$\boxed{\text{Day 2}}$ 6/27/11

Colorings and Homomorphisms for Graphs and Finite Structures

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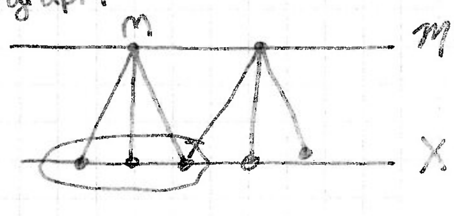
Erdös Thm| from last time: $\forall k \ \forall g \ \exists G$ s.t. $\chi(G) \geq k$ and girth $G > g$.

(Erdös, Hajnal) $\chi(G) \geq c = 2^{\omega} \Rightarrow G \supseteq \square$

$(X, \mathcal{M}) \quad \mathcal{M} \subseteq \binom{X}{3}$

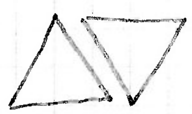
hypergraph



$\chi(X, \mathcal{M}) \gg \not\Rightarrow \exists M \neq M'$
$M \cap M' = 2$

Y large $\quad X = \binom{Y}{2} \qquad M \in \mathcal{M} \Leftrightarrow \exists T \in \binom{Y}{3}$
$M = \binom{T}{2}$

Ramsey $\Rightarrow X$ large.



(Smolíková, N.)

oriented graph

Thm| $\forall g, k \ \exists \vec{G}$: ① sparse (girth g) then
② $\forall \ |\vec{H}| \leq k$, all loops — loops at every vertex
$\vec{G} \not\to \vec{H}$ which is not a constant.

Conjectures Continued from Last Time

③ Conj| (Victor Neumann-Lara) Suppose $\chi(G) \gg$ large
$\exists$ orientation $\vec{G}$ of G so that vertices of $\vec{G}$ cannot be partitioned into k acyclic subgraphs
i.e. $\forall k \ \exists f(k)$ $\chi(G) \geq f(k)$ then $\exists$ an orientation $\vec{G}$ of G so that the vertices of $\vec{G}$ cannot be partitioned into k acyclic subgraphs.

(Müller)
<u>Thm</u> $\forall k\ \forall g\ \exists G$: girth$(G)=g$ and $\chi(G)=k$ and G has a unique k-coloring

<u>Thm</u> (Sparse noncomparability Lemma) (N. Rödl)
$\forall g\ \forall k,\ \forall G$ $\chi(G)>2$ and $\exists\ G' \longrightarrow G$ s.t. girth$(G')\geq g$

$$\begin{tikzcd}
G \arrow[dr] & \\
& H \\
G' \arrow[uu] \arrow[ur] \arrow[u, phantom, "\Updownarrow"] &
\end{tikzcd}$$

$|H|\leq k$ $\quad$ ($G\to H$ iff $G'\to H$)

$$\begin{tikzcd}
& G \arrow[dr] & \\
F \arrow[ur] \arrow[dr, "\text{left distinguishing}"'] & & F \\
& G' \arrow[uu] \arrow[ur, "\text{right distinguishing}"'] &
\end{tikzcd}$$

G and G' cannot be distinguished by small graphs.

<u>Motivation</u>

G, $\chi(G)=k$ does there exist an H such that $H\geq G$, H rigid, $\chi(H)=k$? $\quad G\ncong K_k$

rigid - If there is a homomorphism $f:G\to G$ then f is the identity.

$$\begin{tikzcd}
G \arrow[loop right, "f=id"]
\end{tikzcd}$$

<u>Homomorphism Order</u> countable

All finite graphs write $G_1\leq G_2 \iff G_1\to G_2$ (i.e. there is a homomorphism from G_1 to G_2).

$\leq/\approx$ (homomorphism equivalence) gives a partial order.

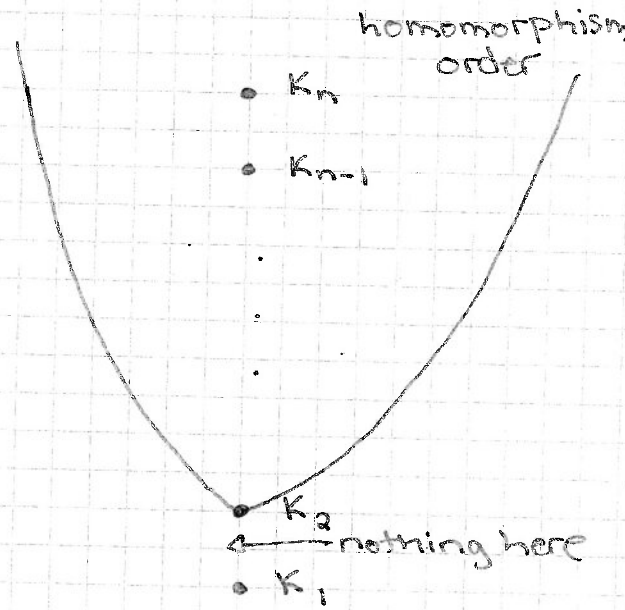
Defn| G is a core of H if G is a minimal retract of H or if G is a subgraph minimal size homomorphic image.
(for finite graphs)

Observation| A core always exists and it is unique up to isomorphism.

Homomorphism order $\mathcal{C}$

$G \longrightarrow K_k \iff G \leq K_k$

- $\mathcal{C}$ is countable universal
- $\mathcal{C}$ is dense
- $\mathcal{C}$ has "fractal look"



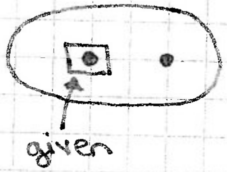
$(P, \leq)$ countable poset $\Rightarrow$ P is induced subposet of $\mathcal{C}$.

$\mathcal{C}$ restricted to the class of
- planar graphs
- cubic graphs (all degrees 3)
- oriented paths

then this restriction is universal

Planar graphs do not represent every group as $Aut(G)$.

Extension Properties of $\mathcal{C}$



independence extension not always possible.

Thm| Given graphs $G_1, G_2, \dots, G_t$ nonbipartite then $\exists\, G$ such that $\forall i\ G \,||\, G_i$ (G is noncomparible to $G_i\ \forall\, i$ $\quad G \not\to G_i$ ...

Pf| $k = \max |V(G_i)|$ let G be graph $\chi(G) > k$

$G \not\to \triangle$, ⬠, $\dots$, $C_{2l+1 \le k}$

$G \not\to G_i$ and $G_i \not\to G$ b/c G_i are nonbipa[rtite] so contain some odd cycle. □

The theorem is not true for oriented graphs. There are infinitely many exceptions. For example

G_1 , G_2

(N., Shelah)

Thm| $\forall\, G$ countable $\exists\, H$: $H \,||\, G$ ← noncomparible

$\cancel{H}$

$\bullet$, $\vert$, K_ω

$t = 1$ true

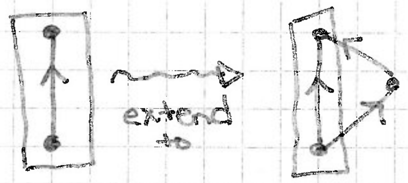
$t = 2$ not true

Example| $t = 2$

$\triangle$, H_3 = universal $\triangle$-free graph

$$\begin{tikzcd} \triangle \arrow[r, "/"] & H_3 \arrow[l, bend left, "/"] \end{tikzcd}$$

Problem| G_1, G_2 (undirected) and $G_1 \not\to G_2$, $G_2 \not\to G_1$ and $\forall\, G$ either $G \to G_i$ or $G_i \to G$ for $i \in \{1,2\}$ (i.e. maximal antichain) then G_1 or G_2 is finite.

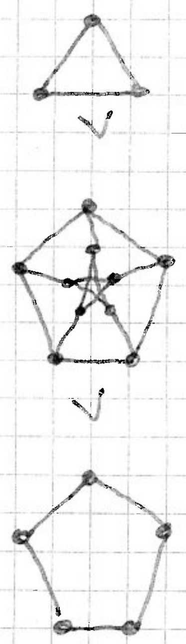


"density"

?

Thm undirected graphs $\mathcal{C}$ is dense / exception except for K_1, K_2

Pf $G_1 < G_2$ G_2 connected find G s.t. $G_1 \lneq G \lneq G_2$



$$G = G_1 + (G_2 \times H) \quad \text{where } \chi(H) >> \text{ and odd girth} >> \text{odd girth of } G_2$$

$+$: disjoint union; $\times$: categorical product

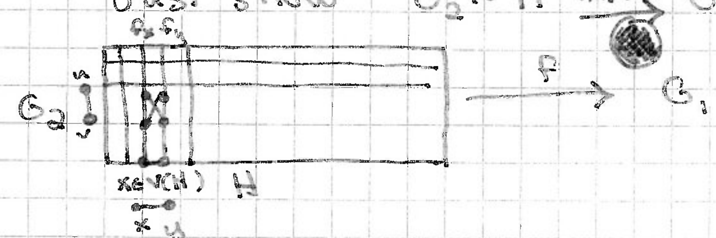
then $G \to G_2$ and $G_1 \to G$

$G_2 \not\to G = G_1 + (G_2 \times H)$ (goes into one or other b/c G_2 connected)

because odd girth of G > odd girth of G_2.

$G = G_1 + (G_2 \times H) \not\to G_1$

Just show $G_2 \times H \not\to G_1$



Consider f_x = restriction of f to xth column.

There are $|V(G_1)|^{|V(G_2)|}$ possibilities.

So choose $\chi(H)$ bigger than $|V(G_1)|^{|V(G_2)|}$

So coloring by fibers f_x is not a good coloring

$\bar{f}: V(G_2) \to V(G_1)$ claim: $\bar{f}$ is a homomorphism can see this from picture