# From Perfect Matchings to the Four Colour Theorem

Aguilar (Cinvestav), Flores (UNAM), Pérez (HP), Santos (Cantabria), Zaragoza (UAM-A)

Universidad Autónoma Metropolitana Unidad Azcapotzalco
Departamento de Sistemas

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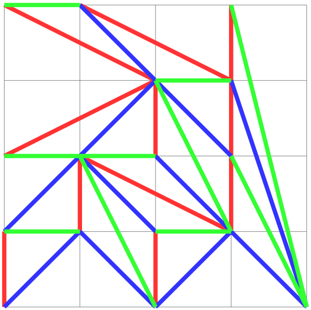
# Unimodular triangulations

Unimodular triangulation

A *unimodular triangulation* $T$ of a polygon $P$ with integer vertices is a partition of $P$ into *unimodular triangles*. Equivalently, into triangles with integer vertices and area one-half.



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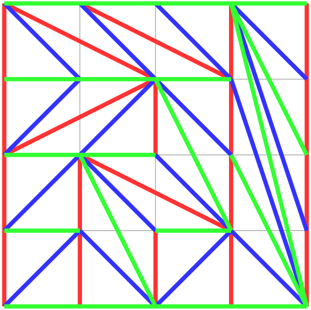
# Unimodular triangulations of rectangles

Perfect matchings

An interesting special case is when the polygon is a rectangle. In this case, the *weak dual* of any unimodular triangulation has a *perfect matching*. Simply choose the right colour!



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# Unimodular triangulations of rectangles

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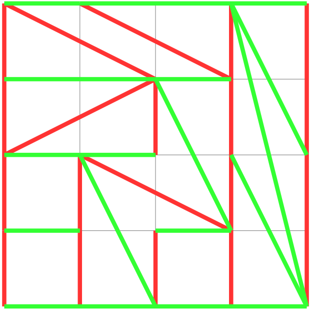
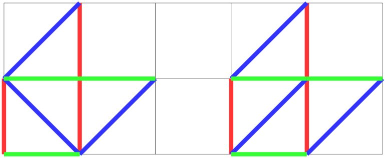


# Open questions

**Maximum matching**

Given a polygon with integer vertices, what is the maximum size of a matching of the weak dual among all of its unimodular triangulations?



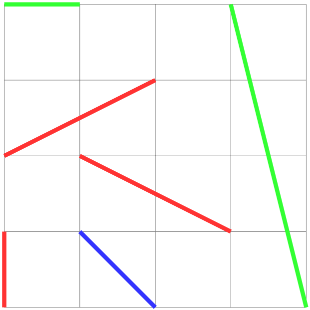
**Characterization**

Is there a nice characterization of the graphs that are weak duals of unimodular triangulations of polygons?

# Primitive segments

Primitive segments

Let $p = (a, b)$ and $q = (c, d)$ be two points with integer coordinates. The segment $pq$ is primitive if it does not contain another point with integer coordinates. Equivalently, if $\gcd(a - c, b - d) = 1$.



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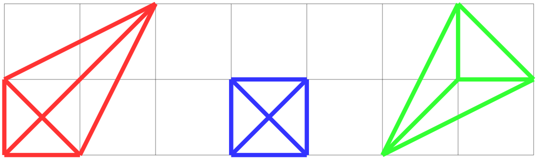
# Primitive drawings and embeddings

**Primitive drawing**: A drawing of a graph is primitive if all its vertices are different and all its edges are primitive segments.

**Primitive embedding**: An embedding of a graph is primitive if all its vertices are different and all its edges are primitive segments.



Three drawings of $K_4$.

# Characterization of primitive drawings

**Theorem (Flores, Z, '09)**

A graph $G$ has a primitive drawing iff $\chi(G) \leq 4$.

**Proof sketch**

($\Rightarrow$) Assume that $G$ has a primitive drawing. Consider the vertex colouring of $G$ given by

$$f(a,b) = (a \mod 2, b \mod 2).$$

Assume that the ends of the edge $pq$ (with $p = (a,b)$ and $q = (c,d)$) receive the same colour. Then $a+c$ and $b+d$ are even, and hence the midpoint

$$r = \left(\frac{a+c}{2}, \frac{b+d}{2}\right)$$

has integer coordinates, a contradiction.
(Kára, Pór, Wood, '05)

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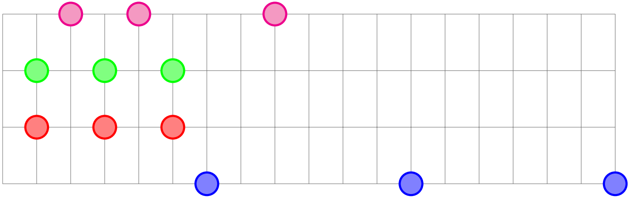
# Construction of primitive drawings

Proof sketch

($\Leftarrow$) The graph $K_{n,n,n,n}$ can be primitively drawn with the vertex set given by $P_0 = \{(6i, 0) : i \in [n]\}$, $P_1 = \{(2i-1, 1) : i \in [n]\}$, $P_2 = \{(2i-1, 2) : i \in [n]\}$, and $P_3 = \{(a_i, 3) : i \in [n]\}$, where $\{a_1, \ldots, a_n\}$ is the set of the smallest $n$ even numbers not divisible by 3.



# Primitive embeddings

Primitive embeddings are:

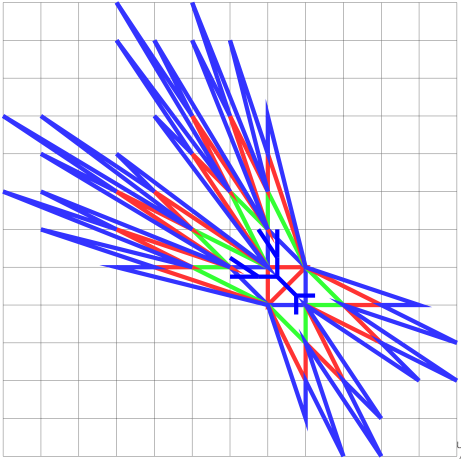
- Plane graphs.
- Primitive drawings.
- 4-chromatic.

Question Which planar graphs have primitive embeddings?

Outerplanar embeddings (Aguilar, Z, '10)

# Outerplanar graphs

Recursive construction. (Nakamoto and Negami, '10).



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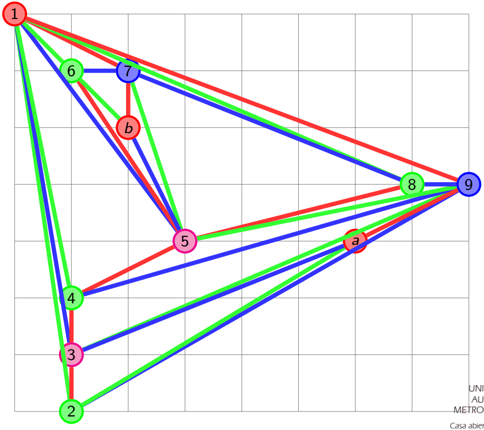
# Small triangulations

Pérez, Z, '11

Every planar triangulation with $n \leq 13$ has a primitive embedding in a square of side $n-1$.

Triangulation with 11 vertices



# Main result

Theorem (Santos, Flores, Z, '12) Every planar graph has a primitive embedding.[1]

Equivalently Planar graphs have primitive embeddings iff 4CT.

[1] This result was obtained independently by Martin Balko and presented in EuroCG 2012.

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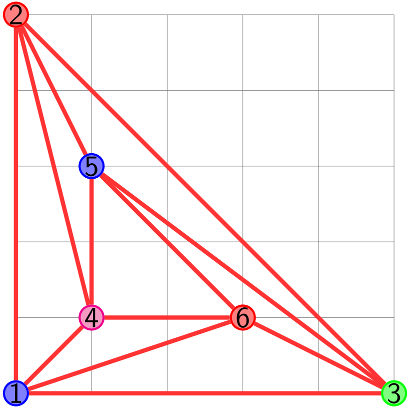
# Proof sketch: Four Colour Theorem

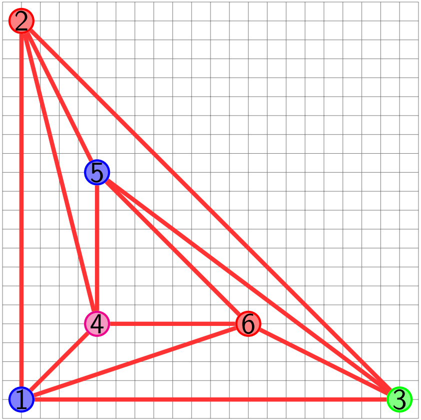
Rectilinear embedding

Let $G$ be a planar, 4-coloured graph and consider any of its rectilinear embeddings.



# Proof sketch: Enlarging the embedding

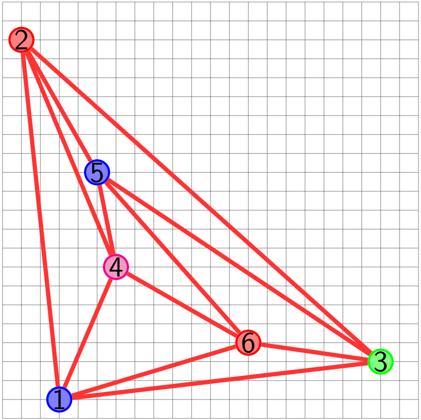
Multiply the coordinates of the embedding by a sufficiently large integer.



# Proof sketch: Perturbing the embedding

Technical
Same row
Same column
Colour class
$(2a, 6i)$
$(2c, 2d+1)$
$(2e+1, 2f)$
$(2g+1, 6j+1)$

Move vertices slightly in order to satisfy some constraints (without changing the embedding).



# Proof sketch: Enlarging again

**Horizontal expansion**

Assume the embedding has height $m$. Then multiply all horizontal coordinates by $M = m!$ and adjust them slightly as follows:

1. $(2a, 6i)$ goes to $(2aM, 6i)$.
2. $(2c, 2d+1)$ goes to $(2cM+2, 2d+1)$.
3. $(2e+1, 2f)$ goes to $((2e+1)M+1, 2f)$.
4. $(2g+1, 6j+1)$ goes to $((2g+1)M+3, 6j+1)$.

**End of the proof**

Now we can verify that all edges are primitive segments.

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# Size of the embedding

**First** If we start with Schnyder's embedding (on a square of side $n-1$), then $m \in O(n^2)$ and the embedding fits on a rectangle of

$$m \times m \cdot m! \approx m2^{m \log m}.$$

**Second** In the last part of the proof it is enough to multiply by $\mathrm{mcm}(1, 2, \ldots, m)$. Using the prime number theorem ($\pi(x) \approx \frac{x}{\ln x}$) we can see that the rectangle is

$$m \times m2^{m}.$$

# Further work

**Trees**: They fit on a square of side $O(n)$ and need $\Omega(\sqrt{n})$. What is the right size?

**Outerplanar graphs**: They fit on a square of side $O(2^n)$ and need $\Omega(\sqrt{n})$. What is the right size? What if we require the outer face to be convex? $\Omega(n\sqrt{n})$.

**Planar**: Is there a polynomial size embedding? For $n \leq 13$ the side is $\leq n - 1$.

**Algorithms**: How fast can we find good embeddings?

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# Thanks!

¡Gracias! ¡Felicidades Bill!