorphic to the middle cubes. (The middle two layers of an odd cube.)
- bipartite.

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- Find the matching preclusion and conditional matching preclusion numbers for these graphs and classify the corresponding optimal solutions. (Want to show that they are maximally matched, super matched, conditionally maximally matched and conditionally super matched except for small/boundary cases.)
- Freebie: Cayley graphs generated by transposition trees and $HS(2k, k)$ are maximally matched since they are bipartite.
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Plesník's Theorem

Theorem: If G is a r-regular $(r-1)$-edge-connected graph with an even number of vertices, then $G - F$ has a perfect matching for every $F \subseteq E$ with $|F| \leq r - 1$.

This immediately tells us that G is maximally matched, that is, $\text{mp}(G) = r$ for r-regular $(r-1)$-edge connected even graphs.

Outline

Replacing the condition $(r-1)$-edge-connected

Strengthen the $(r-1)$-edge-connected condition to get super matched.

r-edge-connected is not enough.
r-connected is not enough.

Beyond connectivity

Assume G is r-regular

- G is *maximally connected* means G has connectivity r.
- G is *loosely super connected* means G is m.c. and every optimal disconnected set is induced by a vertex.
- G is *tightly super connected* means G is m.c. and deleting an optimal disconnected set will disconnect the graph into two components, one of which is a singleton. (Other terms for the same concept: vosperian property, hyper-connectivity, $r\frac{1}{2}$-connected.)
- tightly > loosely. Example: $K_{3,3}$
- Can define maximally edge-connectedness and super edge-connectedness. (Here no distinction between loosely and tightly.)

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Beyond superconnectedness

Assume G is r regular

- G is *super m-connected of order q* if with at most m vertices deleted, the resulting graph is either connected or it has a big component and a number of small components with at most q vertices in total.
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Sufficient conditions for bipartite graphs

- **Theorem:** Let G be a r-regular bipartite graph that is super edge-connected. Then G is super matched.
- **Theorem:** Let G be a r-regular bipartite graph that is super $(3r-6)$-edge-connected of order 2. Then G is conditionally maximally matched, that is, $\mathrm{mp}_1(G) = 2r-2$.
- **Theorem:** Let G be a r-regular bipartite graph with $\mathrm{mp}_1(G) = 2r-2$. If G is super $(3r-4)$-edge-connected of order 3, then it is conditionally super matched.

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- All proofs use Hall's Theorem.
- Each proof requires tighter analysis than the previous one.
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Known structural results

Suppose G is Cayley graph generated by a transposition tree on n vertices.

- Theorem [EC and Lipták 06]: Let $n \geq 4$. Then G is super $(2n-5)$-connected of order 1. (Tight result)
- Theorem [EC and Lipták 06]: Let $n \geq 4$. Then G is super $(3n-9)$-connected of order 2. (Tight result)
- Theorem [EC and Lipták 07]: Suppose G is Cayley graph generated by a transposition tree on n vertices where $n \geq 4$. Let $1 \leq k \leq n-2$. Then G is super $(kn-(k+\frac{k(k+1)}{2})$-connected of order $k-1$. (Asymptotically tight)

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Remarks

- Proofs use the structural theorem plus the fact that "vertex" version in the structural theorem is stronger than the "edge" version in the sufficient condition if the graph is large enough.
- Letting $k = 4$ in the asymptotic result gives super $(4n - 14)$-connected of order 3. We want $4n - 14 \geq 3(n - 1) - 4$.
- The $n \geq 7$ condition can be improved (with additional work) to $n \geq 4$ except for the following graph.

Remarks

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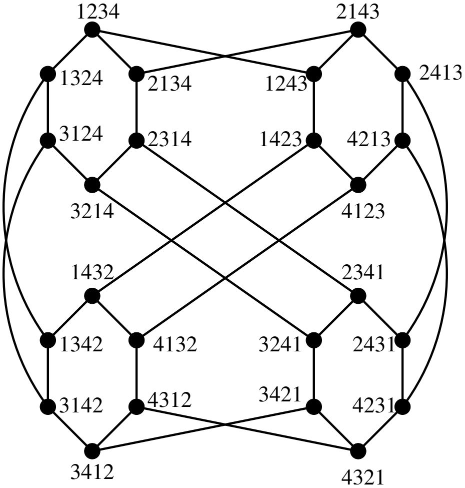
1234
2143
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1243
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4213
3214
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2341
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3241
2431
3142
4312
3421
4231
3412
4321

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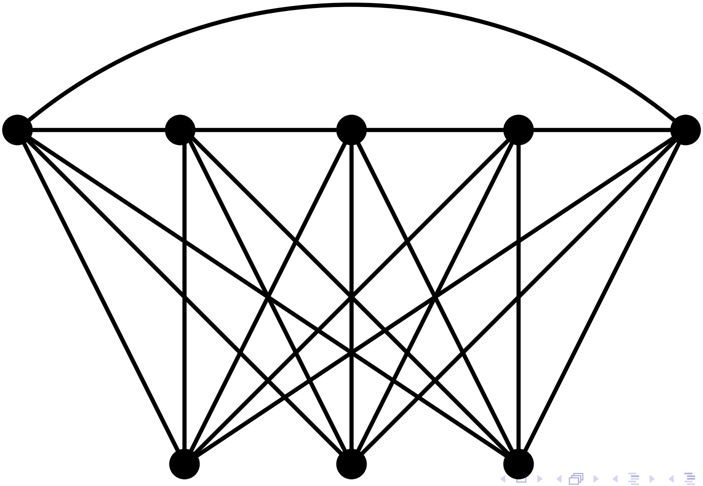
Outline

Are the sufficient conditions true for non-bipartite graphs?

The answer is no. Below is a counterexample.



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Sufficient condition for a graph being super matched

Theorem: Let $G = (V, E)$ be a r-regular graph with an even number of vertices where $r \geq 3$. Suppose that G is super edge-connected and $\alpha(G) < \frac{|V|-2}{2}$ where $\alpha(G)$ is the stability number of G. Then G is super matched.

The proof uses Tutte's Theorem.

Proof

- Let F be a matching preclusion set, $|F| = r$. Let W be the Tutte set in $G - F$.
- $F = \delta_G(X_1, X_2, \ldots, X_p)$ where the X_i's are the odd components in $G - F$.
- $p = |W| + 2$ and W is an independent set in G
- There are no even components in $G - F$.
- $W \neq \emptyset$ gives contradiction. (Forces every X_i to be a singleton as G is super edge-connected.) This violates the stability number condition.)
- $W = \emptyset$ and there are two odd components in $G - F$, one of which is a singleton since G is super edge-connected.

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Sufficient condition for a graph being conditionally maximally matched

- Need extra conditions.
- Let

$$\zeta(G, p, q) = \min\{\alpha(H)\}$$

  where the minimum is take over all induced subgraphs of G with p vertices and at most q edges.
- Let $\gamma_G(X)$ to be the set of edges with both ends in X.

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Theorem: Suppose that G is triangle-free, G is an r-edge connected even graph and G is super $(3r-6)$-edge-connected of order 2. Moreover suppose an additional technical assumption holds. Then G is conditionally maximally matched, that is, $\mathrm{mp}_1(G) = 2r-2$.

Technical assumption:
Either $|\gamma_G(X)| > 2r-3$ for every $X \subseteq V$ and $|X| = \frac{|V|+2}{2}$, or $\alpha(G) < \zeta(G, \frac{|V|-2}{2}, 2r-6)$.

Theorem: Suppose that G contains a 3-cycle, G is an r-edge-connected even graph and G is super $(3r-8)$-edge-connected of order 2. Moreover suppose an additional technical assumption holds. If $r=3$, we require, additionally, that G be super $(3r-7)$-edge-connected of order 2. Then G is conditionally maximally matched, that is, $\mathrm{mp}_1(G) = 2r-3$.

Technical assumption:
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Sufficient condition for a graph being conditionally super matched

Theorem: Suppose that G is triangle-free, $\mathrm{mp}_1(G) = 2r - 2$, G is super edge-connected and G is super $(3r-4)$-edge-connected of order 3. Moreover suppose an additional technical assumption holds. Then G is conditionally super matched.

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Theorem: Let $G = (V, E)$ be a r-regular even graph. Suppose that G has a 3-cycle, $\mathrm{mp}_1(G) = 2r - 3$, $|V| \geq 8$, G is super edge-connected, G is super $(3r-6)$-edge-connected of order 3 and $\alpha(G) < \frac{|V|-4}{2}$. Moreover suppose an additional technical assumption holds. Then G is conditionally super matched.

Technical assumption:
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Structural theorem for Cayley graph generated by a 2-tree

Suppose G is Cayley graph generated by a 2-tree on n vertices.
Theorem [EC, Lipták and Sala 09]

- Let $n \geq 5$. Then G is tightly super connected.
- Let $n \geq 5$. Then G is super $(4n-12)$-connected of order 1. The bound is sharp.
- Let $n \geq 5$. Then G is super $(6n-20)$-connected of order 2. The bound is sharp.
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Cayley graph generated by a 2-tree

Theorem:
Suppose G is Cayley graph generated by a 2-tree on $n \geq 4$ vertices. Then G is maximally matched, that is, $\mathrm{mp}(G) = 2n - 4$ and G is super matched. If $n \geq 5$, then G is conditionally maximally matched, that is, $\mathrm{mp}_1(G) = 4n - 11$. If $n \geq 12$, then G is conditionally super matched.

Can be strengthened to include $n \leq 11$.

Cartesian Product: $G\square C_k$

- **Theorem:** Let G be a r-regular even graph with $r \geq 2$ and $k \geq 3$. If G is maximally matched, then $G\square C_k$ is maximally matched and super matched.
- **Theorem:** Let G be triangle-free r-regular even graph with $r \geq 3$ and $k \geq 4$. Suppose G is super matched and $\text{mp}_1(G) \geq 2r - 3$. Then $G\square C_k$ is conditionally super matched unless k is odd and G is either $K_{r,r}$ or $K_{r+1,r+1}$ minus a perfect matching.

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