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The Approximability of Constraint Satisfaction Problems

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www.cs.cmu.edu/~odonnell/slides/fields-csp-approx.pdf

D domain $\{0, 1, 2, \ldots, q-1\}$

Γ set of relations R on D^r
(predicate $R : D^r \to \{0,1\}$)

Instance $\mathcal{I}$ of CSP(Γ) is variables $|V| = n$
list of m constraints $C = (R, S)$ $R \in \Gamma$
S (scope) is an ordered list of $ar(R)$ distinct variables.

Defn Assignment $F : V \to D$ satisfies $C = (R, S)$ if $R(F(S)) = 1$.

Defn Value $\mathrm{val}_{\mathcal{I}}(F)$ = fraction of constraints that F satisfies $= \underset{C=(R,S)\in \mathcal{I}}{\mathrm{avg}} \{R(F(S))\} \in [0,1]$.

Defn $\mathrm{Opt}(\mathcal{I}) = \max_F \{\mathrm{val}_{\mathcal{I}}(F)\}$

Examples
- k-Sat: $D = \{0,1\}$ Γ = {disjunctions on $\leq k$ literals}
- Ek-Sat: $D = \{0,1\}$ Γ = {disjunctions on exactly k literals}
- Cut: $D = \{0,1\}$, $\Gamma = \{\neq\}$
- q-Cut: $|D| = q$
- (E)k-Lin(q): $D = \mathbb{Z}_q$
  constraints: $\pm v_1 \pm v_2 \pm \ldots \pm v_k = c$
- Bijection(q) aka Unique-Games(q):
  $|D| = q$ any binary bijective relation.
- Projection(q) aka Label-Cover(q)
  $|D| = q$ any binary projections

  $\exists\, \pi : D \to D$ s.t. $R = \{(a, \pi(a)) : a \in D\}$

Thm Given instance $\mathcal{I}$ of Max-Cut with $\mathrm{Opt}(\mathcal{I}) \geq 3/4$, NP-hard to find assignment F with $\mathrm{Val}_{\mathcal{I}}(F) \gtrsim 3/4$.

Defn For $0 \leq \alpha \leq \beta \leq 1$ we say algorithm <u>(α, β)-approximates</u> $\mathrm{CSP}(\Gamma)$ if

$\forall\ \mathcal{I}$ with $\mathrm{Opt}(\mathcal{I}) \geq \beta$ finds assignment of value $\geq \alpha$.

and <u>(α, β)-distinguishes</u> if

$\mathrm{Opt}(\mathcal{I}) \geq \beta \Rightarrow$ alg. outputs YES
$\mathrm{Opt}(\mathcal{I}) < \alpha \Rightarrow$ alg. outputs NO

Thm $(3/4, 3/4)$-distinguishing Max-Cut is NP-hard.

Sat ? $\Leftrightarrow$ $(1, 1)$-dist. ?

For Max-Cut, $\beta = 3/4$ ...

- $\exists\ \varepsilon_0 > 0$ s.t. $(3/4 - \varepsilon_0, 3/4)$-distinguishing is NP-hard (consequence of "PCP Theorem")
- (GW '94) $(0.878 \cdot 3/4, 3/4)$-approx is in P
  $0.878 \cdot 3/4 = 0.659$

  $$0.878 = \frac{2}{\pi \sin\theta^*}, \quad \theta^* = \tan\left(\frac{\theta^*}{2}\right)$$

- $\left(\frac{11}{16} + \varepsilon, 3/4\right)$-distinguishing NP-hard $\forall\ \varepsilon > 0$
  $\frac{11}{16} = 0.6875$ [Håstad, '97]

- $\left(\frac{2}{3}, \frac{3}{4}\right)$-distinguishing Max-Cut not known in P or NP-hard.
- $\left(0.878 \cdot \frac{3}{4} + \varepsilon, \frac{3}{4}\right)$-dist. is NP-hard $\forall\ \varepsilon > 0$ assuming "Unique Games Conjecture" (2002)

Dream Goal For each $CSP(\Gamma)$ and $\beta \in [0,1]$ find $\alpha_\Gamma(\beta)$ and efficient alg. $(\alpha_\Gamma(\beta), \beta)$-approx. $CSP(\Gamma)$ and show $(\alpha_\Gamma(\beta)+\varepsilon, \beta)$-dist. is NP-hard.

→ Accomplished for k-Sat
Cut
(E)k-Lin(q) — assuming Unique Games conjecture.

Defn Max-$CSP(\Gamma)$ is APX-hard if $\exists \beta$ $\exists \varepsilon_0 > 0$ s.t. $(\beta - \varepsilon_0, \beta)$-dist. is NP-hard.

PCP Thm $\Leftrightarrow$ 3Sat is $(1-\varepsilon_0, 1)$-hard.

Open $\forall$ $CSP(\Gamma)$ exact optimization in P or NP-hard.

→ Known in many cases

Trivial Random Algorithm
Pick $F(v) \sim D$ randomly indep. for each $v \in V$

Prop Trivial Random $(1-2^{-k}, \beta)$-approxes Max-EK-Sat $\forall \beta$.

Pf For a fixed constraint $C = v_1 \vee v_2 \vee v_3$

$$\Pr_F [F \text{ satisfies } C] = 7/8$$

$$\mathop{E}_F [\text{Val}(F)] = E\left[\text{frac of constrs. sat.}\right] = 7/8$$

Thm (Håstad) $\left(\frac{1}{2}+\varepsilon, 1-\varepsilon\right)$-distinguishing E3Lin(2) is NP-hard $\forall \varepsilon > 0$.

$\left(\frac{7}{8}+\varepsilon, 1\right)$-distinguishing E3-Sat is NP-hard $\forall \varepsilon > 0$.

[Raz] $\forall \varepsilon > 0 \ \exists q = q(\varepsilon)$ s.t. Label-cover(q) is $(\varepsilon, 1)$-dist. is NP-hard.

Unique Games Conj (UGC) $\forall \varepsilon > 0 \ \exists q = q(\varepsilon)$ s.t. Unique-Games(q) is $(\varepsilon, 1-\varepsilon)$-dist. is NP-hard

Integer Programming (I.P.)

CSP(Γ) Let $\mathcal{U}$ be an instance

Variable indicators $\forall v \in V \ \forall \ell \in D \ \mu_v[\ell] \in \{0,1\}$

Condition: $\forall v \in V \ \sum_{\ell \in D} \mu_v[\ell] = 1$.

Constraint indicators $\forall C = (R, S) \in \mathcal{U}$

$\forall L : S \to D \quad \lambda_C[L] \in \{0,1\}$.

$\forall C = (R, S) \quad \sum_{L : S \to D} \lambda_C[L] = 1$.

"Consistency" $\forall C = (R, S) \ \forall v \in S \ \forall \ell \in D$

$\sum \{\lambda_C[L] : L(v) = \ell\} = \mu_v[\ell]$.

Max $\underset{C=(R,S)}{\text{avg}} \sum \{\lambda_C[L] : L(S) \text{ satisfies } R\}$

$(R(L(S)) = 1)$

Example Sat $C_1 = (x \vee y \vee z), (\bar{x} \vee w), \ldots$

$D = \{0,1\} \quad \mu_x[0], \mu_x[1]$

$\mu_y[0], \mu_y[1]$

$\vdots$

$\lambda_{C_1}[x \leftarrow 0, y \leftarrow 0, z \leftarrow 0]$

$\lambda_{C_1}[x \leftarrow 0, y \leftarrow 0, z \leftarrow 1]$,

$\vdots$

$\lambda_{C_1}[x \leftarrow 1, y \leftarrow 1, z \leftarrow 1]$

<u>Linear Programming</u> (L.P.) relaxation for instance $\mathcal{I}$ of CSP(Γ)

L.P. solution $\mathcal{L}$: [illegible]

<u>variable distributions</u>: $\forall v \in V$ μ_v prob. distribution on D.

<u>constraint distributions</u>: $\forall C = (R, S)$ λ_C is prob. dist. on assignments $L : S \to D$

LP Opt($\mathcal{I}$) := maximize $\mathrm{LPVal}(\mathcal{L}) = \underset{C=(R,S)\in\mathcal{I}}{\mathrm{avg}} \left\{ \Pr_{L \sim \lambda_C} [L(S) \text{ satisfies } R] \right\}$

s.t. <u>consistency condition</u>
$\forall C \ \forall v \in S \ \forall \ell \in D \quad \Pr_{L \sim \lambda_C} [L(v) = \ell] = \Pr_{\mu_v} [\ell]$

$\forall \mathcal{I} \quad \mathrm{Opt}(\mathcal{I}) \leq \mathrm{LP\,Opt}(\mathcal{I}) \leq 1$

<u>Example</u>) Max-kSat. $D = \{0, 1\}$ Γ Conj $\leq k$.

"Randomized Rounding"
Pick <u>actual</u> assignment $F : V \to \{0, 1\}$
draw $F(v) \sim \mu_v$ <u>indep.</u> $\forall v \in V$.

<u>Thm</u>) $\mathop{\mathbf{E}}_F [\mathrm{Val}_{\mathcal{I}}(F)] \geq \left(1 - \frac{1}{e}\right) \mathrm{LP\,Opt}(\mathcal{I})$

$\geq \left(1 - \frac{1}{e}\right) \mathrm{Opt}(\mathcal{I})$

<u>Cor</u>) This alg. $\left(\left(1 - \frac{1}{e}\right)\beta, \beta\right)$-approximates Max-Sat $\forall \beta$.