Graphic Matroids

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Problem: Given a binary matroid M, is M graphic?

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Theorem [Whitney 1933]
A graph G is planar $\Leftrightarrow$ $M(G)^*$ is graphic

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- $O(r^2 n)$-time [Bixby, Cunningham 1980]

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- $O(r^2 n)$-time [Bixby, Cunningham 1980]
- $O(r^7)$-time [Geelen, Gerards 2011]

$$
\begin{array}{r c l}
& \begin{array}{cc} \qquad\quad B \qquad\qquad\qquad & \qquad\quad B^* \end{array} & \\
& \begin{array}{cccccc|ccccc} a & b & c & d & e & f & 1 & 2 & 3 & 4 & 5 \end{array} & \\
M & \left[\begin{array}{cccccc|ccccc}
1 & & & & & & 1 & 0 & 0 & 0 & 0 \\
 & 1 & & & & & 1 & 0 & 0 & 0 & 0 \\
 & & 1 & & & & 0 & 1 & 0 & 0 & 1 \\
 & & & 1 & & & 0 & 1 & 1 & 1 & 1 \\
 & & & & 1 & & 0 & 0 & 1 & 1 & 0 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 1
\end{array}\right] & \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} \quad B
\end{array}
$$

Fundamental matrix

$$
\begin{array}{ccc}
 & \begin{array}{cc} \color{red}{B} & \qquad\qquad \color{blue}{B^*} \end{array} & \\
 & \begin{array}{ccccccccccc} a & b & c & d & e & f & 1 & 2 & 3 & 4 & 5 \end{array} & \\
M & \left[\begin{array}{cccccc|ccccc}
1 & & & & & & 1 & 0 & 0 & 0 & 0 \\
 & 1 & & & & & 1 & 0 & 0 & 0 & 0 \\
 & & 1 & & & & 0 & 1 & 0 & 0 & 1 \\
 & & & 1 & & & 0 & 1 & 1 & 1 & 1 \\
 & & & & 1 & & 0 & 0 & 1 & 1 & 0 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 1
\end{array}\right] & \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} \quad \color{red}{B}
\end{array}
$$

Fundamental circuits

$$C(4) = \{d, e, f, 4\}$$

$$\begin{array}{cc}
 & \begin{array}{ccccccccccc} \multicolumn{6}{c}{B} & \multicolumn{5}{c}{B^*} \\ a & b & c & d & e & f & 1 & 2 & 3 & 4 & 5 \end{array} \\
M & \left[\begin{array}{cccccc|ccccc}
1 & & & & & & 1 & 0 & 0 & 0 & 0 \\
 & 1 & & & & & 1 & 0 & 0 & 0 & 0 \\
 & & 1 & & & & 0 & 1 & 0 & 0 & 1 \\
 & & & 1 & & & 0 & 1 & 1 & 1 & 1 \\
 & & & & 1 & & 0 & 0 & 1 & 1 & 0 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 1
\end{array}\right]\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} \quad B
\end{array}$$

Fundamental cocircuit

$$C^*(c) = \{c, 2, 5\}.$$

Main Theorem [G., Gerards]

Let B be a basis in a binary matroid M.

M is graphic $\Leftrightarrow$ the following system has a solution over $GF(2)$:

(G1) $\quad x_{ab} + x_{ac} = 0,$

$$\begin{array}{c|c|} & e \\ \hline a & 0 \\ b & 1 \\ c & 1 \\ \hline \end{array}$$

(G2) $\quad x_{ab} + x_{ac} + x_{ba} + x_{bc} + x_{ca} + x_{cb} = 1,$

$$\begin{array}{c|c|} & e \\ \hline a & 1 \\ b & 1 \\ c & 1 \\ \hline \end{array}$$

Main Theorem [G., Gerards]

Let B be a basis in a binary matroid M.

M is graphic $\Leftrightarrow$ the following system has a solution over $GF(2)$:

(G1) $x_{ab} + x_{ac} = 0$, if $C^*(b) \cap C^*(c) - C^*(a) \neq \emptyset$

(G2) $x_{ab} + x_{ac} + x_{ba} + x_{bc} + x_{ca} + x_{cb} = 1$, if $C^*(a) \cap C^*(b) \cap C^*(c) \neq \emptyset$

Algorithm

$n = |M|, \quad r = r(M).$

$O(r^2)$ variables

$O(r^3)$ constraints

$\Rightarrow \quad O(r^7)$-time to solve

$$\begin{array}{cc}
\begin{array}{cccccc|ccccc} a & b & c & d & e & f & 1 & 2 & 3 & 4 & 5 \end{array} & \\
\left[\begin{array}{cccccc|ccccc}
1 & & & & & & 1 & 0 & 0 & 0 & 0 \\
& 1 & & & & & 1 & 0 & 0 & 0 & 0 \\
& & 1 & & & & 0 & 1 & 0 & 0 & 1 \\
& & & 1 & & & 0 & 1 & 1 & 1 & 1 \\
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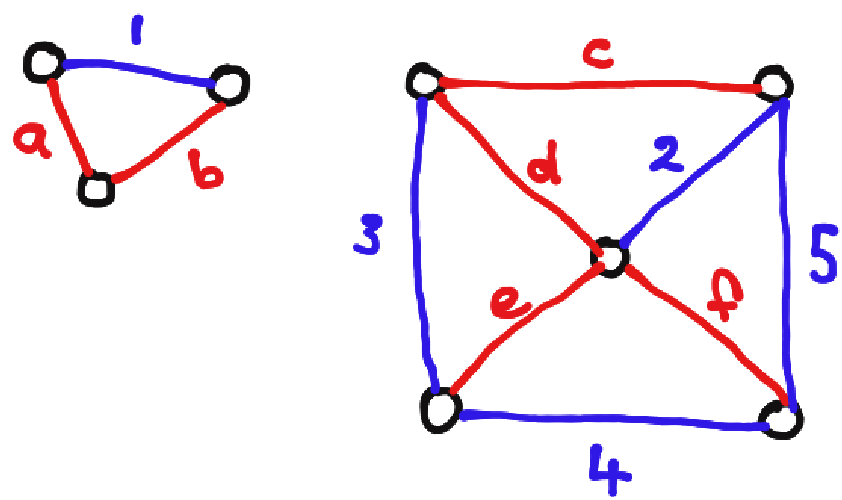
\end{array}\right] & \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array}
\end{array}$$



$$
\begin{array}{cccccccccccc}
a & b & c & d & e & f & 1 & 2 & 3 & 4 & 5 & \\
\left[\begin{array}{cccccc|ccccc}
1 & & & & & & 1 & 0 & 0 & 0 & 0 \\
& 1 & & & & & 1 & 0 & 0 & 0 & 0 \\
& & 1 & & & & 0 & 1 & 0 & 0 & 1 \\
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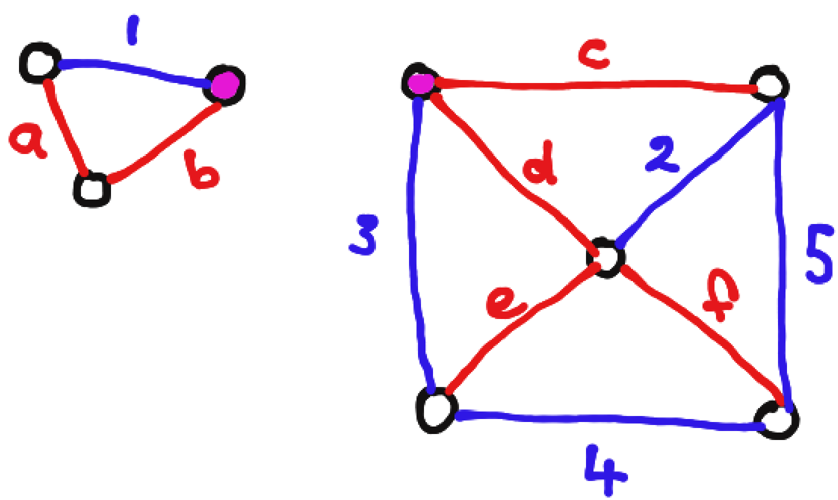
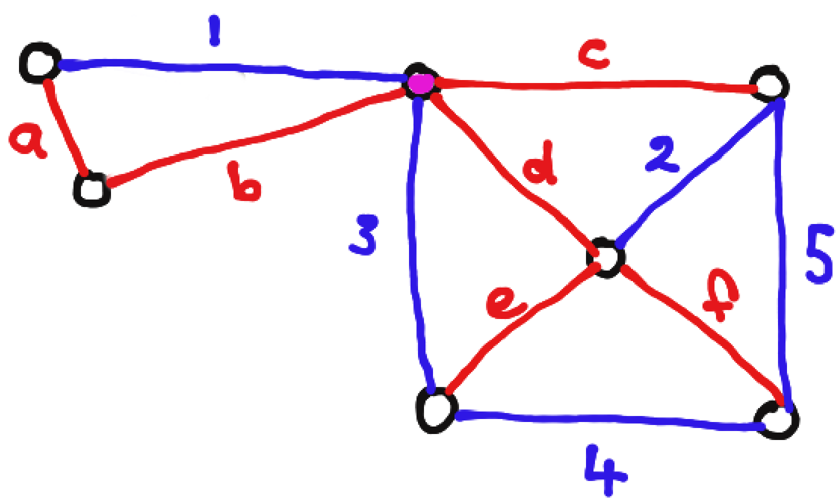
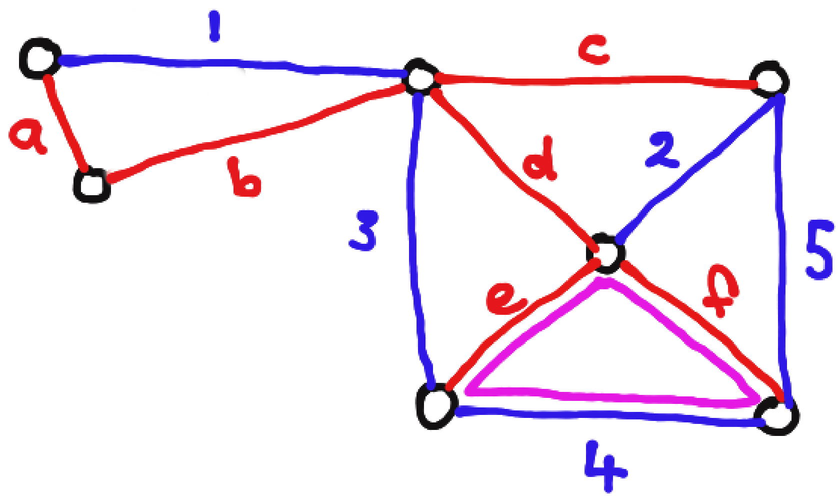
\end{array}\right]
\end{array}
\begin{array}{c}
a \\ b \\ c \\ d \\ e \\ f
\end{array}
$$



$$\begin{array}{cc} \begin{array}{ccccccccccc} a & b & c & d & e & f & 1 & 2 & 3 & 4 & 5 \end{array} & \\ \left[\begin{array}{cccccc|ccccc} 1 & & & & & & 1 & 0 & 0 & 0 & 0 \\ & 1 & & & & & 1 & 0 & 0 & 0 & 0 \\ & & 1 & & & & 0 & 1 & 0 & 0 & 1 \\ & & & 1 & & & 0 & 1 & 1 & 1 & 1 \\ & & & & 1 & & 0 & 0 & 1 & 1 & 0 \\ & & & & & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right] & \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} \end{array}$$

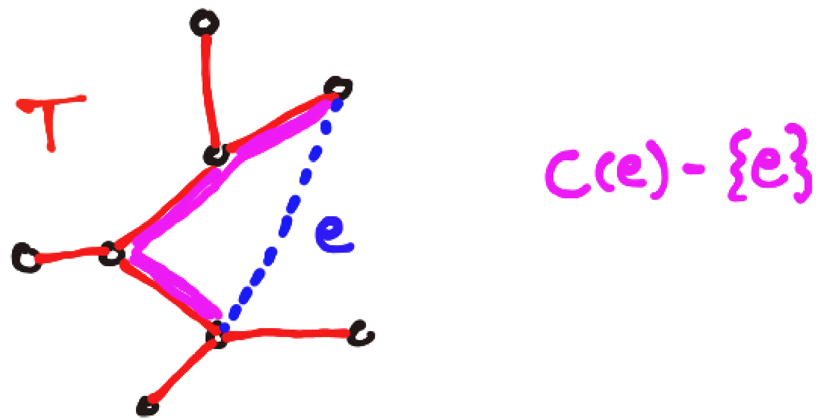


$$
\begin{array}{cccccccccccc}
a & b & c & d & e & f & 1 & 2 & 3 & 4 & 5 & \\
\left[\begin{array}{cccccc} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{array}\right. & & & & & &
\left|\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] & & & & & \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array}
\end{array}
$$



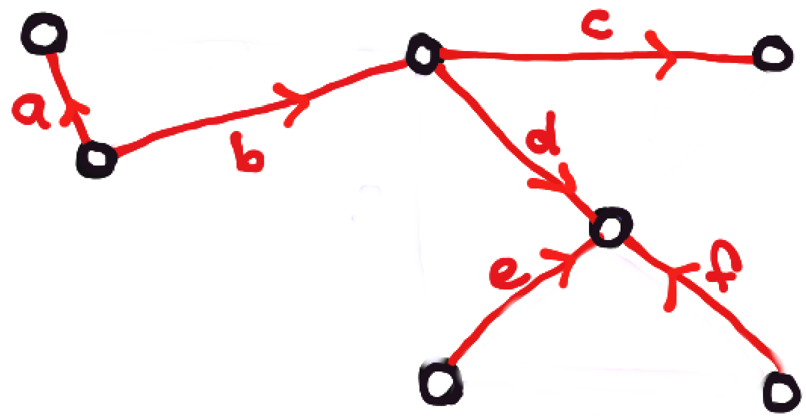
$C(4) = \{4, e, f\}$

Easy Lemma. Let B be a basis in a binary matroid M. Then M is graphic $\Leftrightarrow$ there is a tree T with $E(T) = B$ such that $C(e) - \{e\}$ is a path in T for each $e \in B^*$.



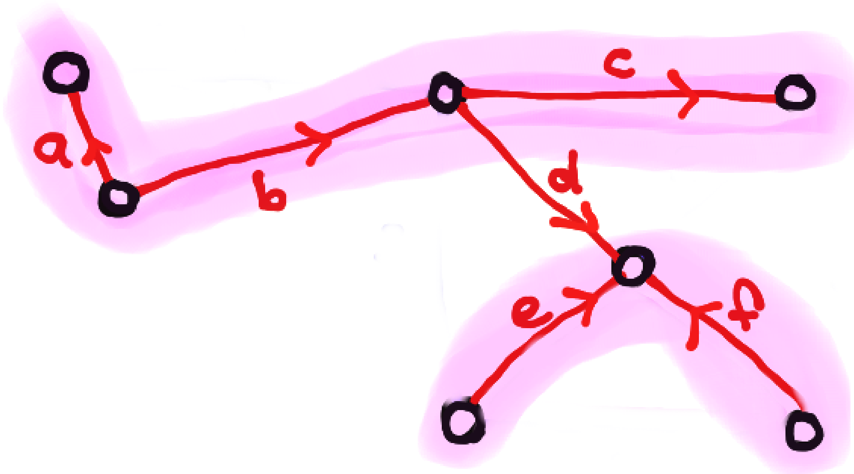
Encoding a tree

T



Encoding a tree

T

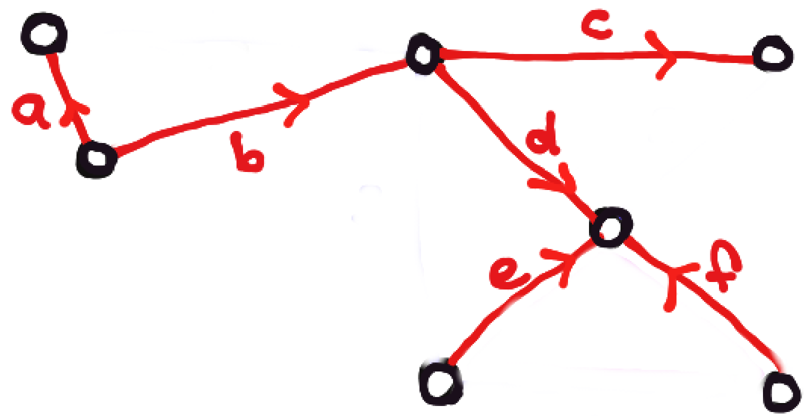


$$x^T_{da} = x^T_{db} = x^T_{dc} = 0$$

$$x^T_{de} = x^T_{df} = 1$$

Encoding a tree

T



Remark. $x^T_{pq} + x^T_{pr} = 1 \iff$ p separates q from r in T.

Main Theorem [G., Gerards]

Let B be a basis in a binary matroid M.

M is graphic $\Leftrightarrow$ the following system has a solution over $GF(2)$:

$$\begin{array}{c|c|} & e \\ a & 0 \\ b & 1 \\ c & 1 \end{array}$$

(G1) $\boxed{x_{ab} + x_{ac}} = 0,$

(G2) $\boxed{x_{ab} + x_{ac}} + \boxed{x_{ba} + x_{bc}} + \boxed{x_{ca} + x_{cb}} = 1,$

$$\begin{array}{c|c|} & e \\ a & 1 \\ b & 1 \\ c & 1 \end{array}$$

Encoding a path in a tree

$P \subseteq E(T)$ is a path $\Leftrightarrow$

(P1) P is connected

(P2) P is contained in a path

Encoding a path in a tree

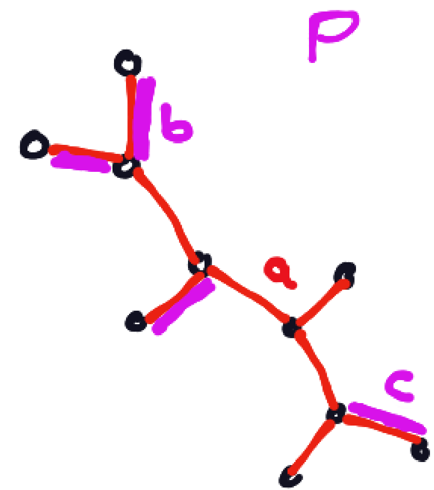
$P \subseteq E(T)$ is a path $\Longleftrightarrow$

(P1) P is connected

For $b, c \in P$ and $a \in E(T) - P$,

$$x^T_{ab} + x^T_{ac} = 0$$

(P2) P is contained in a path



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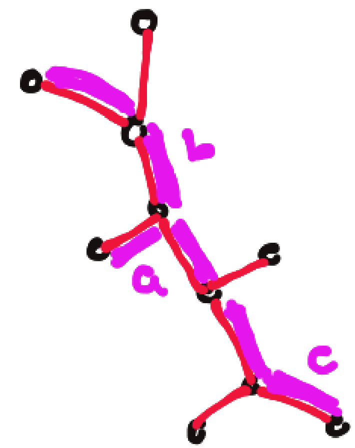


(P2) P is contained in a path

For $a, b, c \in P$,

$$x^T_{ab} + x^T_{ac} + x^T_{ba} + x^T_{bc} + x^T_{ca} + x^T_{cb} = 1$$

Summary.

Let B be a basis of a binary matroid M.

Then

(1) If M is graphic, then there is a solution to (G1) & (G2).

(2) If there is a directed tree T such that x^T satisfies (G1) & (G2), then M is graphic.

Summary.

Let B be a basis of a binary matroid M.

Then

(1) If M is graphic, then there is a solution to (G1) & (G2).

(2) If there is a directed tree T such that x^T satisfies (G1) & (G2), then M is graphic.

Required to prove: If (G1) & (G2) has a solution, then it has a solution of the for x^T for some tree T.

Lemma A. If (M, B) is a minimal counterexample, then M is 3-connected.

Lemma B. If M is 3-connected and x satisfies (G1) & (G2), then $x = x^T$ for some tree T.

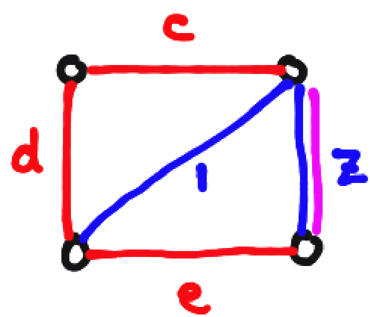
Connectivity reduction

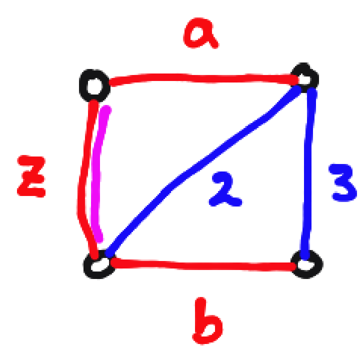
$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{ccc} 1 & 2 & 3 \\ \left[\begin{array}{c} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{array}\right. & \begin{array}{c} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{array} & \left.\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}\right] \end{array} \longleftrightarrow \begin{array}{c|cc} & 1 & z \\ \hline c & 1 & 1 \\ d & 1 & 1 \\ e & 0 & 1 \end{array} \qquad \begin{array}{c|cc} & 2 & 3 \\ \hline a & 1 & 1 \\ b & 0 & 1 \\ z & 1 & 1 \end{array}$$

Connectivity reduction

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline a & 0 & 1 & 1 \\ b & 0 & 0 & 1 \\ c & 1 & 1 & 1 \\ d & 1 & 1 & 1 \\ e & 0 & 1 & 1 \end{array} \quad \longleftrightarrow \quad \begin{array}{c|cc} & 1 & z \\ \hline c & 1 & 1 \\ d & 1 & 1 \\ e & 0 & 1 \end{array} \qquad \begin{array}{c|cc} & 2 & 3 \\ \hline a & 1 & 1 \\ b & 0 & 1 \\ z & 1 & 1 \end{array}$$

$$\updownarrow \qquad\qquad\qquad \updownarrow$$

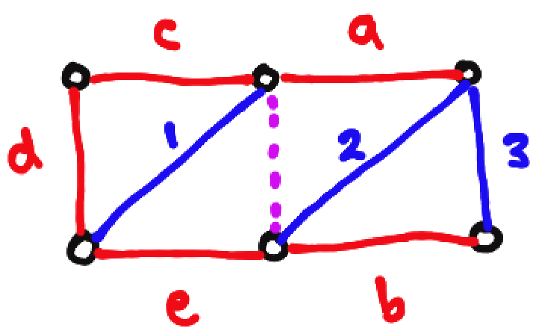




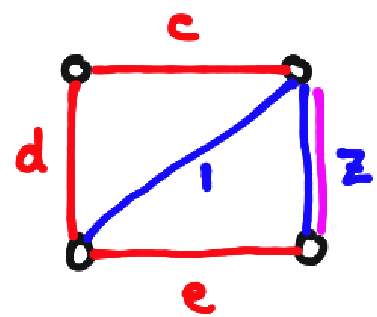
Connectivity reduction

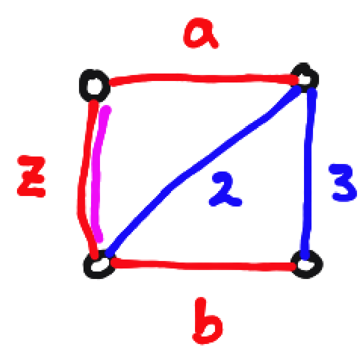
$$\begin{array}{cc} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \longleftrightarrow \begin{array}{cc} & \begin{matrix} 1 & z \end{matrix} \\ \begin{matrix} c \\ d \\ e \end{matrix} & \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{matrix} 2 & 3 \end{matrix} \\ \begin{matrix} a \\ b \\ z \end{matrix} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

$$\updownarrow \qquad\qquad \updownarrow$$



$\longleftrightarrow$

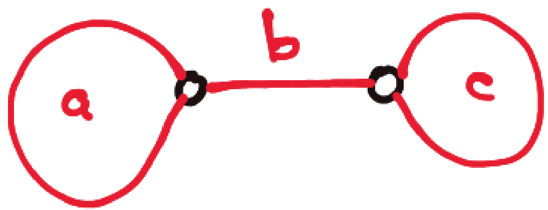




Lemma A. If (M,B) is a minimal counterexample, then M is 3-connected.

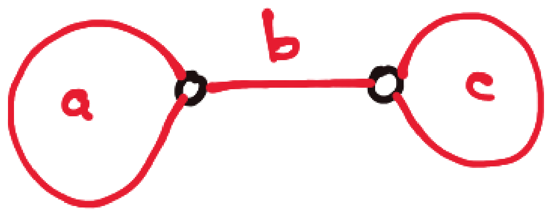
Lemma B. If M is 3-connected and x satisfies (G1) & (G2), then $x = x^T$ for some tree T.

Recognizing a tree



Remark: For $a, b, c \in E(T)$, if b separates a from c, then a does not separate b from c.

Recognizing a tree



Remark: For $a,b,c \in E(T)$, if b separates a from c, then a does not separate b from c.

Lemma. $x: B^{(2)} \to \{0,1\}$ encodes a tree $\Leftrightarrow$

(T) For $a,b,c \in B$,

$$x_{ba} + x_{bc} = 1 \Rightarrow x_{ab} + x_{ac} = 0.$$

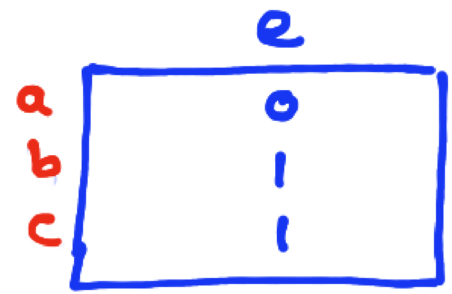
Recall

(G1) $\quad x_{ab} + x_{ac} = 0,$

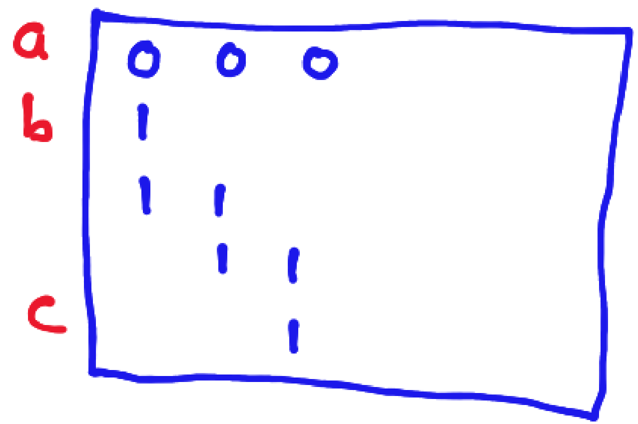
$$\begin{array}{c|c|} & e \\ \hline a & 0 \\ b & 1 \\ c & 1 \\ \hline \end{array}$$

Recall

(G1) $\quad x_{ab} + x_{ac} = 0,$



(G1') $\quad x_{ab} = x_{ac}$



b & c are in the same component of $M \setminus C^*(a)$.

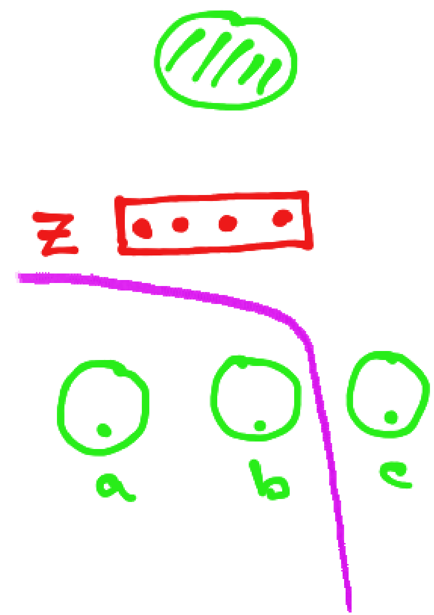
Lemma B. If M is 3-connected and x satisfies (G1) & (G2), then $x = x^T$ for some tree T.

Suppose $x_{ab} + x_{ac} = 1$ and $x_{ba} + x_{be} = 1$

Suppose $x_{ab} + x_{ac} = 1$ and $x_{ba} + x_{bc} = 1$

Let $Z = C^*(a) \cap C^*(b)$

(GI') $\Rightarrow$ a, c are in different components of $M \setminus Z$



Suppose $x_{ab}+x_{ac}=1$ and $x_{ba}+x_{bc}=1$

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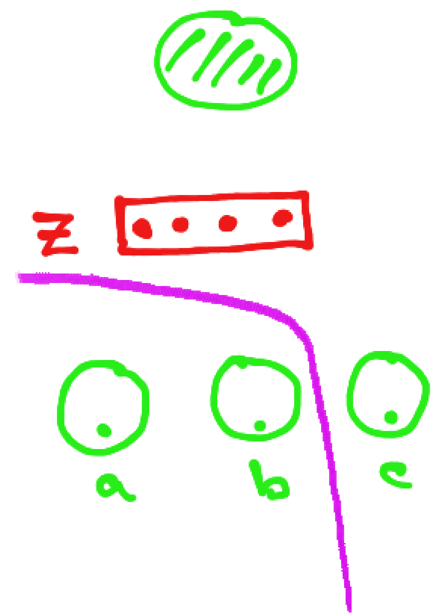

We may assume: $C^*(c)\cap Z \neq \emptyset$.

Suppose $x_{ab}+x_{ac}=1$ and $x_{ba}+x_{bc}=1$

Let $Z = C^*(a) \cap C^*(b)$

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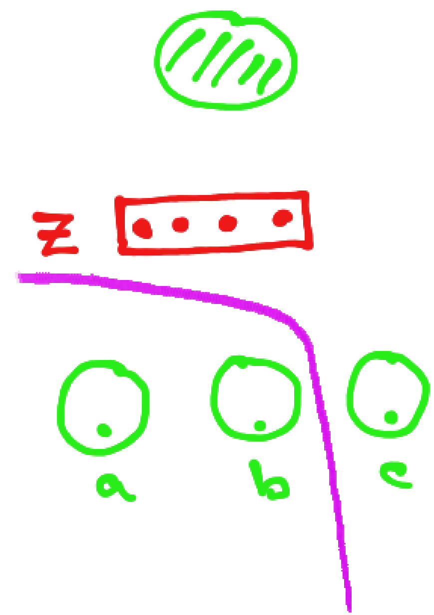


We may assume: $C^*(c) \cap Z \neq \emptyset$.

(G2) $\Rightarrow$ $x_{ca}+x_{cb}=1$.

Suppose $x_{ab} + x_{ac} = 1$ and $x_{ba} + x_{bc} = 1$

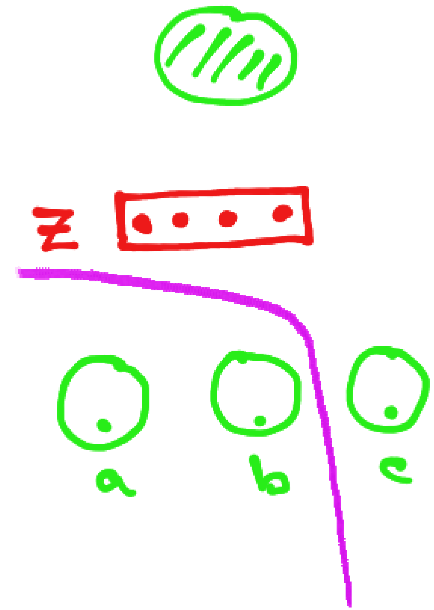
Let $Z = C^*(a) \cap C^*(b)$

(G1') $\Rightarrow$ a, c are in different components of $M \setminus Z$

We may assume: $C^*(c) \cap Z \neq \emptyset$.

(G2) $\Rightarrow$ $x_{ca} + x_{cb} = 1$.

Symmetry $\Rightarrow$ $Z \subseteq C^*(c)$.



Suppose $x_{ab}+x_{ac}=1$ and $x_{ba}+x_{bc}=1$

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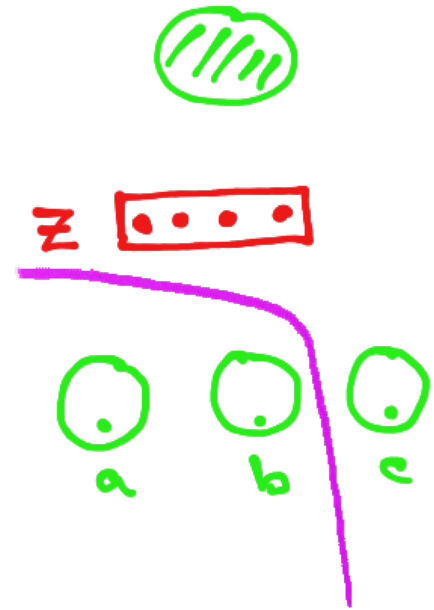


We may assume: $C^*(c) \cap Z \neq \emptyset$.

(G2) $\Rightarrow$ $x_{ca} + x_{cb} = 1$.

Symmetry $\Rightarrow$ $Z \subseteq C^*(c)$.

Symmetry $\Rightarrow$ M not 3-connected.

Related Work

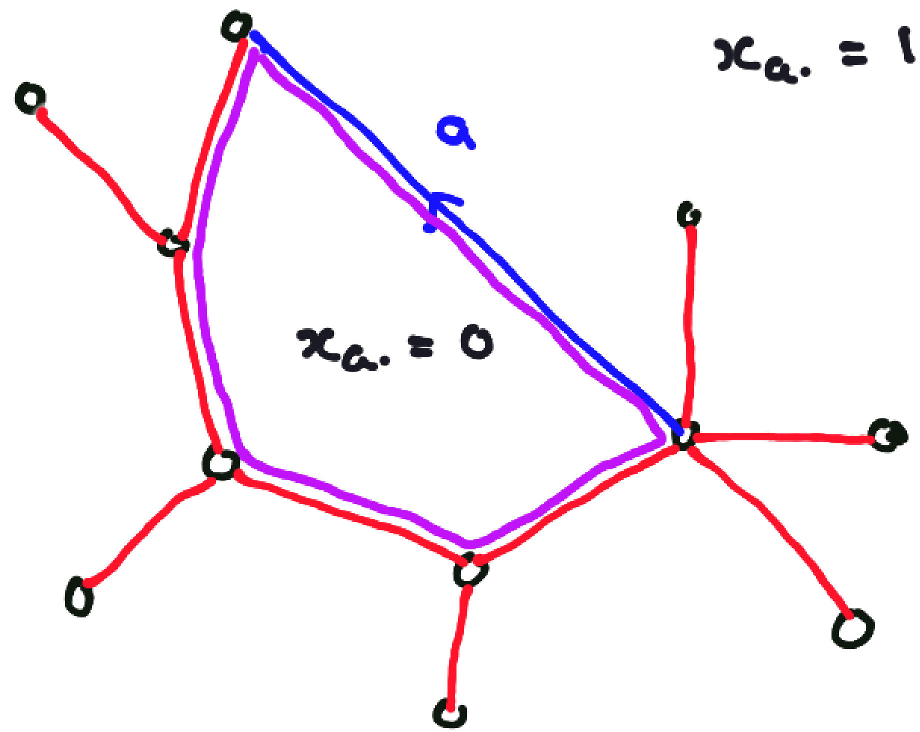
Corollary Let T be a spanning tree of a graph G. Then G is planar $\Leftrightarrow$ the following system has a solution over $GF(2)$

For each $a, b, c \in E(G) - E(T)$,

(P1) $x_{ab} + x_{ac} = 0$, $\quad C_b \cap C_c - C_a \neq 0$

(P2) $x_{ab} + x_{ac} + x_{ba} + x_{bc} + x_{ca} + x_{cb} = 1$,

$C_a \cap C_b \cap C_c \neq 0.$



$x_{a\cdot} = 1$
a
$x_{a\cdot} = 0$

Naji's Theorem. Circle graphs are characterized by a system of linear equations over GF(2).

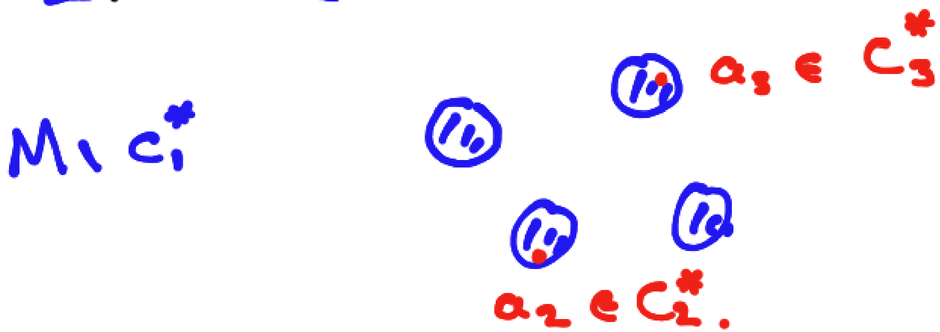
De Fraysseix's Theorem. A bipartite graph is a circle graph $\Leftrightarrow$ if it is the fundamental graph of a planar graph.

$\Rightarrow$ Characterization of cycle matroids of planar graphs.

Open Problem. Does our algebraic characterization imply other characterizations of graphic matroids?

Fournier's Condition.

If C_1^*, C_2^*, C_3^* are cocircuits and $C_1^* \cap C_2^* \cap C_3^* \neq \emptyset$, then one separates the other two.



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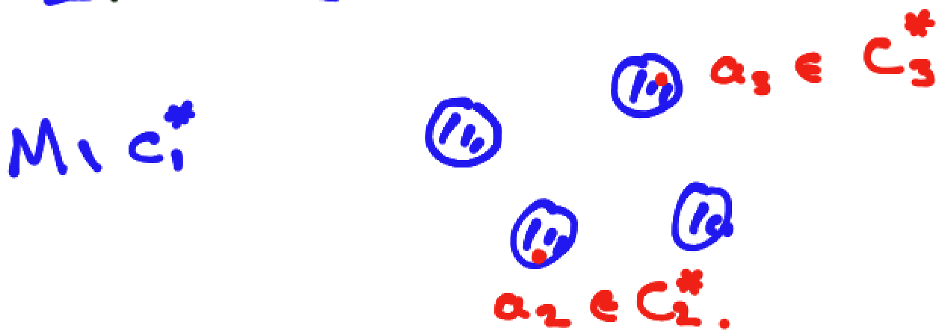
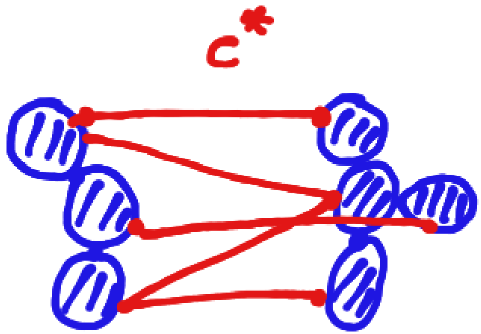


(G2) $$x_{ab} + x_{ac} + x_{ba} + x_{bc} + x_{ca} + x_{cb} = 1,$$

$$C_a^* \cap C_b^* \cap C_c^* \neq \emptyset$$

Tutte's Condition For a cocircuit C^*, the overlap diagram of its bridges is bipartite.



Mightons Theorem (2008).

Let B be a basis in a binary matroid M.

Then M is graphic $\Leftrightarrow$

(1) Tutte's condition holds for each fundamental cocircuit

(2) Fournier's condition holds for each triple of fundamental cocircuits.

Mightons Theorem (2008).

Let B be a basis in a binary matroid M.

Then M is graphic $\Leftrightarrow$

(1) Tutte's condition holds for each fundamental cocircuit

(2) Fournier's condition holds for each triple of fundamental cocircuits.

$\Rightarrow$ our theorem

Happy Birthday

Bill