Weighted linear matroid matching

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Definitions

- V is a vectorspace
- $V_1, V_2, \cdots, V_k < V$ are called **skew subspaces** ("independent subspaces") if they satisfy $r(V_1 \vee V_2 \vee \cdots \vee V_k) = \sum_{i=1}^{k} r(V_i)$
- $e < V$ is a **line** if $r(e) = 2$
- V, E is an **instance of linear matroid matching** if E is a set of lines
- $M \subseteq E$ is a **matching** if it consists of skew lines, i.e. $r(sp(M)) = 2|M|$

Linear Matroid Matching

(Unweighted) Linear Matroid Matching Problem

Given: vectorspace V, set of lines E
Find: matching M to maximize $|M|$

$$\nu(V, E) := max\{|M| : M \text{ a matching}\}$$

Weighted Linear Matroid Matching Problem

Given: vectorspace V, set of lines E, weights $w : E \to \mathbb{R}$
Find: matching M to maximize $w(M)$

$$\nu(V, E, w) := max\{w(M) : M \text{ a matching}\}$$

Matroid matching:

- definition (Lawler, 1976)
- exponential, oracle model (Lovász, 1981; Jensen, Korte, 1982)
- NP-hard (Schrijver, 2003)

Arbitrary matroids:

- 2/3-approximation, unweighted (Fujito, 1993)
- PTAS, unweighted (Lee, Sviridenko, Vondrák, 2010)

Linear matroid matching is tractable:

- min-max, polytime algorithm (Lovász, 1980)
- fastest polytime (Gabow, Stallmann, 1986; Orlin, 2008)
- different polytime (Orlin, Vande Vate, 1990)
- fastest randomized (Cheung, 2011)

Applications / special cases:

- graph matching (Edmonds, 1965)
- matroid intersection (Edmonds, 1970)
- Mader's node-disjoint S-paths (Lovász, 1980; Schrijver, 2000)
- maximum genus embedding (Nebesky, 1981; Furst, Gross, McGeoch, 1988)
- matchoid (Lovász, Plummer, 1986)
- polymatroid matching
- parity-constrained rooted-connected orientation (Frank, Jordán, Szigeti, 2001; Király, Szabó, 2003)
- maximum triangle cactus, graphic matroid matching (Szigeti, 2003)
- minimum generically rigid pinning-down in the plane

Variations:

- algebraic matroids (Dress, Lovász, 1987)
- pseudomodular matroids (Hochstättler, Kern, 1987)
- double circuit property (Björner, Lovász, 1987)
- ntcdc-free polymatroid matching (Makai, Pap, Szabó, 2007)

Generalization:

- linear delta-matroid parity (Geelen, Iwata, Murota, 1997)

Related:

- fractional matroid matching (Vande Vate, 1992)
- unweighted algorithm (Vande Vate, Chang, Llewellyn, 2001)
- weighted algorithm (Gijswijt, Pap, 2008)

Weighted matroid matching:

- graphic matching, matroid intersection
- gammoids (Tong, Lawler, Vazirani, 1984)
- linear matroid, randomized pseudopolynomial (Camerini, Galbiati, Maffioli, 1992)
- fractional matching (Gijswijt, Pap, 2008)
- PTAS, strongly base orderable (Soto, 2011)
- linear matroid, randomized polynomial (Cheung, 2011)

This talk:

Theorem (Iwata 2011 — and independently — P 2011)

Weighted linear matroid matching is solvable in strongly polynomial time. $*$

$*$ (assuming "nice" linear representation of input lines)

(Linear) Matroid Intersection

- S is the groundset
- $\phi_i : S \to V_i$ $(i = 1, 2)$, where V_i is a vectorspace
- $U \subseteq S$ is a **common independent set** if $\phi_i(U)$ is independent
- FIND $\max |U|$, or $\max w(U)$ for some $w : S \to \mathbb{R}$
- representation: $\psi(s) := sp(\phi_1(s), \phi_2(s)) \in V_1 \times V_2$

Claim

$U \subseteq S$ is a common independent set iff $\psi(U)$ is a matching in $V_1 \times V_2$

(Linear) Matroid Intersection

- [Edmonds, 1979]

$$\mathcal{P} := conv\{\chi_U : U \text{ common indep.}\} =$$

$$= \{x \in \mathbb{R}_+^S : x(Z) \le r(\phi_i(Z)) \text{ for all } Z \subseteq S, i = 1, 2\}$$

- $\mathcal{P}$ is determined by an LP that is
  — integral, TDI, polytime optimization
  — 0-1 inequalities ("RANK" inequalities)

Graph Matching

- Let $G = (V_G, E_G)$ be a graph
- $$V := \bigotimes_{v \in V_G} sp(\mathbf{1}_v),$$
  where $\mathbf{1}_v$ is a unit vector introduced for node $v \in V$
- $$E := \{sp(\{\mathbf{1}_u, \mathbf{1}_v\}) : uv \in E_G\}$$

Claim

$M_G \subseteq E_G$ is a graph matching iff $\{sp(\{\mathbf{1}_u, \mathbf{1}_v\}) : uv \in M_G\}$ is a linear matroid matching in V, E

Graph Matching

- [Edmonds, 1968]

$$\mathcal{P} := conv\{\chi_M : M \text{ matching in } G\} =$$

$$= \{x \in \mathbb{R}_+^E : x(E[Z]) \leq \left\lfloor \tfrac{1}{2}|Z| \right\rfloor \text{ for all } Z \subseteq V, \text{ and}$$

$$x(\delta_v) \leq 1 \text{ for all } v \in V_G\}$$

- $\mathcal{P}$ is determined by an LP that is
  — integral, TDI, polytime optimization
  — 0-1 inequalities ("RANK" inequalities)

Example for linear matroid matching polytope

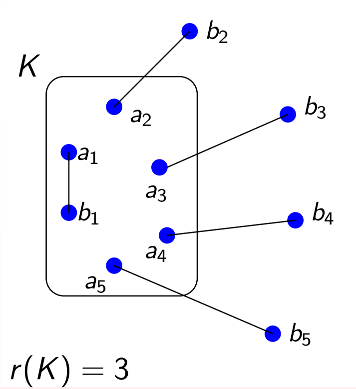
$$a_1 = (1, 0, 0, 0, 0, 0, 0)$$
$$b_1 = (1, 1, 0, 0, 0, 0, 0)$$

$$a_2 = (1, 0, 1, 0, 0, 0, 0)$$
$$b_2 = (1, 0, 0, 1, 0, 0, 0)$$

$$a_3 = (1, 2, 2, 0, 0, 0, 0)$$
$$b_3 = (1, 0, 0, 0, 1, 0, 0)$$

$$a_4 = (1, 2, 1, 0, 0, 0, 0)$$
$$b_4 = (1, 0, 0, 0, 0, 1, 0)$$

$$a_5 = (1, 1, 2, 0, 0, 0, 0)$$
$$b_5 = (1, 0, 0, 0, 0, 0, 1)$$



$$E := \{sp(a_i, b_i) : i = 1, 2, 3, 4, 5\}$$

Claim

No rank-inequality separates $x = (1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ *from* $\mathcal{P}$.

Example for linear matroid matching polytope

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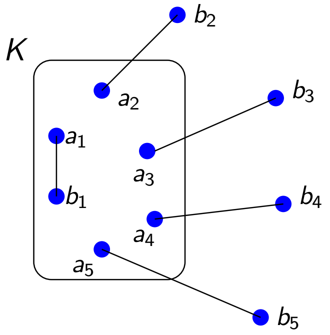


$r(K) = 3$

$E := \{sp(a_i, b_i) : i = 1, 2, 3, 4, 5\}$

Claim

$2x_1 + x_2 + x_3 + x_4 + x_5 \leq 3$

$\text{for all } x \in \mathcal{P} = conv(\{\chi_M : M \text{ a matching }\})$

Unweighted matroid matching min-max (Lovász)

Theorem (Lovász, 1980)

$$\nu(V,E) = \min_{K,\pi} \; r(K) + \sum_i \left\lfloor \frac{1}{2} r_{V/K}(E_i) \right\rfloor$$

where $K < V$ *and* $\pi = \{E_1, E_2, \cdots\}$ *is a partition of* E.

Necessity follows from:

- $\nu(V,E) \le \nu(V/K,E) + r(K)$ for any $K < V$
- $\nu(V,E) \le \sum_i \left\lfloor \frac{1}{2} r(E_i) \right\rfloor$ for any partition

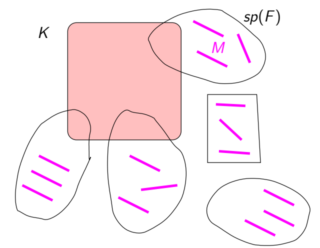
Necessity in Lovász' min-max

For a matching M, define $x = x^M, y = y^M$ by

$$x(e) := \begin{cases} 1 \text{ if } e \in M \\ 0 \text{ otherwise,} \end{cases}$$

and for all $K < V$ and $F \subseteq E$, let

$$y_K(F) := r(K \wedge sp(M \cap F)).$$



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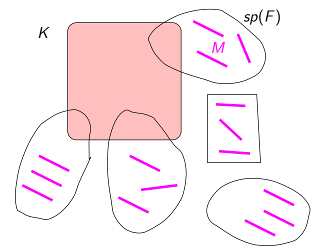

$$x(e) := \begin{cases} 1 \text{ if } e \in M \\ 0 \text{ otherwise,} \end{cases}$$

and for all $K < V$ and $F \subseteq E$, let

$$y_K(F) := r(K \wedge sp(M \cap F)).$$

The following inequalities hold:

(1) $x(F) - y_K(F) \leq \left\lfloor \frac{1}{2} r_{V/K}(sp(F)) \right\rfloor$, ("Parity Constraint")

(2) $\sum_{F \in \pi} y_K(F) \leq r(K)$, ("Partition Constraint")

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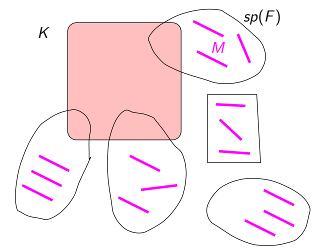

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Combining these inequalities we get

$$|M| = x(E) \leq r(K) + \sum_{F \in \pi} \left\lfloor \frac{1}{2} r_{V/K}(F) \right\rfloor .$$

The following inequalities hold:

$$\begin{array}{lll} (1) & x(F) - y_K(F) \leq \left\lfloor \frac{1}{2} r_{V/K}(sp(F)) \right\rfloor, & \text{("Parity Constraint")} \\ (2) & \sum_{F \in \pi} y_K(F) \leq r(K), & \text{("Partition Constraint")} \\ (3) & x(E) \leq \frac{1}{2}(V), & \end{array}$$

Assign dual variables:

$$
\begin{array}{lrl}
(1) & x(F) - y_K(F) \le \left\lfloor \frac{1}{2} r_{V/K}(sp(F)) \right\rfloor, & \delta(K,F) \\
(2) & \sum_{F\in\pi} y_K(F) \le r(K), & \gamma(K,\pi) \\
(3) & x(E) \le \frac{1}{2}(V), & \alpha
\end{array}
$$

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\begin{array}{lrl}
(1) & x(F) - y_K(F) \leq \left\lfloor \frac{1}{2} r_{V/K}(sp(F)) \right\rfloor, & \delta(K,F) \\
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(3) & x(E) \leq \frac{1}{2}(V), & \alpha
\end{array}
$$

STEP 0.

Initially, set

$$\alpha := w_{max}$$

and set all other dual variables 0.

$$(1)\quad x(F) - y_K(F) \leq \left\lfloor \tfrac{1}{2} r_{V/K}(sp(F)) \right\rfloor, \qquad \delta(K,F)$$

$$(2)\quad \textstyle\sum_{F\in\pi} y_K(F) \leq r(K), \qquad \gamma(K,\pi)$$

$$(3)\quad x(E) \leq \tfrac{1}{2}(V), \qquad \alpha$$

STEP 0.

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and set all other dual variables 0.

Complementary slackness $\Leftrightarrow$ FIND perfect matching in $V, E^{=}$

$$
\begin{array}{llr}
(1) & x(F) - y_K(F) \le \left\lfloor \frac{1}{2} r_{V/K}(sp(F)) \right\rfloor, & \delta(K,F) \\
(2) & \sum_{F \in \pi} y_K(F) \le r(K), & \gamma(K,\pi) \\
(3) & x(E) \le \frac{1}{2}(V), & \alpha
\end{array}
$$

STEP 1.

CASE 1. There is a perfect matching M in $V, E^{=}$. RETURN M.

CASE 2. Otherwise, take K, π from Lovász' min-max, and change dual by

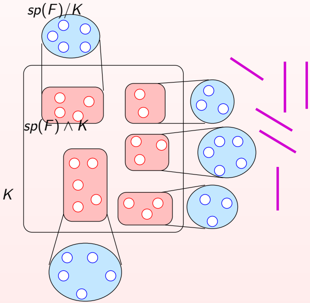
— $\delta(K, F) := \epsilon$ for all $F \in \pi$,

— $\gamma(K, \pi) := \epsilon$,

— $\alpha := w_{max} - \epsilon$,

taking ϵ maximal, subject to dual feasibility.

STEP 2. Complementary slackness conditions equivalent with:



FIND $M_{purple} \subseteq E_{purple}$, B_{blue}, B_{red} SUCH THAT

— For all F, either B_{blue} has a basis of $sp(F)/K$ and B_{red} contains one element from $sp(F) \wedge K$, OR B_{blue} has a near-basis of $sp(F)/K$ and B_{red} contains no element from $sp(F) \wedge K$

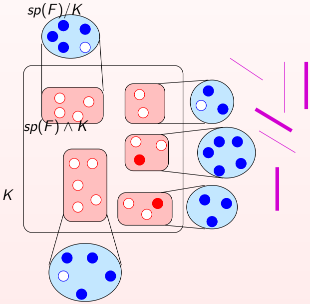
— B_{red} is a basis of K

— $B_{blue} \cup B_{red} \cup M_{purple}$ spans V/K

— $B_{blue} \cup B_{red} \cup M_{purple}$ are skew

This, in turn, is equivalent with an instance of unweighted linear matroid matching.

STEP 2. Complementary slackness conditions equivalent with:



FIND $M_{purple} \subseteq E_{purple}$, B_{blue}, B_{red}

SUCH THAT

— For all F, either B_{blue} has a basis of $sp(F)/K$ and B_{red} contains one element from $sp(F) \wedge K$, OR B_{blue} has a near-basis of $sp(F)/K$ and B_{red} contains no element from $sp(F) \wedge K$

— B_{red} is a basis of K

— $B_{blue} \cup B_{red} \cup M_{purple}$ spans V/K

— $B_{blue} \cup B_{red} \cup M_{purple}$ are skew

Extended Formulation of the Linear Matroid Matching Polytope

- For $e \in E$, introduce variable

$$x(e) \geq 0.$$

- For subspaces $V > K > L$ and $F \subseteq E$, introduce variable

$$y_{K,L}(F) \geq 0.$$

For a matching M, we put

- $x^M(e) := 1$ if $e \in M$, and 0 otherwise,
-

$$y^M_{K,L}(F) := y^M_K(F) - y^M_L(F) = r(K \wedge sp(M \cap F)) - r(L \wedge sp(M \cap F)).$$

Extended Formulation of the Linear Matroid Matching Polytope

$$\sum_{i=1}^{j} \sum_{F \in \mathcal{F}} y_{D_{i-1},D_i}(F) \leq r(D_j) \qquad \text{"Partition Constraint"}$$

$$x(F) - \sum_{i \in I_2} y_{D_{i-1},D_i}(F) \leq \left\lfloor \frac{1}{2} \sum_{i \notin I_2} r_{V/D_{i-1}}(D_i \wedge sp(F)) \right\rfloor \qquad \text{"Parity Constraint"}$$

$$2x(e) - \sum_{i \leq j} y_{D_{i-1},D_i}(e) \leq 0 \qquad \text{"Line Constraint"}$$

where $0 = D_0 < D_1 < D_2 < \cdots < D_k = V$ is a chain of subspaces, $F \subseteq E$, $\mathcal{F}$ a partition of E, $j \leq k$, $I_2 \subseteq \{1, 2, \cdots, k\}$, and $e < D_j$.

Necessity in the extended formulation

1. Consider $\mathcal{D}, j, \mathcal{F}$ for the degree constraint. Then

$$\sum_{i=1}^{j} \sum_{F \in \mathcal{F}} y_{D_{i-1}, D_i}(F) = \sum_{F \in \mathcal{F}} r(sp(M \cap F) \wedge D_j) \leq r(sp(M) \wedge D_j) \leq r(D_j)$$

implying the Partition Constraint.

2. Assume $e \in M$. Then

$$\sum_{i \leq j} y_{D_{i-1}, D_i}(e) = \sum_{i \leq j} (r(D_{i-1} \wedge e) - r(D_i \wedge e)) = r(D_j \wedge e) = r(e) = 2 = 2x(e)$$

implying the Line Constraint.

Necessity in the extended formulation

3. By

$$y_{D_{i-1},D_i}(F) \le r_{V/D_{i-1}}(D_i \wedge sp(F))$$

we get that

$$2x(F) = 2|M \cap F| = \sum_{i=1}^{k} y_{D_{i-1},D_i}(F) \le \sum_{i \in I_2} y_{D_{i-1},D_i}(F) + \sum_{i \notin I_2} r_{V/D_{i-1}}(D_i \wedge sp(F)).$$

Claim

For $a, b, c \in \mathbb{N}$,

$$2a \le b + c \quad \text{implies} \quad a \le b + \left\lfloor \frac{1}{2} c \right\rfloor.$$

Thus

$$x(F) \le \sum_{i \in I_2} y_{D_{i-1},D_i}(F) + \left\lfloor \frac{1}{2} \sum_{i \notin I_2} r_{V/D_{i-1}}(D_i \wedge sp(F)) \right\rfloor,$$

implying the parity constraint.

Extended Formulation of the Linear Matroid Matching Polytope

$$\sum_{i=1}^{j} \sum_{F \in \mathcal{F}} y_{D_{i-1},D_i}(F) \leq r(D_j) \qquad \text{"Partition Constraint"}$$

$$x(F) - \sum_{i \in I_2} y_{D_{i-1},D_i}(F) \leq \left\lfloor \tfrac{1}{2} \textstyle\sum_{i \notin I_2} r_{V/D_{i-1}}(D_i \wedge sp(F)) \right\rfloor \qquad \text{"Parity Constraint"}$$

$$2x(e) - \sum_{i \leq j} y_{D_{i-1},D_i}(e) \leq 0 \qquad \text{"Line Constraint"}$$

where $0 = D_0 < D_1 < D_2 < \cdots < D_k = V$ is a chain of subspaces, $F \subseteq E$, $\mathcal{F}$ a partition of E, $j \leq k$, $I_2 \subseteq \{1, 2, \cdots, k\}$, and $e < D_j$.

Extended Formulation of the Linear Matroid Matching Polytope

$$\sum_{i=1}^{j} \sum_{F \in \mathcal{L}} \lambda(F) y_{D_{i-1},D_i}(F) \leq \kappa(\mathcal{L}, \lambda) r(D_j)$$ "Laminar Constraint"

$$x(F) - \sum_{i \in I_2} y_{D_{i-1},D_i}(F) \leq \left\lfloor \tfrac{1}{2} \textstyle\sum_{i \notin I_2} r_{V/D_{i-1}}(D_i \wedge sp(F)) \right\rfloor$$ "Parity Constraint"

$$2x(e) - \sum_{i \leq j} y_{D_{i-1},D_i}(e) \leq 0$$ "Line Constraint"

where $0 = D_0 < D_1 < D_2 < \cdots < D_k = V$ is a chain of subspaces, $F \subseteq E$, $\mathcal{L}$ is a weighted laminar family of subsets of E, with weights $\lambda : \mathcal{L} \to \mathbb{R}_+$, $j \leq k$, $I_2 \subseteq \{1, 2, \cdots, k\}$, and $e < D_j$.

$$\sum_{i=1}^{j} \sum_{F \in \mathcal{L}} \lambda(F) y_{D_{i-1},D_i}(F) \leq \kappa(\mathcal{L}, \lambda) r(D_j)$$ "Laminar Constraint"

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$$2x(e) - \sum_{i \leq j} y_{D_{i-1},D_i}(e) \leq 0$$ "Line Constraint"

$$x(E) \leq r(V)/2$$

Assign dual variables:

- $\gamma(\mathcal{D}, j, \mathcal{L}, \lambda)$ for Laminar Constraints
- $\delta(\mathcal{D}, I, F)$ for Parity Constraints
- $\beta(\mathcal{D}, j, e)$ for Line Constraints
- α for constraint $x(E) \leq r(V)/2$

Consider $\alpha, \mathcal{D}, \mathcal{L}, \delta, \lambda_i, I_2^F$, where

- $\mathcal{D} = \{D_1, D_2, \cdots, D_k\}$ is a chain of subspaces
- $\mathcal{L}$ is a laminar family of subsets of E
- $\delta : \mathcal{L} \to \mathbb{R}_+$
- $\lambda_i : \mathcal{F} \to \mathbb{R}_+$ for $i = 1, 2, \cdots, k$
- $I_2^F \subseteq I_{D,F} \subseteq \{1, 2, \cdots, k\}$ (such that $I_{D,F} - I_2^F$ are laminar) for all $F \in \mathcal{L}$

A **Laminar Dual Solution** is given by

- $\gamma(\mathcal{D}, i, \mathcal{L}, \lambda_i) := 1$ for $i = 1, 2, \cdots k$
- $\delta(\mathcal{D}, I, F) := \delta_F$ for $F \in \mathcal{L}$
- $\beta(\mathcal{D}, j, e)$ maximal – subject to dual feasibility – for $e \in E$
- α

Min-max for weighted linear matroid matching

The maximum weight of a matching is equal to the minimum value of a laminar dual feasible solution, that is,

$$\nu(V, E, w) = \min \alpha r(V) + \sum_{i=1}^{k} \kappa(\mathcal{L}, \lambda_i) r(D_i) +$$
$$+ \sum \delta(F) \left\lfloor \tfrac{1}{2} \textstyle\sum_{i \notin I_2} r_{V/D_{i-1}}(D_i \wedge sp(F)) \right\rfloor$$

where $\alpha, \mathcal{D}, \mathcal{L}, \delta, \lambda_i, I_2^F$ is a laminar dual solution.

Algorithm.

- We maintain a laminar dual solution.
- Start with $\alpha = w_{max}$, $\mathcal{D} = \mathcal{L} = \emptyset$.
- Given a laminar dual solution $\alpha, \mathcal{D}, \mathcal{L}, \delta, \lambda_i, I_2^F$, construct auxiliary unweighted instance as follows.
- Auxiliary unweighted instance is equivalent with complementary slackness conditions
- $V_D := \bigotimes_{i=1}^{k} (D_i / D_{i-1})$
- For $F \in \mathcal{F}$, let G_F be a basis of $sp(F) \cap \bigcup_{i \in I_2} (D_i / D_{i-1})$, and let H_F be a basis of $sp(F) \cap \bigcup_{i \notin I_2} (D_i / D_{i-1})$
- Let $m_F := \left\lfloor \frac{1}{2} \sum_{i \notin I_2^F} r_{V/D_{i-1}}(D_i \wedge sp(F)) \right\rfloor$
- $B := \{h_F : F \in \mathcal{L}_{max}\} \cup \{g_{F,p} : F \in \mathcal{L}_{max}, p = 1, 2, \cdots, m_F\}$
- $V' := V_D \otimes \bigotimes_{b \in B} \mathbf{1}_b$
- $E' := E^{=} - \bigcup_{F \in \mathcal{F}} F \cup \bigcup_{F \in \mathcal{L}} (E[G_F, h_F] \cup E[h_F, B_F] \cup E[B_F, H_F])$

Algorithm.

- SOLVE maximum matching in V', E'
- IF $\exists$ perfect matching M' in V', E', expand M' to M, and RETURN M
- OTHERWISE, take K', π' from Lovász' min-maximal
- K' is separable, that is, it has the form of

$$K' = \bigotimes_{i=1}^{k} K'_i \otimes \bigotimes_{b \in B'} \mathbf{1}_b$$

where $K'_i < D_i / D_{i-1}$ and $B' \subseteq B$.

Algorithm.

- DUAL CHANGE using K', π', constructed as follows.
- Let $D'_{2i} := D_i$, and $D_{2i-1} := K'_i \otimes D_{i-1}$, and put $\mathcal{D}' := \{D'_i : i = 1, 2, \cdots, 2k\}$.
- $\alpha' := \alpha - \epsilon$
- For $F' \in \pi'$, let $F'' := (F' \cap E^{=}) \cup \bigcup \mathcal{L}_{F'}$.
- Put $\mathcal{L}' := \mathcal{L} \cup \{F'' : F' \in' pi\}$
- Let $\mathcal{L} = \mathcal{L}_+ \cup \mathcal{L}_0 \cup \mathcal{L}_-$ based on π.
- $\delta'_F := \begin{cases} \delta(F) \text{ if } F \in \mathcal{L}_0 \\ \delta(F) - \epsilon \text{ if } F \in \mathcal{L}_- \\ \delta(F) + \epsilon \text{ if } F \in \mathcal{L}_+ \\ \epsilon \text{ if } F = F''. \end{cases}$
- For $i = 1, 2, \cdots, k$, let $J^i = J^i_+ \cup J^i_0 \cup J^i_-$ based on i and π.
- For $i = 1, 2, \cdots, k$, put $\lambda'_{2i}(F) := \begin{cases} \lambda_{2i}(F) \text{ if } F \in J^i_0 \\ \lambda_{2i}(F) - \epsilon \text{ if } F \in J^i_- \\ \lambda_{2i}(F) + \epsilon \text{ if } F \in J^i_+ \\ 0 \text{ otherwise,} \end{cases}$

  while $\lambda'_{2i-1}(F'') := \epsilon$.

Running time:

— either deficiency of auxiliary instance decreases, or the rank of its kernel decreases, thus we obtain a bound of $r(V)^2$ on the number of dual changes

— we can determine a basis of every subspace D_i, if, for example, $V = GF(q)^n$ or $\mathbb{Q}^n$ in polynomial time

Conclusion

Theorem (Iwata 2011 — and independently — P 2011)

Weighted linear matroid matching is solvable in strongly polynomial time.

Questions:

— weighted linear delta-matroid parity

— bound the coefficients in a facet

Thank you for your attention!