Regular polyhedra through time

The greeks were the first to study the symmetries of polyhedra. Euclid, in his *Elements* showed that there are only five regular solids (that can be seen in Figure 1). In this context, a polyhedron is regular if all its polygons are regular and equal, and you can find the same number of them at each vertex.

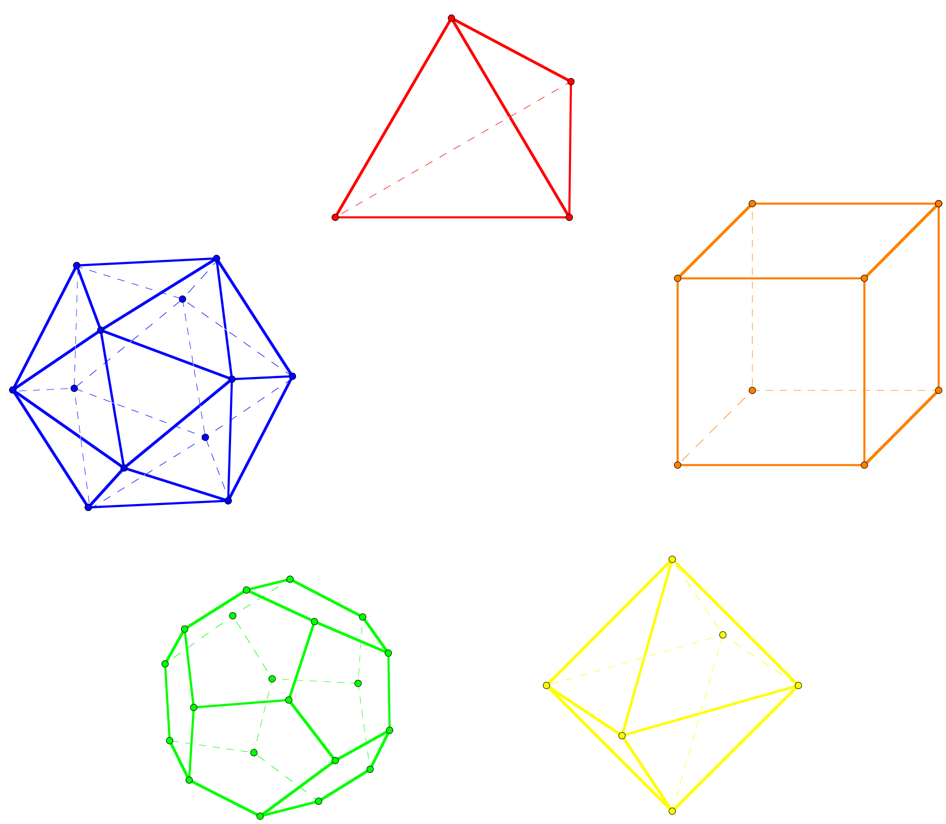

Figure 1: Platonic Solids.

It is until 1619 that Kepler finds other two regular polyhedra: the great dodecahedron and the great icosahedron (on Figure 2. To do so, he allows "false" vertices and intersection of the (convex) faces of the polyhedra at points that are not vertices of the polyhedron, just as the

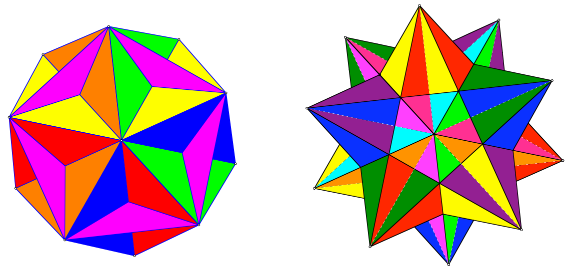

Figure 2: Kepler polyhedra. 1619.

pentagram allows intersection of edges at points that are not vertices of the polygon. In this way, the vertex-figure of these two polyhedra are pentagrams (see Figure 3).

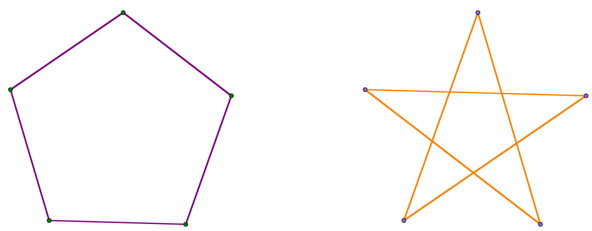

Figure 3: A regular convex pentagon and a pentagram, also regular!

In 1809 Poinsot re-discover Kepler's polyhedra, and discovers its duals: the small stellated dodecahedron and the great stellated dodecahedron (that are shown in Figure 4). The faces of such duals are pentagrams, and are organized on a "convex" way around each vertex.

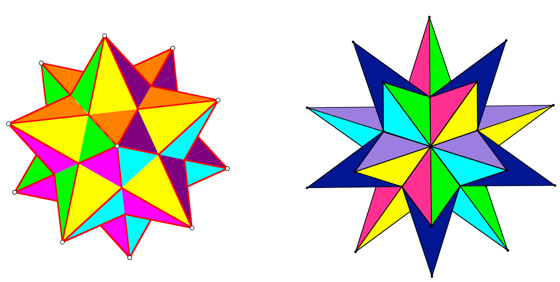

Figure 4: The other two Kepler-Poinsot polyhedra. 1809.

A couple of years later Cauchy showed that these are the only four regular "star" polyhedra. We note that the convex hull of the great dodecahedron, great icosahedron and small stellated dodecahedron is the icosahedron, while the convex hull of the great stellated dodecahedron is the dodecahedron.

In the early 1920's, Petrie realized that regular polyhedra can be constructed organizing the faces going "up and down". He found two infinite such polyhedra (one of each is shown in Figure 7); Coxeter found a third one and showed there were no more.

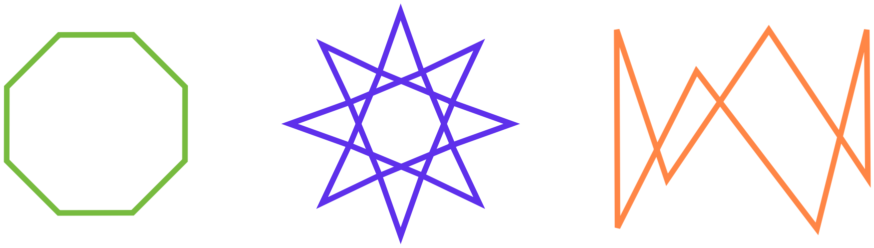

Figure 5: Three geometrically different regular 8-gons.

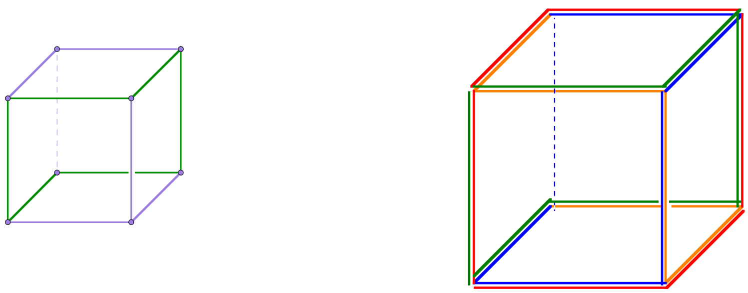

Figure 6: A Petrie polygon of the cube and the Petrial (or Petrie-dual) of the cube.

Grünbaum found 47 regular polyhedra in the seventies. In the early 80's, Danzer found the last one and showed that there are 48 regular polyhedra in Euclidean three space, 18 of which are finite. Note that in Euclidean three space there are four types of regular polygons: convex, star, zig-zag or skew and helicoidal (that are infinite!). Hence a regular polyhedron can have, in principle, faces of any of these types and the arrangement of edges around each vertex can be of any of the finite types.

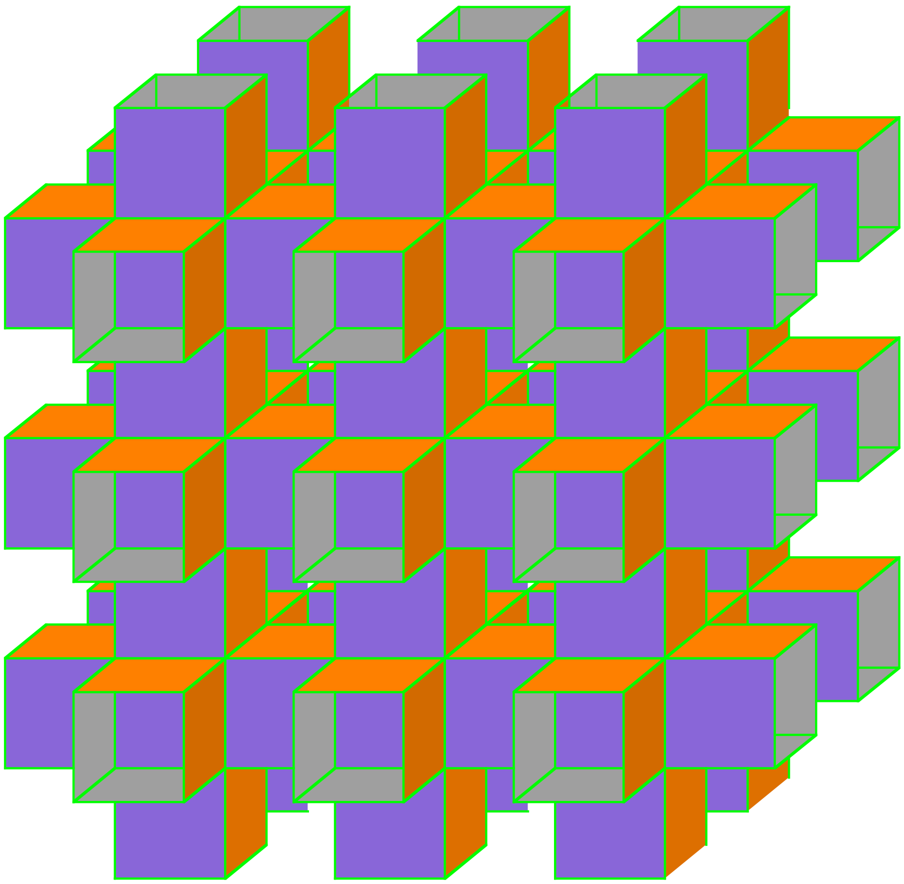

Figure 7: The $\{4, 6|4\}$ (infinite) Petrie-Coxeter polyhedron.

Abstract polytopes

When dealing with abstract polytopes we always try to keep in mind the geometry behind them. Given a "classical" polytope $\mathcal{P}$, we consider it as the set containing all its vertices, edges, 2-faces, ..., facets, together with $\leq$ relation between them, given by the incidence. Such relation is easily seen to be a partial order on all the "faces". Furthermore, since a vertex, say, cannot be in a polygon of the polytope without being in an edge of them, and two vertices cannot be incident to each other, then the maximal chains of the order have one element of each kind. A diagram of how would we see the tetrahedron in this way is shown in Figure 8.

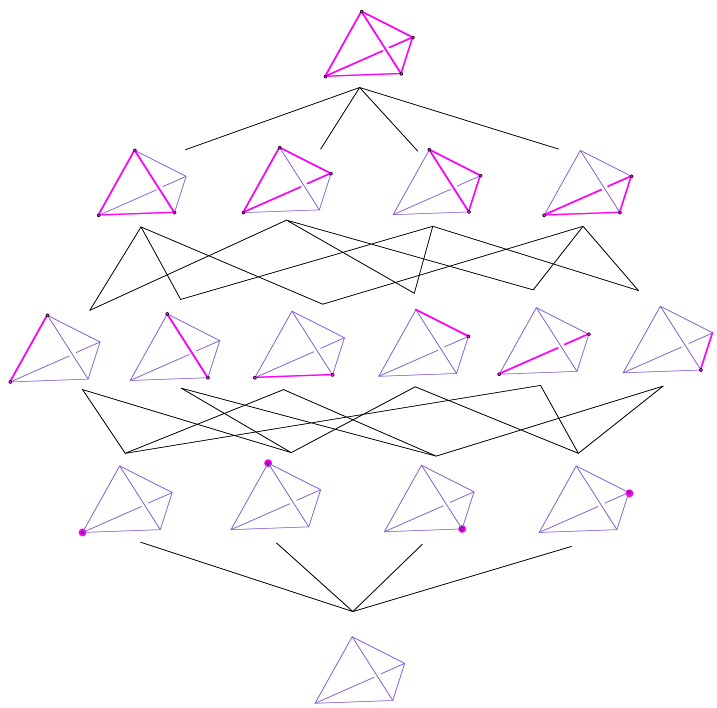

Figure 8: A tetrahedron seen as a poset

An *(abstract) polytope* of rank n or an n*-polytope* is a partially ordered set $\mathcal{P}$ endowed with a strictly monotone rank function having range $\{-1, \ldots, n\}$. For $0 \leq j < n$, the elements of $\mathcal{P}$ of rank j are called j*-faces*, and often a j-face is denoted by F_j. The faces of rank $0, 1$ and $n-1$ are usually called the *vertices*, *edges* and *facets* of the polytope, respectively. We require that $\mathcal{P}$ has a smallest face F_{-1}, and a greatest face F_n (called the *improper faces* of $\mathcal{P}$), and that each maximal chain (called a *flag*) of $\mathcal{P}$ contains exactly $n+2$ faces. We denote by $\mathcal{F}(\mathcal{P})$ the set of all flags of $\mathcal{P}$.

Figure 9 shows a way in which we can see the minimal and maximal face of the tetrahedron and has one flag of it in the partial order. As we shall see later, the minimal and maximal face of a polytope are required for technical reasons. Since a flag is a maximal chain of the order, as well as the minimal and maximal faces, it contains a vertex, an edge, a polygon, etc., in such a way that the vertex is incident to the edge, the edge is incident to the polygon and so on. In Figure 10 a flag of a tetrahedron is displayed in two different geometrical ways, as an incident triplet vertex-edge-face.

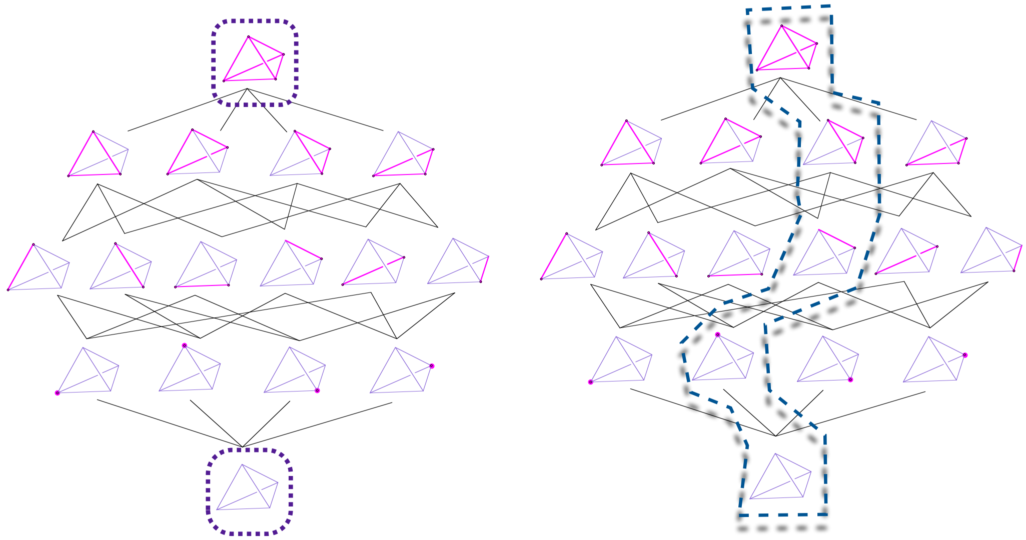

Figure 9: Minimal and maximal faces of a tetrahedron, and a flag of it.

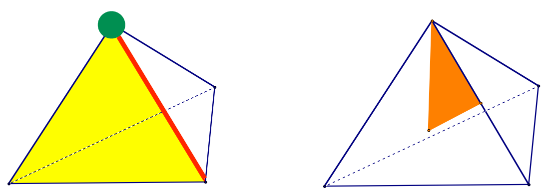

Figure 10: A flag of a tetrahedron, seen in a geometric way

Two flags are said to be *adjacent* if they differ by exactly one face. Figure 11 has a geometric view of a flag and its 0-adjacent, in two different ways, and Figure 12 shows the Hasse diagram (or face lattice) of the tetrahedron emphasizing a flag and its 1-adjacent flag.

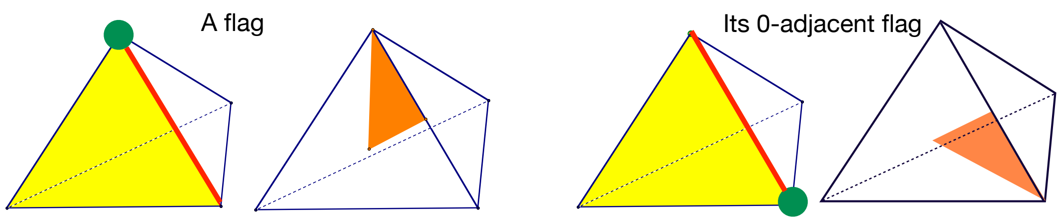

Figure 11: A flag of a tetrahedron and its 0-adjacent.

Also, we require that $\mathcal{P}$ be strongly flag-connected, that is, any two flags $\Phi, \Psi \in \mathcal{F}(\mathcal{P})$ can be joined by a sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$ such that each two successive flags Φ_{i-1} and Φ_i are adjacent, with $\Phi \cap \Psi \subseteq \Phi_i$ for all i. Finally, we require the homogeneity property

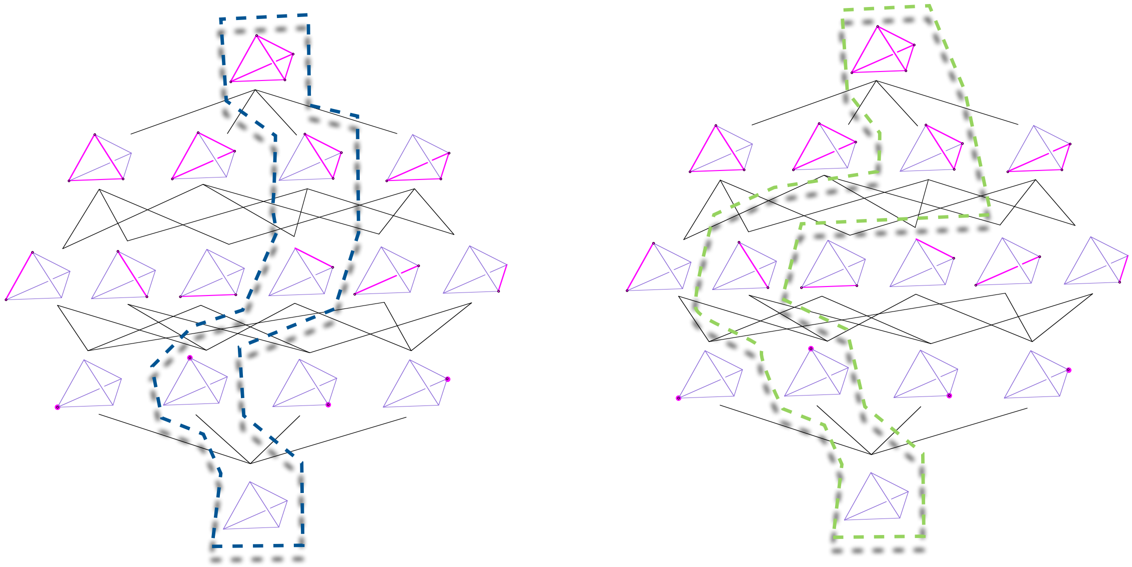

Figure 12: A flag on the Hasse diagram of the tetrahedron and its 1-adjacent one.

(often called the *diamond condition*), that is, whenever $F \leq G$, with $\text{rank}(F) = j-1$ and $\text{rank}(G) = j+1$, there are exactly two faces H of rank j such that $F \leq H \leq G$.

Figure 13 represents the diamond condition on a tetrahedron (for $i = 0, 1, 2$) in a geometric way. Figure 14 shows the same thing in the Hasse diagram.

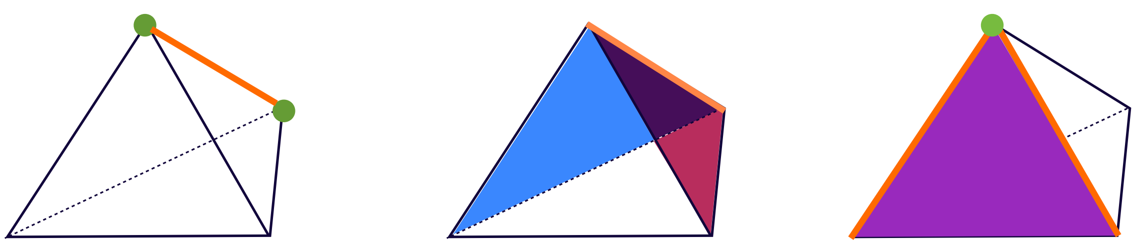

Figure 13: Geometric visualization of the diamond condition of a tetrahedron.

Exercise 1 *Let $\mathcal{P}$ an abstract polytope and consider V and E, it sets of vertices and edges, respectively. Then each $e \in E$ can be seen as a subset of V by identifying e with the set $\{v \in V \mid v \leq e\}$. Therefore $G := (V, E)$ is a hypergraph with vertex set V and hyperedge set E.*

1. *Show that G is a graph.*
2. *Can G have loops?*
3. *What about multiple edges?*

The graph described in Exercise 1 is often refer to as the *1-skeleton* of $\mathcal{P}$.

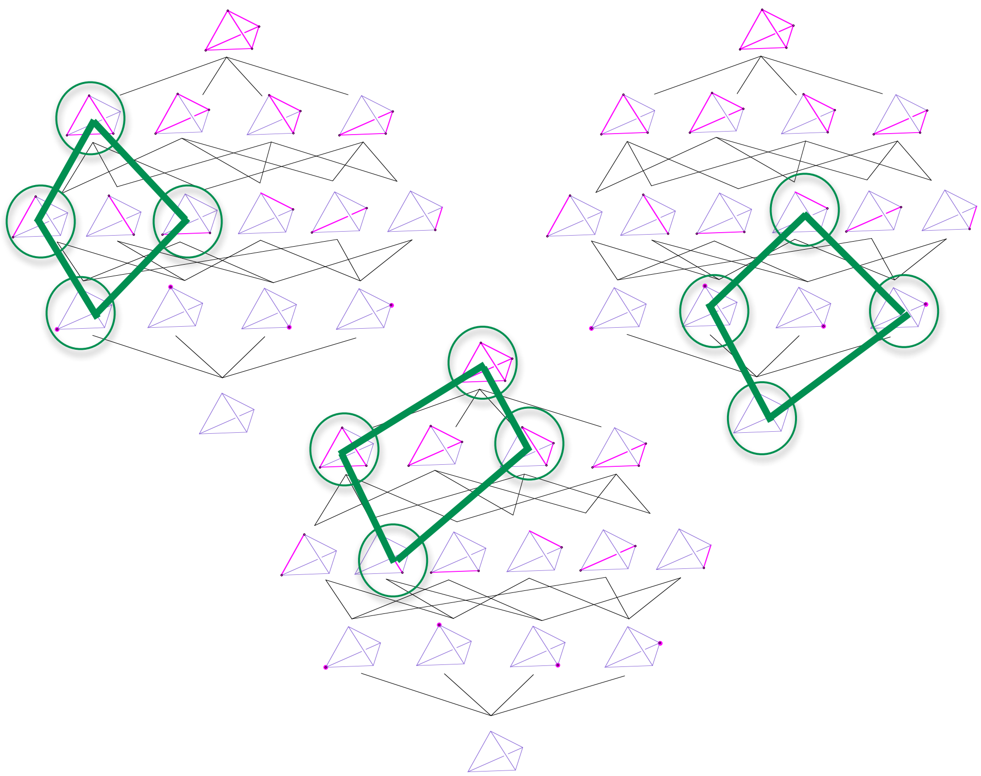

Figure 14: The diamond condition on the Hasse diagram of the tetrahedron

A 0-polytope contains only two (incident) elements, F_{-1} and F_0; hence, up to isomorphism, there is only one 0-polytope, and it can be thought of as a single point or vertex. A 1-polytope must have a diagram with diamond shape (see Figure 15), and we can think of it as an edge with its two end-vertices.

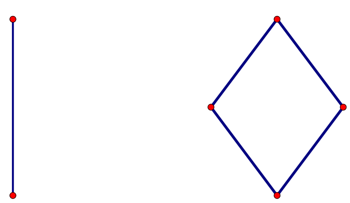

Figure 15: Hasse diagrams of rank 0 and 1 polytopes, respectively.

If $\mathcal{P}$ is a 2-polytope, it is easy to see that the number of vertices and edges of $\mathcal{P}$ are exactly the same. Furthermore, every vertex is incident to exactly two edges and every edge is incident to exactly two vertices (see Figure 16). For this reason, a 2-polytope is called a *polygon*, or if it is finite and has p vertices (and hence also p edges), a *p-gon*. Finally, 3-polytopes are also often called *polyhedra*.

Note that every two *geometric* p-gons are combinatorially equivalent. For example, a convex pentagon and a pentagram (in Figure 3) are different representations of the same abstract polytope, the 5-gon (in Figure 16).

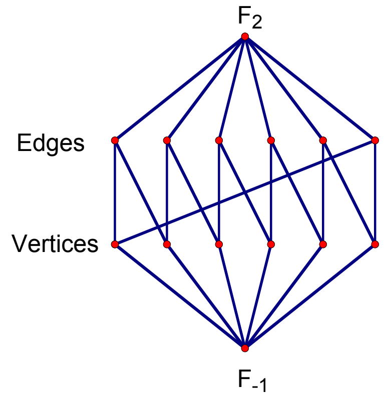

Figure 16: The Hasse diagram of a 5-gon

Exercise 2 *Can a 0-, 1- or 2-polytope be infinite? If not, explain why. If so, describe such infinite polytopes, geometrically and their Hasse diagram.*

Exercise 3 *Which of the geometric structures in Figure 17 are abstract polytopes?*

Exercise 4 *Choose one of the structures of Figure 17 that satisfies to be a polytope and draw its Hasse diagram.*

Let $\mathcal{P}$ be an n-polytope and Φ be a flag of $\mathcal{P}$. The diamond condition tells us that for $i = 0, \ldots, n-1$ there is exactly one flag that differs from Φ in the i-face. Such a flag is called the *i-adjacent* flag to Φ and it is denoted by Φ^i. Furthermore, we define $\Phi^{i,j} := (\Phi^i)^j$ and extend such notation by induction. We shall denote by $(\Phi)_i$ the i-face of the flag Φ. For convenience, we often omit the improper faces when describing a flag, thus, a flag Φ can be denoted as $\{(\Phi)_0, (\Phi)_1, \ldots, (\Phi)_{n-1}\}$. Two i-faces of $\mathcal{P}$, F and F', are said to be *adjacent* if there exists a flag Φ such that $(\Phi)_i = F$ and $(\Phi^i)_i = F'$.

Proposition 1 *For $i, j \in \{0, 1, ..., n-1\}$,*

1. $(\Phi^i)^i = \Phi$.
2. *if* $|i-j| > 1$, $\Phi^{i,j} = \Phi^{j,i}$.
3. $(\Phi)_i = (\Phi^j)_i$ *if and only if* $i \neq j$.

Exercise 5 *Similarly as in Exercise 1, given an abstract polytope $\mathcal{P}$, for each $i \in \{1, \ldots, n-1\}$ one can consider $\mathcal{F}_i$, $\mathcal{F}_{i+1}$, the sets of i- and $i+1$-faces, respectively, and identify every $g \in \mathcal{F}_{i+1}$ with the subset $\{f \in \mathcal{F}_i \mid f \leq g\}$ of $\mathcal{F}_i$. Is $\mathcal{H} := (\mathcal{F}_i, \mathcal{F}_{i+1})$ a graph?*

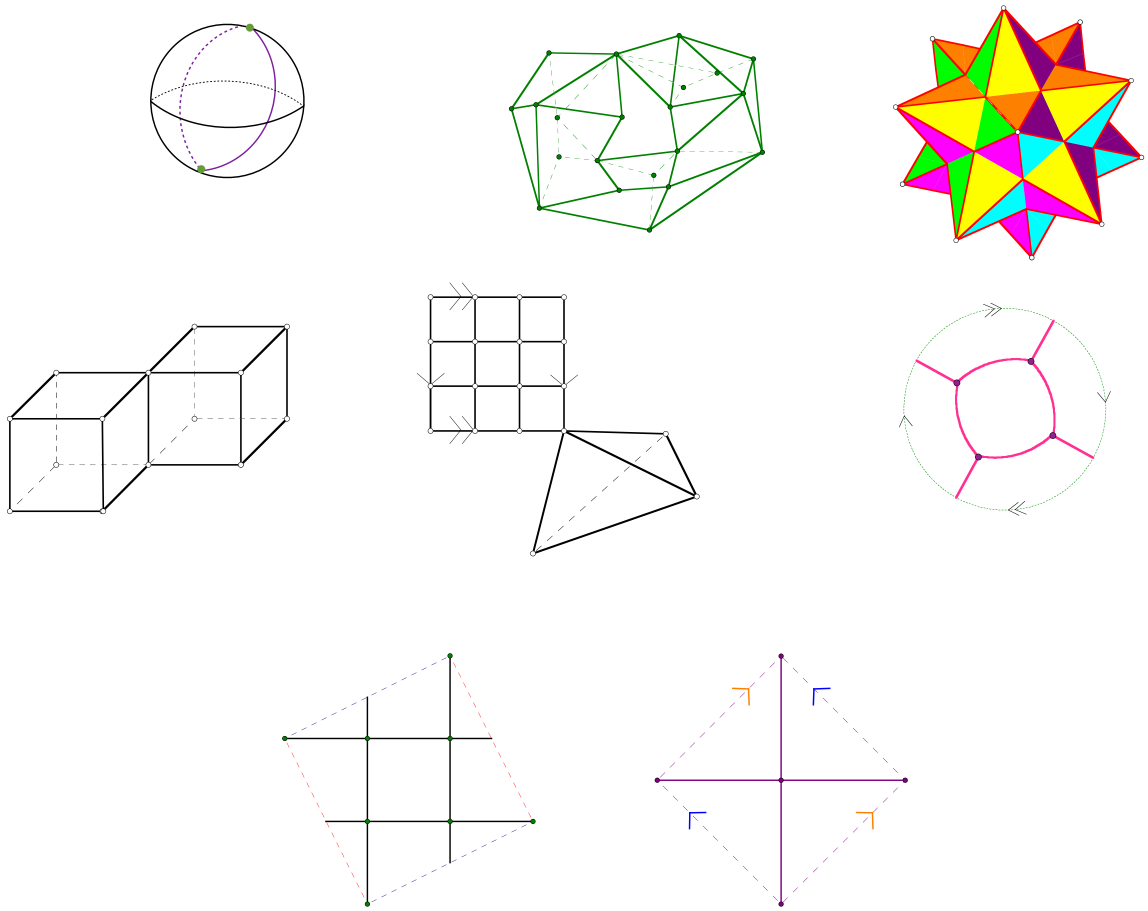

Figure 17: Are they polytopes?

Given two faces F and G of a polytope $\mathcal{P}$ such that $F \leq G$, the *section* G/F of $\mathcal{P}$ is the set of faces $\{H | F \leq H \leq G\}$. If F_0 is a vertex, then the section F_n/F_0 is called the *vertex-figure* of F_0.

Exercise 6 *Consider all the geometric structures that are polytopes of Figure 17. For each of them, pick a vertex and describe the vertex figure geometrically. For the polytope that you chose in Exersice 4, identify the vertex figure in the Hasse diagram.*

Exercise 7 *Show that every section G/F of a polytope $\mathcal{P}$ (where $F, G \in \mathcal{P}$ such that $F \leq G$) is an abstract polytope of rank* $\text{rank}(G/F) = \text{rank}(G) - \text{rank}(F) - 1$.

Exercise 8 *Describe the sections of rank* 1 *of an abstract polytope* $\mathcal{P}$ *of rank* n.

An n-polytope $\mathcal{P}$, $n \geq 2$, is said to be *equivelar* if, for each $j = 1, \ldots, n-1$, there exists an integer p_j, such that, for each flag $\Phi \in \mathcal{F}(\mathcal{P})$, the section $(\Phi)_{j+1}/(\Phi)_{j-2}$ is a p_j-gon. In this case, we say that $\mathcal{P}$ has *Schläfli type* (or sometimes only *type*) $\{p_1, p_2, \ldots p_{n-1}\}$.

All 2-polytopes are equivelar; furthermore, a p-gon has Schläfli type $\{p\}$. We note that an infinite 2-polytope or *aperiogon* has Schläfli type $\{\infty\}$.

Exercise 9 *Show that every two 2-polytopes with the same Schläfli type are isomorphic abstract polytopes.*

Exercise 10 *If we take a cube and identify opposite vertices, edges and faces, we obtain a* hemi-cube. *The hemi-cube can be represented in the projective plane as in Figure 18. What is the Schläfli type of the hemi-cube? Can you find an abstract polytopes that is not isomorphic to the hemi-cube, but has the same Schläfli type?*

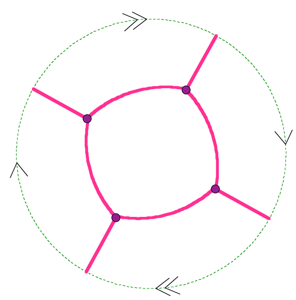

Figure 18: A hemi-cube.

A polyhedron is equivelar of Schläfli type $\{p, q\}$ if and only if all its facets are p-gons and all its vertex-figures are q-gons.

Exercise 11 *Find the Schläfli type of each of the Platonic Solids.*

An example of a 4-polytope is a hypercube or 4-cube. It has cubes as facets, three of them around each edge (see Figure 19) implying that its vertex-figures are tetrahedra. Hence, its Schläfli type is $\{4, 3, 3\}$.

Alternatively, the Schläfli type of an equivelar polytope can be defined as follows. For rank 2, we say that the Schläfli type of a p-gon is $\{p\}$. For higher rank, an $n-$polytope $\mathcal{P}$ is said to have Schläfli type $\{p_1, p_2, \ldots, p_{n-2}\}$ if all its facets have Schläfli type $\{p_1, p_2, \ldots, p_{n-1}\}$ and all its vertex-figures have Schläfli type $\{p_2, p_3, \ldots, p_{n-1}\}$.

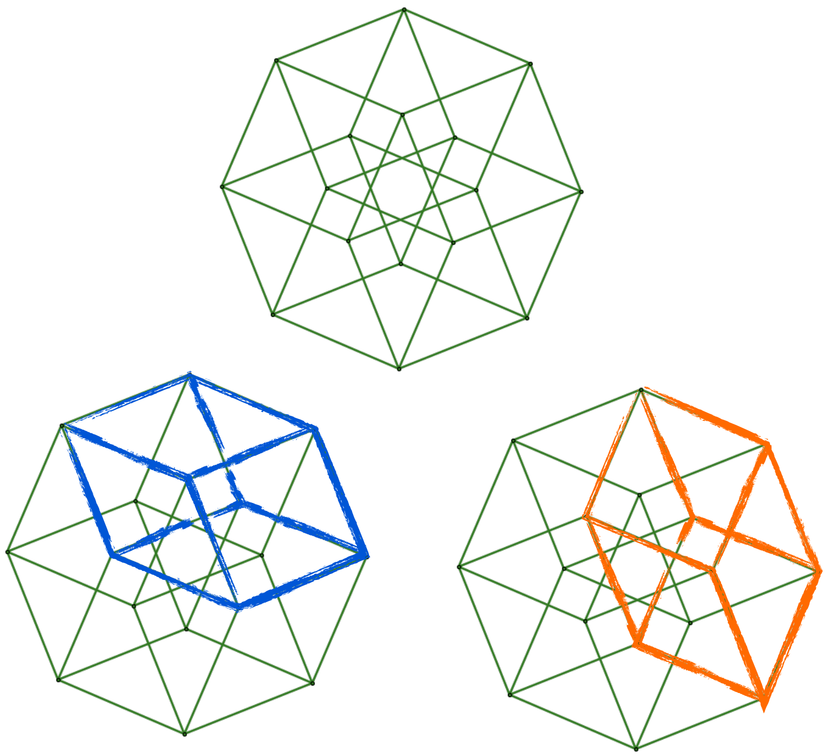

Figure 19: A hyperbuce

We now reexamine the third condition in our definition of a polytope. A poset $\mathcal{P}$ is said to be *connected* if for any two proper faces (elements) F and G of $\mathcal{P}$, there is a sequence of proper faces $F = F^0, F^1, \ldots, F^k = G$ such that F^i and F^{i+1} are *incident* (i.e $F^i \leq F^{i+1}$ or $F^{i+1} \leq F^i$), for every $i = 0, \ldots, k-1$. A poset $\mathcal{P}$ is said to be *strongly connected* if every section of $\mathcal{P}$, including itself, is connected.

Proposition 2 *A poset $\mathcal{P}$ with a strictly monotone rank function having range $\{-1, \ldots, n\}$, a smallest (-1)-face F_{-1}, a greatest n-face F_n and such that each flag of $\mathcal{P}$ contains exactly $n+2$ faces is strongly connected if and only if it is strongly flag-connected.*

Petrie polygons, holes and zig-zags

A *Petrie polygon* of an n-polytope $\mathcal{P}$ is a path among the edges of $\mathcal{P}$ which has exactly k edges in the same k-face, for each $k = 2, 3, \ldots, n-1$. In particular, for 3-polytopes, a Petrie polygon has exactly two edges on a face but any third must belong to a differtent face. (See Figure 20 for an example).

For a 3-polytope $\mathcal{P}$, we can define the *Petrial* of $\mathcal{P}$ as a polyhedron that has the same vertices and edges than $\mathcal{P}$, but its faces are the Petrie polygons of $\mathcal{P}$. The Petrial of $\mathcal{P}$ is denoted by $\Pi(\mathcal{P})$. Note that $\Pi(\Pi(\mathcal{P})) \cong \mathcal{P}$. The Petrial of a polyhedron need not to be a polytope, since

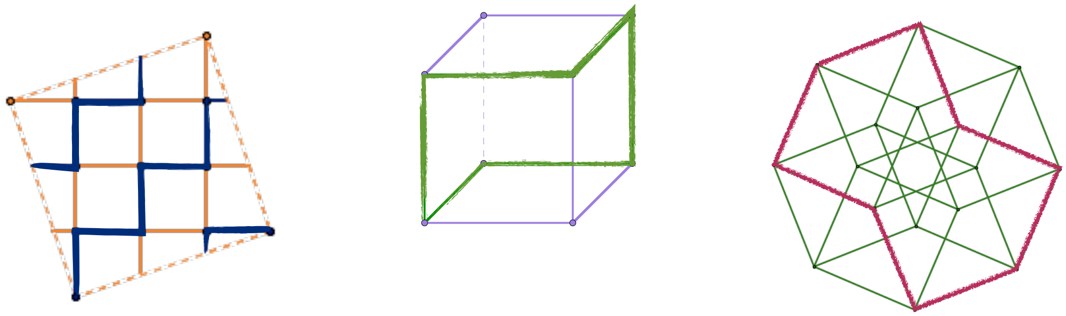

Figure 20: Petrie polygons of a 3-polytope on a torus, of a cube and of a hypercube, respectively.

the diamond condition might not be satisfied.

Exercise 12 *Consider all the geometric structures that are polytopes of Figure 17. For each of them, find its Petrial and decide if it is a polytope or not. Do the same with the Platonic Solids.*

We say that the *right (left) Petrie motion* of a polyhedron $\mathcal{P}$ is the path that takes a base flag Φ of $\mathcal{P}$ to $\Phi^{2,1,0}$ ($\Phi^{1,0,2}$). Accordingly, we can define the *right (left) Petrie polygons* of an n-polytope. Note that in general the right and left Petrie polygons of an n-polytopes need not have the same length.

Exercise 13 *Find examples of polytopes such that their right and left Petrie polygons have the same length. Find examples of polytopes such that their right and left Petrie polygons do not have the same length.*

A *j-hole* of a 3-polytope $\mathcal{P}$ is a path among the edges of the 1-skeleton of $\mathcal{P}$ which leaves a vertex by the j-th edge which it entered, in the same sense (that is, keeping always the right, say, in some local orientation). A *j-zig-zag* of $\mathcal{P}$ is a path among the edges of the 1-skeleton of $\mathcal{P}$ which leaves a vertex by the j-th edge which it entered, changing the sense (say, right, left, right, left, in some local orientation). Figure 21 shows two 2-holes and two 2-zig-zags of the icosahedron.

Exercise 14 *Find all the 2- and 3-holes and all the 2- and 3-zig-zags of all the polytopes considered in Exercise 12.*

Exercise 15 *What are the 1-holes of a 3-polytope? And the 1-zig-zags?*

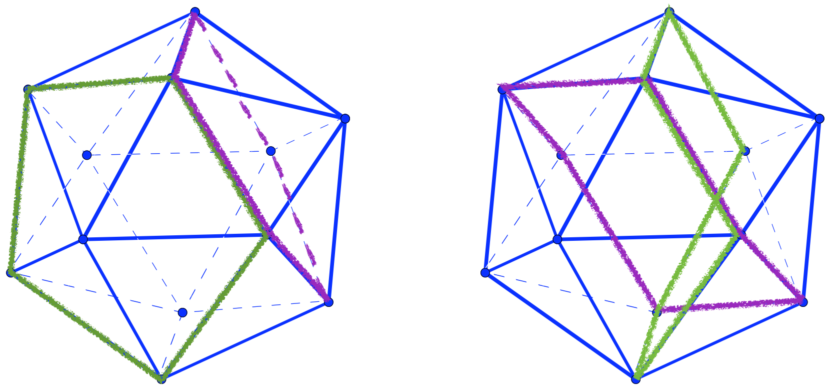

Figure 21: An icosahedron and some 2-holes and 2-zig-zags of it, respectively.

In a similar way as for the Petrie polygons, one can construct a new poset of rank 3 using the j-holes or the j-zig-zags of a 3-polygon. That is, we consider the 1-sketeton of the new structure to be the same as that of $\mathcal{P}$, and define the faces to be the j-holes (or j-sig-zags) of $\mathcal{P}$.

Exercise 16 *Use Exercise 14 and find the structures describes above for each 3-polytope. Determine which of them is again a polytope.*

Symmetries of polytopes

Let $\mathcal{P}$ be an n-polytopes. An *automorphism* of $\mathcal{P}$ is a bijection $\gamma : \mathcal{P} \to \mathcal{P}$ such that γ and γ^{-1} preserve the order. It is straightforward to see that the set of all automorphisms of a polytope $\mathcal{P}$ is a group, the automorphism group $\Gamma(\mathcal{P})$ of $\mathcal{P}$.

Exercise 17 *Find all the automorphisms of a p-gon, with $p \in \{2, 3, \ldots\}$.*

Exercise 18 *Describe all the automorphisms of a tetrahedron as permutations of its four vertices. Thinking now the tetrahedron as a geometric object, divide the automorphisms the you just described in:* a) *the rotations,* b) *the reflections,* c) *the rotatory reflections.*

One can think of the automorphisms of a polytope $\mathcal{P}$ acting on different sets of $\mathcal{P}$. For example, one can consider the action of the automorphism group on the flags, vertices, edges, etc, of $\mathcal{P}$.

Lemma 3 *Let $\mathcal{P}$ be an n-polytope, let $\gamma \in \Gamma(\mathcal{P})$ and let Φ be a flag of $\mathcal{P}$. Then,*

$$(\Phi^i)\gamma = (\Phi\gamma)^i \quad \textit{and} \quad (\Phi)_i\gamma = (\Phi\gamma)_i,$$

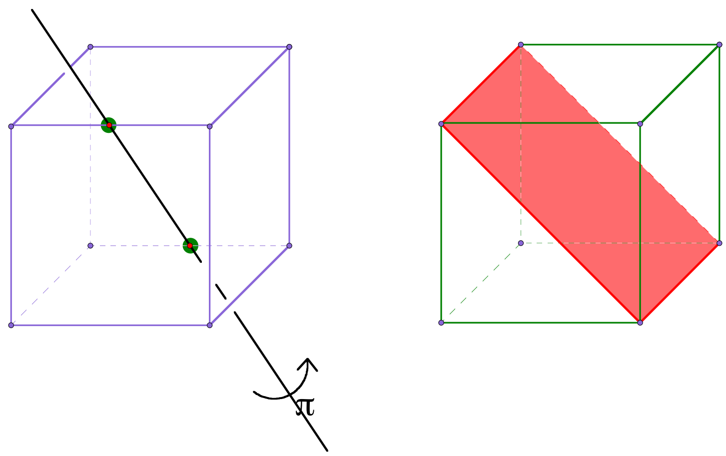

Figure 22: Two symmetries of the cube seen geometrically.

for every $i = 0, 1, \ldots, n-1$.

Corollary 4 *Let* $\mathcal{P}$ *be an* n*-polytope, then* $\Gamma(\mathcal{P})$ *acts freely on* $\mathcal{F}(\mathcal{P})$*, that is, every automorphism of* $\mathcal{P}$ *that fixes one flag is the identity.*

Lemma 5 *Let* $\mathcal{P}$ *be an* n*-polytope, and let us denote by* $\mathrm{Orb}(\mathcal{P})$ *the set of all flag orbits of* $\mathcal{P}$ *under the action of* $\Gamma(\mathcal{P})$*. Let* $\mathcal{O}_1, \mathcal{O}_2 \in \mathrm{Orb}(\mathcal{P})$ *and* $\Phi \in \mathcal{O}_1$*. If for some* $i \in \{0, \ldots n-1\}$*,* $\Phi^i \in \mathcal{O}_2$*, then for any* $\Psi \in \mathcal{O}_1$*,* $\Psi^i \in \mathcal{O}_2$*.*

If the action of $\Gamma(\mathcal{P})$ has k orbits on the flags, we shall say that $\mathcal{P}$ is a *k-orbit polytope*. Since $\Gamma(\mathcal{P})$ acts freely on $\mathcal{F}(\mathcal{P})$, for each $\mathcal{O} \in \mathrm{Orb}(\mathcal{P})$ and each $\Phi \in \mathcal{O}$, there exists a bijection $\phi : \Gamma(\mathcal{P}) \to \mathcal{O}$, $\phi : \gamma \mapsto \Phi\gamma$. Therefore there exists a bijection between every two flag orbits of $\mathcal{P}$. In particular, if $\mathcal{P}$ is finite, $|\Gamma(\mathcal{P})| = |\mathcal{O}| = \frac{|\mathcal{F}(\mathcal{P})|}{|\mathrm{Orb}(\mathcal{P})|}$. Hence, $|\Gamma(\mathcal{P})| \leq |\mathcal{F}(\mathcal{P})|$.

We say that an n-polytope $\mathcal{P}$ is *i-face transitive*, for some $i \in \{0, \ldots, n-1\}$ if $\Gamma(\mathcal{P})$ acts transitively on the faces of rank i. We say that $\mathcal{P}$ is *fully-transitive* if it is i-face transitive for all $i \in \{0, \ldots, n-1\}$ if $\Gamma(\mathcal{P})$.

Exercise 19 *Let* $\mathcal{P}$ *be a 3-polytope such that its Petrial* $\Pi(\mathcal{P})$ *is also a 3-polytope.*

1. *What can you say about the automorphism group of* $\Pi(\mathcal{P})$ *in terms of the automorphism group of* $\mathcal{P}$*?*
2. *If* $\mathcal{P}$ *is a* k*-orbit polytope, how many orbits on flags would* $\Pi(\mathcal{P})$ *have?*

3. *Show that $\mathcal{P}$ is 0-face transitive if and only if $\Pi(\mathcal{P})$ is 0-face transitive. Show that $\mathcal{P}$ is 1-face transitive if and only if $\Pi(\mathcal{P})$ is 1-face transitive. Is the same statement true for 2-face transitive? (Give a proof or a counterexample.)*

Exercise 20 *a) Mark the vertices of an octahedron $1, 2, \ldots, 6$. List all the rotations of the octahedron by the permutations they induce on the vertices. What are their orders?*

b) The octahedron has four axes, a, b, c, d running through the centres of opposite faces. Any rotation induces a permutation of a, b, c, d. Thus we get a map $\psi : R \to S_4$ from the set of rotations to the symmetric group on the four letters a, b, c, d. Show that R has at least 24 elemets, show that the map ψ is injective, and conclude that R is a group isomorphic to S_4.

c) Find subgroups of the group of rotations of the octahedron isomorphic to $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{D}_2, \mathcal{D}_3 = S_3, \mathcal{D}_4$, and describe them in terms of the geometry of the octahedron. (Where $\mathcal{D}_n$ denotes the dihedral group *of order $2n$, that is, the group of symmetries of a regular n-gon, and $\mathcal{C}_n$ denotes the* cyclic group *of order n, that is also the rotational subgroup of a regular n-gon.)*

d) Show that the group of all symmetries of the octahedron is a group of order 48.

e) Conclude that the automorphism group of the octahedron acts transitively on its flags.

Let $\mathcal{P}$ and $\mathcal{Q}$ be two n-polytopes. If there exists a bijection $\delta : \mathcal{P} \to \mathcal{Q}$ reversing the order, we say that $\mathcal{P}$ and $\mathcal{Q}$ are *duals* of each other, and the usual convention is to denote $\mathcal{Q}$ by $\mathcal{P}^*$. (Note that $(\mathcal{P}^*)^* \cong \mathcal{P}$.) For example, the cube and the octahedron are dual to each other.

Proposition 6 *Let $\mathcal{P}$ and $\mathcal{Q}$ be two dual n-polytopes. Then $\Gamma(\mathcal{P}) \cong \Gamma(\mathcal{Q})$.*

Exercise 21 *If $\mathcal{P}$ is a k-orbit polytope and $\mathcal{Q}$ is its dual, how many orbits on flags does $\Gamma(\mathcal{Q})$ have?*

If there exists a bijection δ that reverses the order from $\mathcal{P}$ onto itself, then $\mathcal{P}$ is called a *self-dual* polytope, and δ is a *duality of $\mathcal{P}$* (sometimes also *self-duality*). The set of all automorphisms and dualities of a polytope $\mathcal{P}$ forms a group, the *extended group $\mathcal{D}(\mathcal{P})$ of $\mathcal{P}$*, which contains $\Gamma(\mathcal{P})$, the subgroup of all automorphisms of $\mathcal{P}$, as a subgroup of index at most 2. When a polytope $\mathcal{P}$ is not self-dual, then its extended group coincides with its automorphism group.

Lemma 7 *If $\mathcal{P}$ is a self-dual n-polytope, Φ a flag and δ is a duality of $\mathcal{P}$ then*

$$(\Phi^i)\delta = (\Phi\delta)^{n-1-i} \quad \text{and} \quad (\Phi)_i\delta = (\Phi\delta)_{n-1-i},$$

for every $i = 0, 1, \ldots, n-1$.

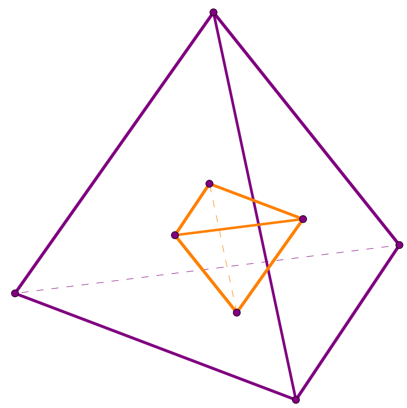

Figure 23: The tetrahedron is self-dual.

Regular polytopes

We started this course with a brief historical introduction to regular polytopes. We now formally define the concept of regularity. An (abstract) polytope is said to be *(combinatorially) regular* if its automorphism group acts transitively on the flags of $\mathcal{P}$.

Every regular convex polytope is a regular abstract polytope. The Kepler-Poinson polyhedra and the Petrie-Coxeter polyhedra are regular abstract polytopes.

Proposition 8 *Let $\mathcal{P}$ be a regular polytope, let $\Phi, \Psi \in \mathcal{F}(\mathcal{P})$ and $i, j \in \{0, \dots, n-1\}$ with $i < j$. Then, the sections $(\Phi)_j/(\Phi)_i$ and $(\Psi)_j/(\Psi)_i$ are isomorphic and regular.*

Corollary 9 *Every regular polytope is equivelar.*

Exercise 22 *Show that all* 0-, 1- *and* 2-*polytopes are regular.*

Exercise 23 *Consider the following definition: Every* 2-*polytope is regular; a n-polytope ($n \geq 3$) is regular if its facets and vertex figures are regular and isomorphic. Is this definition equivalent to our concept of regularity?*

Let $\mathcal{P}$ be an regular n-polytope of Schläfli type $\{p_1, p_2, \dots p_{n-1}\}$ and let Φ be a base flag of $\mathcal{P}$. Since $\Gamma(\mathcal{P})$ is transitive on the flags, for every $i = 0, 1, \dots, n-1$, there exists $\rho_i \in \Gamma(\mathcal{P})$ such that $\Phi\rho_i = \Phi^i$.

Then,

$$\Phi\rho_i^2 = (\Phi^i)\rho_i = (\Phi\rho_i)^i = \Phi^{i,i} = \Phi;$$

also, if $|i - j| > 1$, then

$$\Phi\rho_i\rho_j = (\Phi^i)\rho_j = (\Phi\rho_j)^i = \Phi^{j,i} = \Phi^{i,j} = \Phi\rho_j\rho_i.$$

Since $\Gamma(\mathcal{P})$ acts freely on $\mathcal{F}(\mathcal{P})$, this implies that

$$\rho_i^2 = (\rho_i\rho_j)^2 = \epsilon, \quad \text{if } |i - j| > 1. \tag{1}$$

Lemma 10 *Suppose that $\mathcal{P}$ is a polytope such that for some base flag Φ there exist automorphisms ρ_i such that $\Phi\rho_i = \Phi^i$ for every $i \in \{0, \ldots, n-1\}$. Then $\mathcal{P}$ is a regular polytope. Furthermore, the automorphisms ρ_i are unique and they generate the automorphism group $\Gamma(\mathcal{P})$.*

We shall refer to $\rho_0, \rho_1, \ldots, \rho_{n-1}$ as the *distinguished generators* of $\mathcal{P}$ with respect to Φ.

Toroids

Before we continue the study of regular polytopes, we shall analyze one family of examples in detail.

We define a *tessellation* of Euclidean n-space $\mathbb{E}^n$ as a collection $\mathcal{U}$ of n-dimensional (not self-intersecting) polytopes, called *cells*, which cover $\mathbb{E}^n$ and tile it in a face-to-face manner. That is, these cells cover $\mathbb{E}^n$ and if two cells of $\mathcal{U}$ have non-empty intersection, then they have disjoint interiors and meet in a common face of each. We shall only consider Euclidean tessellations with convex isomorphic cells which are regular polytopes.

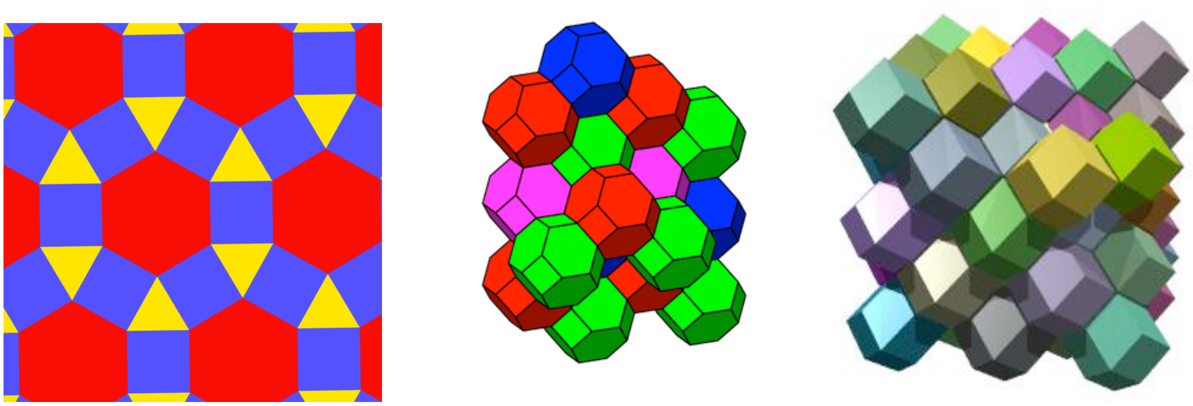

Figure 24: A tessellation of the plane and two tessellations of the Euclidean 3-space.

Exercise 24 *Show that an Euclidean tessellation is an abstract polytope.*

In this section, particular attention will be given to regular tessellations. Since a tessellation is an abstract polytope, if $\mathcal{U}$ is a regular Euclidean tessellation, and Φ is a base flag of $\mathcal{U}$, then $\Gamma(\mathcal{U})$ is a Coxeter group, generated by reflections $R_0, \dots, R_n$, where R_i maps the base flag Φ to its i-adjacent Φ^i. Note that given a base flag, the reflections $R_0, \dots, R_n$ are unique.

For each $n \geq 2$, there is a regular tessellation by n-cubes with Schläfli type $\{4, 3^{n-2}, 4\}$. In the plane, there are also triangular and hexagonal tessellations $\{3, 6\}$ and $\{6, 3\}$, respectively. For $n = 4$, there are two exceptional tessellations $\{3, 3, 4, 3\}$ and its dual $\{3, 4, 3, 3\}$, with 4-cross-polytopes and 24-cells as cells, respectively.

We denote by $\mathbf{T} \leq \mathbf{\Gamma}(\mathcal{U})$ the group of all translations of the Euclidean tessellation $\mathcal{U}$. We identify this group with the orbit of the origin o of $\mathbb{E}^n$ under $\mathbf{T}$ and note that the set of these points is an integer vector lattice. Each rank n subgroup $\mathbf{\Lambda}$ of $\mathbf{T}$ generated by n linearly independent translations $t_1, \dots t_n$ yields a lattice $\Lambda := o\mathbf{\Lambda}$, and the corresponding translation vectors $v_1, \dots v_n$ determine a fundamental region for $\mathbf{\Lambda}$, that is the parallelepiped $P(\{v_1, \dots, v_n\})$.

A *toroid of rank n+1* or an *(n+1)-toroid* is the quotient of a Euclidean tessellation $\mathcal{U}$ over a rank n subgroup $\mathbf{\Lambda} \leq \mathbf{T}$, or using the identification presented above a lattice Λ, and it is denoted by $\mathcal{U}/\Lambda$. It is natural to define an *i-face* of a toroid $\mathcal{U}/\Lambda$ to be an orbit of an i-face of $\mathcal{U}$ and a *flag* of $\mathcal{U}/\Lambda$ as an orbit of a flag of $\mathcal{U}$.

Exercise 25 *Show that the toroids for which all the vertices of each cell of $\mathcal{U}$ are different under $\mathbf{\Lambda}$, with the induced partial order, are abstract $(n+1)$-polytopes.*

We now turn our attention to the regular $\{4, 3^{n-2}, 4\}$ tessellations. The vertex set of the regular tessellation $\mathcal{U} = \{4, 3^{n-2}, 4\}$ may be taken to be $\mathbb{Z}^n$, the set of points in $\mathbb{E}^n$ with integer cartesian coordinates. The translation subgroup $\mathbf{T}$ of $\Gamma(\mathcal{U})$ is then generated by the translations on each of the vectors of the canonical base $e_1, \dots e_n$. Every sublattice Λ can be described by n generating translations $t_1, \dots, t_n$ with vectors $v_1, \dots v_n$, respectively. The corresponding toroid $\mathcal{U}/\Lambda$ is denoted by $\{4, 3^{n-2}, 4\}_{v_1, \dots, v_n}$.

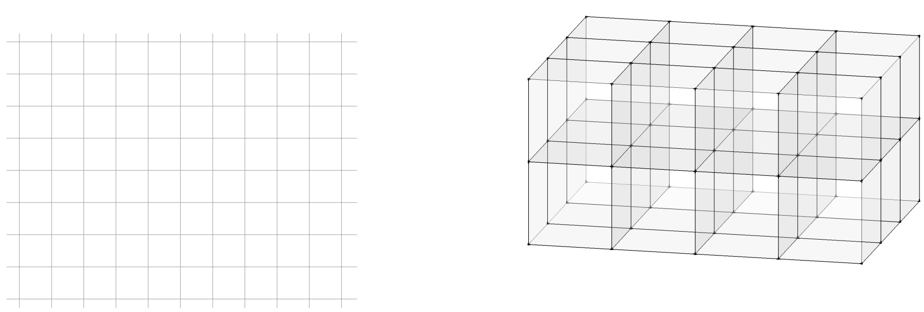

Figure 25: Square and cubical tessellations.

Note that since the translation subgroup $\mathbf{T}$ of $\mathcal{U}$ is vertex-transtive and $\mathbf{S} \cap \mathbf{T} = \{Id\}$, where $\mathbf{S}$ is the stabilizer of the origin o in $\Gamma(\mathcal{U})$, then $\Gamma(\mathcal{U}) = \mathbf{T} \rtimes \mathbf{S}$. Furthermore, a symmetry $\gamma \in \Gamma(\mathcal{U})$

projects to an automorphism of a toroid $\mathcal{U}/\Lambda$ if and only if γ fixes Λ as a lattice. Hence, to see which symmetries of $\mathcal{U}$ project to toroids it suffices to consider the symmetries in the stabilizer of o that fix the sublattice Λ.

Regular toroids of type $\{4, 4\}$

We shall now classify all regular toroids arising from the $\{4, 4\}$ tessellation of the Euclidean plane by squares. For this, we let $\mathcal{U}$ be the square tessellation $\{4, 4\}$ of the Euclidean plane, and Λ a sublattice of the integer lattice. We start by setting the base flag Φ to have vertex at the origin $o = (0, 0)$, edge from $(0, 0)$ to $(0, 1)$ and 2-face $(0, 0), (0, 1), (1, 0), (1, 1)$. If we want a $\mathcal{P} = \mathcal{U}/\Lambda$ to be regular, then there should be automorphisms of $\mathcal{P}$ sending Φ to its adjacent flags. That is, Λ is such that R_1 and R_2 fix it.

It is not difficult to see that for any point (x, y) the reflections R_1 and R_2 send it to (y, x) and $(x, -y)$, respectively. hen, $R_1R_2 \in \mathbf{S}$ is the rotation for $\frac{\pi}{2}$ around o.

Assuming $\lambda \in \Lambda \setminus \{o\}$ is such that $d(\lambda, o) \leq d(\lambda', o)$ for any $\lambda' \in \Lambda \setminus \{o\}$, the orbit of λ under $\langle R_1R_2\rangle$ is a set of four points forming the vertices of a square. Clearly, there can not be other points in Λ at the distance $d(\lambda, o)$ as the distance between any such point and $\lambda\langle R_1R_2\rangle$ would be smaller than $d(\lambda, o)$. In addition, as $R_1, R_2 \in \mathbf{S}$, these four points must be either on the coordinate axes x and y or on the lines $y = x$ and $y = -x$, yielding the two well known possible cases of regular toroids $\{4, 4\}_{(a,0),(0,a)}$ and $\{4, 4\}_{(a,a),(a,-a)}$, respectively. These two toroids are often denoted by $\{4, 4\}_{(a,0)}$ and $\{4, 4\}_{(a,a)}$, respectively.

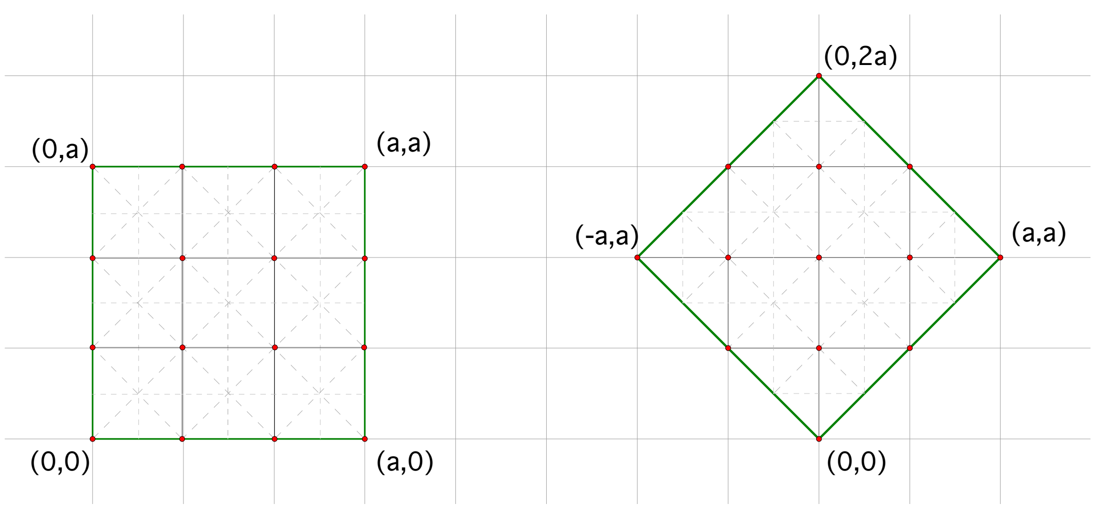

Figure 26: Regular 3-todoids of type $\{4, 4\}$.

C-groups

In this section, unless otherwise stated we let $\mathcal{P}$ be a regular n-polytope, Φ be a base flag of $\mathcal{P}$ and let $\rho_0, \rho_1, \dots, \rho_{n-1}$ be the distinguished generators of $\mathcal{P}$ with respect to Φ.

Lemma 11 *Let j and k such that $-1 \leq j \leq k \leq n$, and consider the section $\mathcal{Q} := (\Phi)_k/(\Phi)_j$ of $\mathcal{P}$. Then $\mathcal{Q}$ is regular and*

$$\Gamma(\mathcal{Q}) \cong \langle \rho_{j+1}, \rho_{j+2}, \dots, \rho_{k-1} \rangle.$$

This lemma implies that for each i, $0 < i < n$, we have that $\Gamma((\Phi)_{i+1}/(\Phi)_{i-2}) \cong \langle \rho_{i-1}, \rho_i \rangle$. Thus, if $\{p_1, p_2, \dots, p_{n-1}\}$ is the Schläfli type of $\mathcal{P}$, we have

$$(\rho_{i-1}\rho_i)^{p_i} = \varepsilon.$$

For any $J \subseteq \{0, \dots, n-1\}$, we set $\Phi_J := \{(\Phi)_j | \ j \in J\}$. We then have that the stabilizer (under the action of $\Gamma(\mathcal{P})$) of $\Phi_{\bar{J}}$ is precisely the group $\Gamma_J := \langle \rho_j | \ j \notin J \rangle$.

Given two subsets $I, J \in \{0, \dots, n-1\}$, we consider their complements $\bar{I} := \{0, \dots, n-1\} \setminus I$ and $\bar{J} := \{0, \dots, n-1\} \setminus J$. We then note that

$$\Gamma_{\bar{I}} \cap \Gamma_{\bar{J}} = \Gamma_{\bar{I} \cup \bar{J}} = \Gamma_{\overline{I \cap J}}.$$

Therefore, the distinguished generators of $\mathcal{P}$ satisfy the *intersection condition*

$$\langle \rho_i | \ i \in I \rangle \cap \langle \rho_j | \ j \in J \rangle = \langle \rho_i | \ i \in I \cap J \rangle, \text{ for every } I, J \subseteq \{0, \dots n-1\}. \tag{2}$$

A group that is a quotient of a sting Coxeter group and whose generators satisfy the intersection condition (2) is called a *string C-group*. It is now straightforward to see the following theorem.

Theorem 12 *The automorphism group of a regular polytope is a string C-group.*

One can also show that the converse of the above theorem is also true. That is, every string C-group is the automorphism group of a regular polytope. The idea for the converse goes as follows.

Let $\Gamma = \langle \rho_0, \dots, \rho_{n-1} \rangle$ be a string C-group. We shall make use of Γ to construct a regular abstract polytope whose automorphism group is isomorphic to Γ. Define $\Gamma_i := \langle \rho_j \mid j \neq i \rangle$, and define the set of i-faces $\mathcal{F}_i := \{\Gamma_i\gamma \mid \gamma \in \Gamma\}$. We then say that

$$\Gamma_i\psi \leq \Gamma_j\varphi \text{ if and only if } i \leq j \text{ and } \Gamma_i\psi \cap \Gamma_j\varphi \neq \emptyset.$$

One then has to show that this definition gives a poset that is a regular polytope.