Regular Maps and Polytopes

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Outline of topics for next six lectures:

1 Symmetries of discrete structures, with lots of examples

2 Regular maps – properties, constructions, classifications

3 Computational and group-theoretic methods

4 Recent developments in the study of regular maps

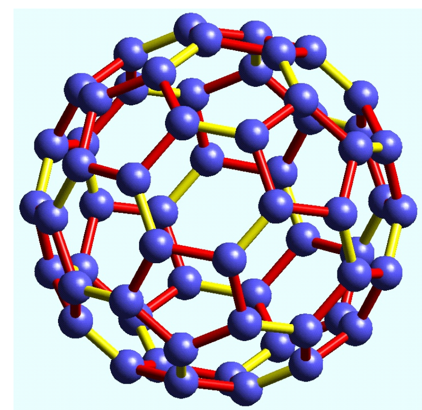
5 Abstract polytopes - especially regular & chiral examples

6 Recent developments and open questions in the study of regular and chiral abstract polytopes

PDF copies of the slides and summaries of the main points (with examples/exercises) will be available after each lecture

What is symmetry?

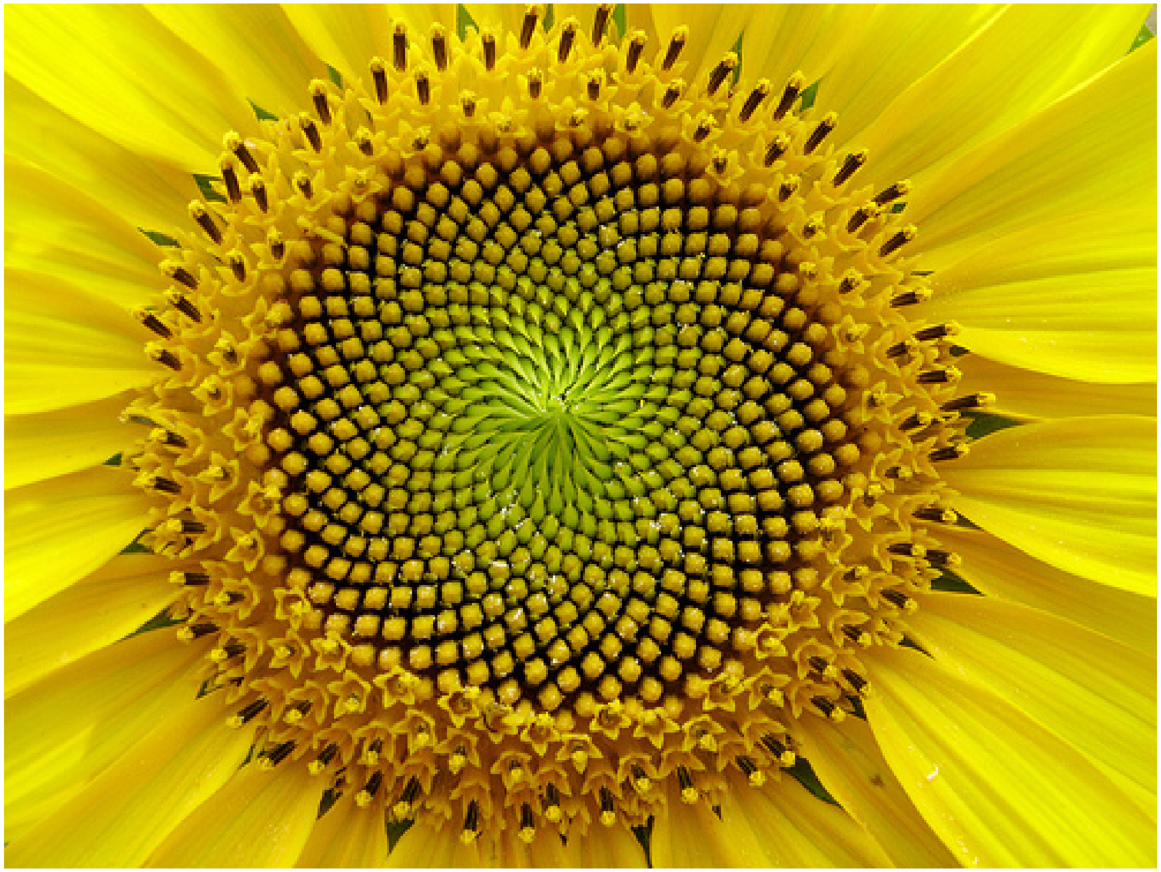
Symmetry can mean many different things, such as balance, uniform proportion, harmony, or congruence

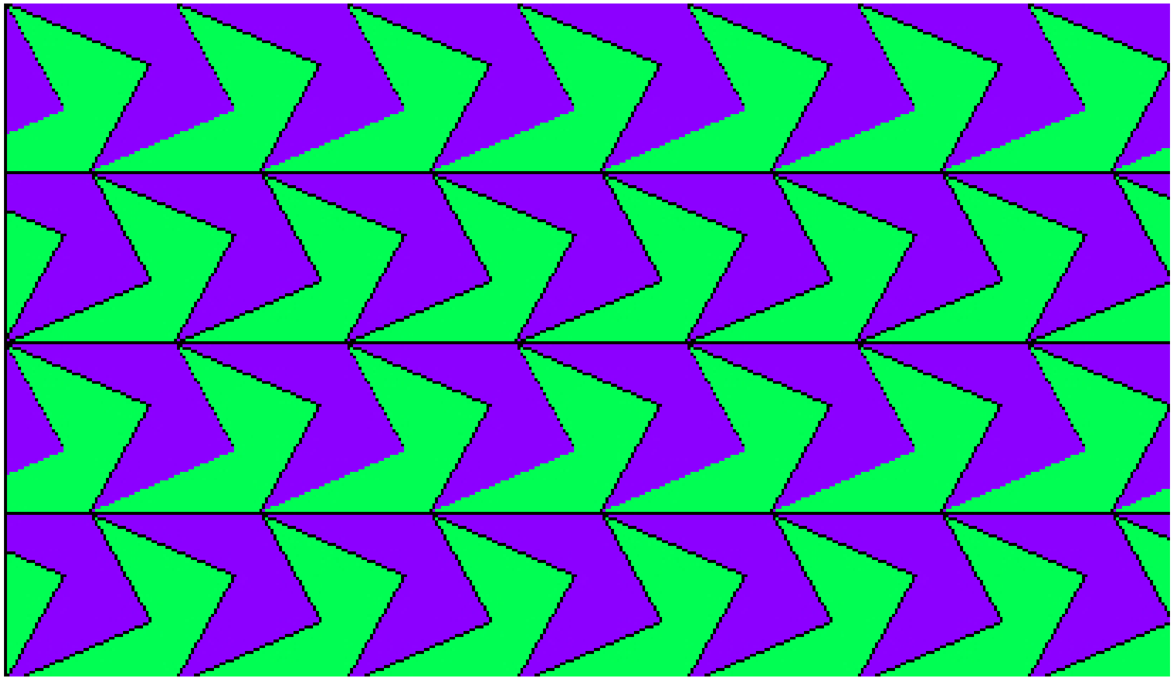
Generally, an object has symmetry if it can be transformed in way that leaves it looking the same as it did originally.

Symmetry can be reflective:

... or rotational:



... or translational:

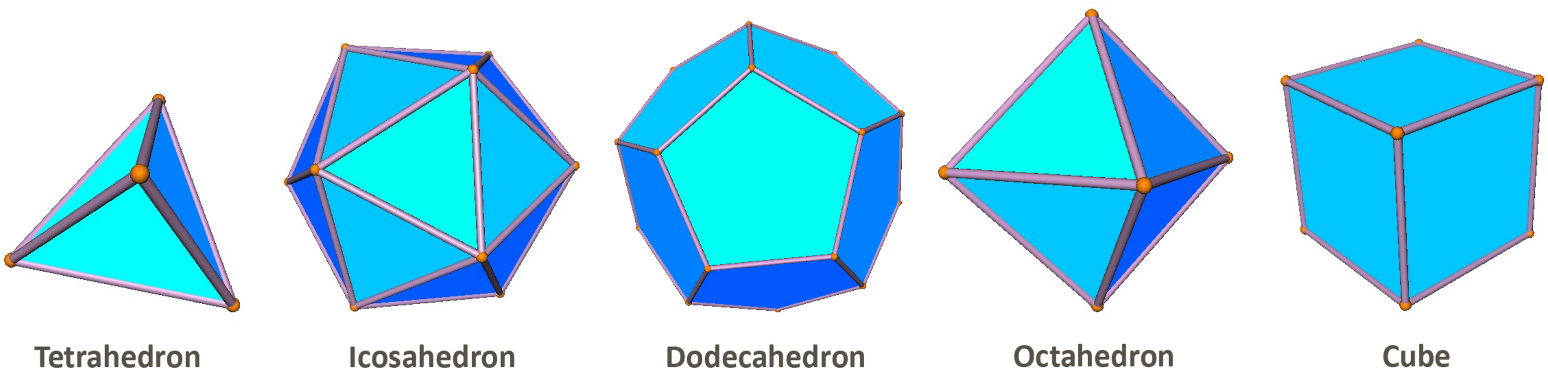


... or a combination of these types

Examples of these kinds of symmetry abound in nature

... but have also been manufactured by human fascination and enterprise

e.g. the Platonic solids (c. 360BC)



or earlier ... the 'Neolithic Scots' (c. 2000BC)

... as publicised by Atiyah and Sutcliffe (2003)

... but unfortunately a hoax!

The claim that the Scots knew about these five regular solids over 1000 years before Plato was based on the above five 'Scottish stones' at the Ashmolean Museum in Oxford — but one has 14 faces, and none of them is an icosahedron

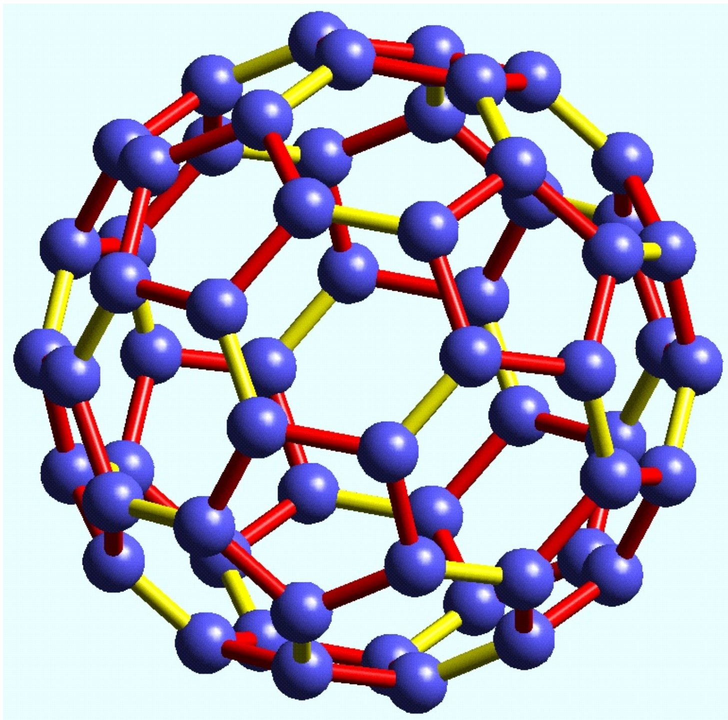
[See John Baez's website for the full story]

Tilings at the Alhambra Palace – on its walls, floors, ceilings, and even some of the furniture – amazingly exhibit all of the 17 "wallpaper symmetries" (in two dimensions)

[Rafael Pérez Gómez and José Mara Montesinos, 1980s]

Symmetry can induce strength and stability:

... or its more contemporary version, the C_{60} molecule Buckminsterfullerene ("buckyball"):



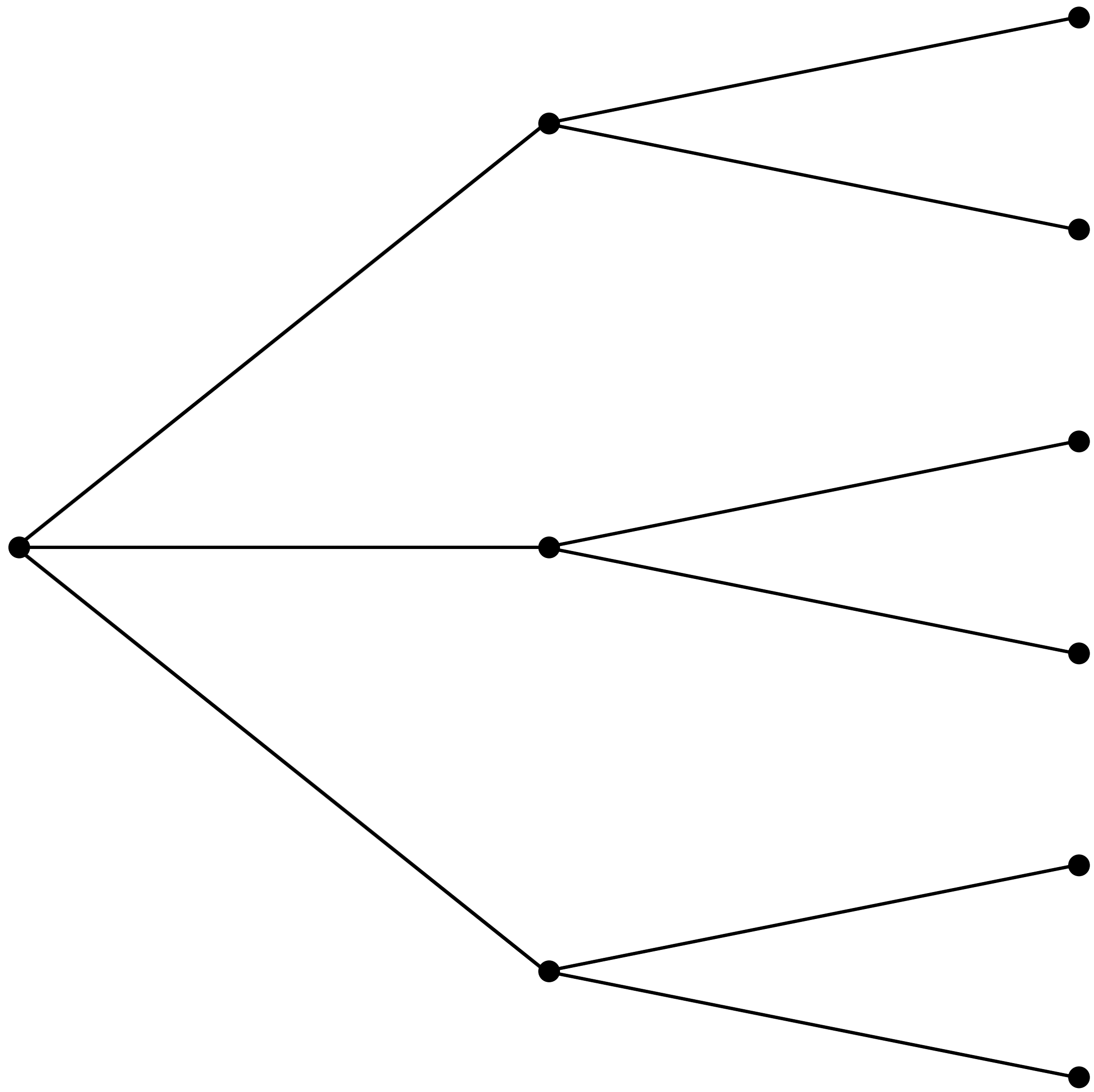
Symmetry can also arise unexpectedly ...

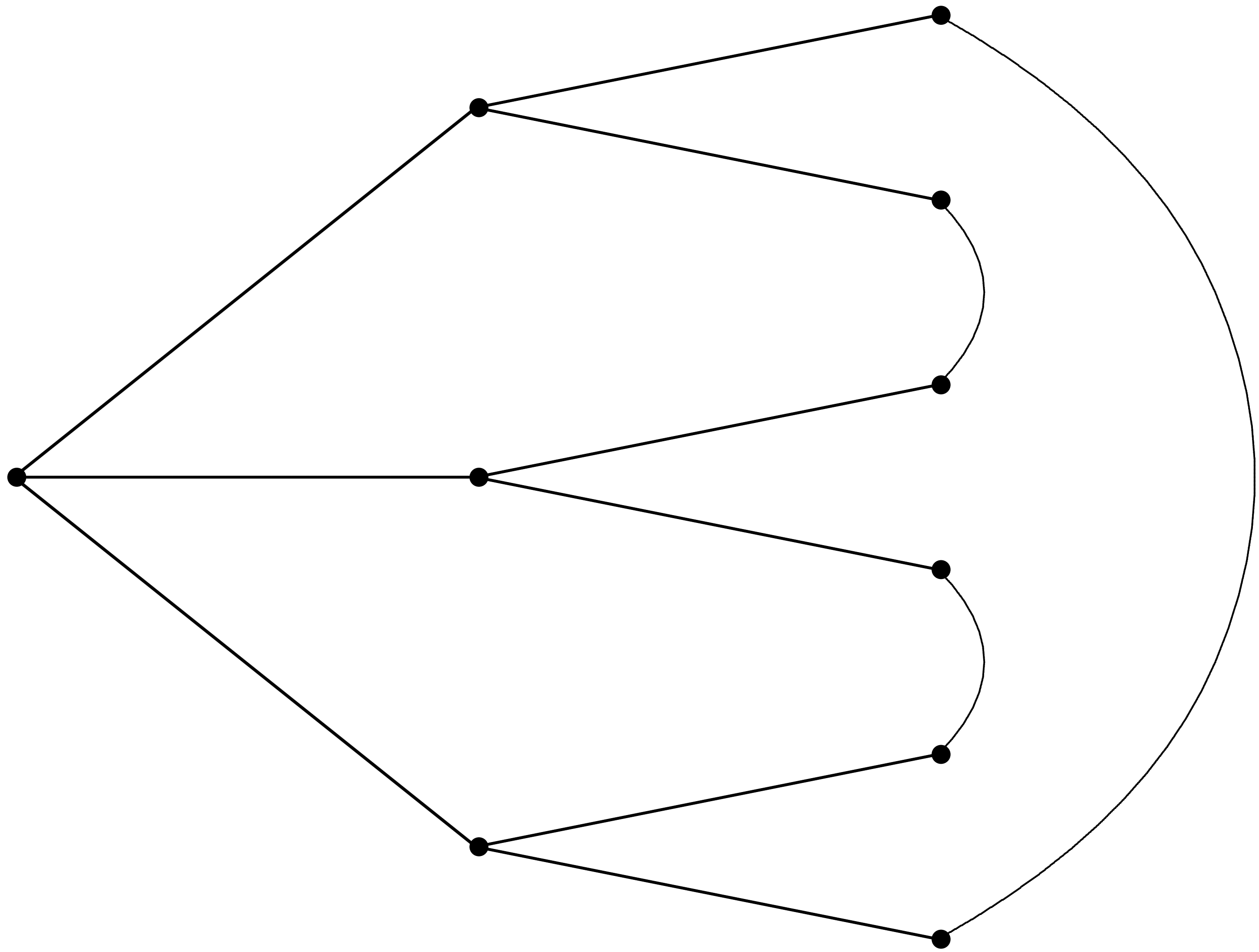
Consider a network in which

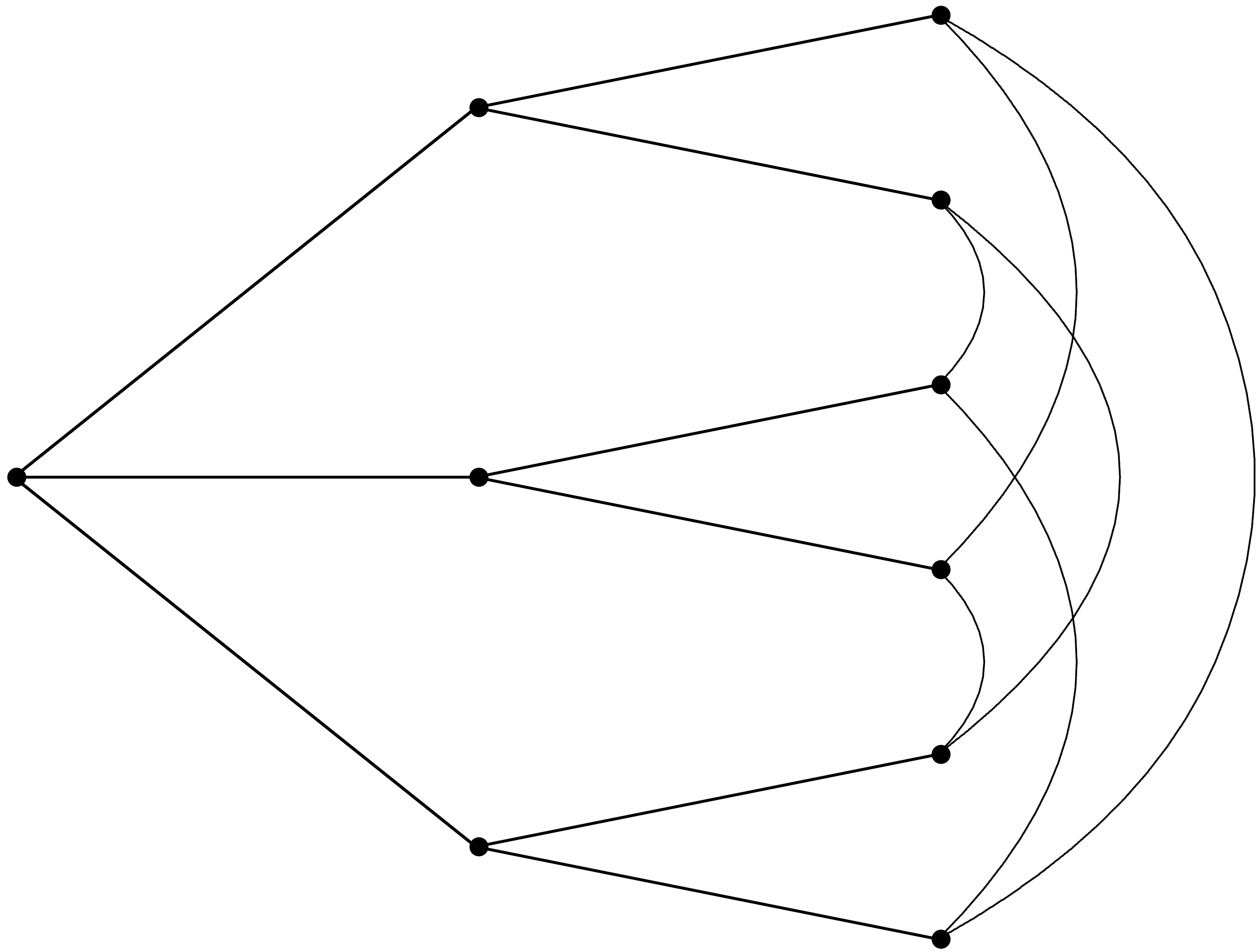
- each node is directly connected to (at most) 3 others
- any two nodes are connected by a path of length ≤ 2

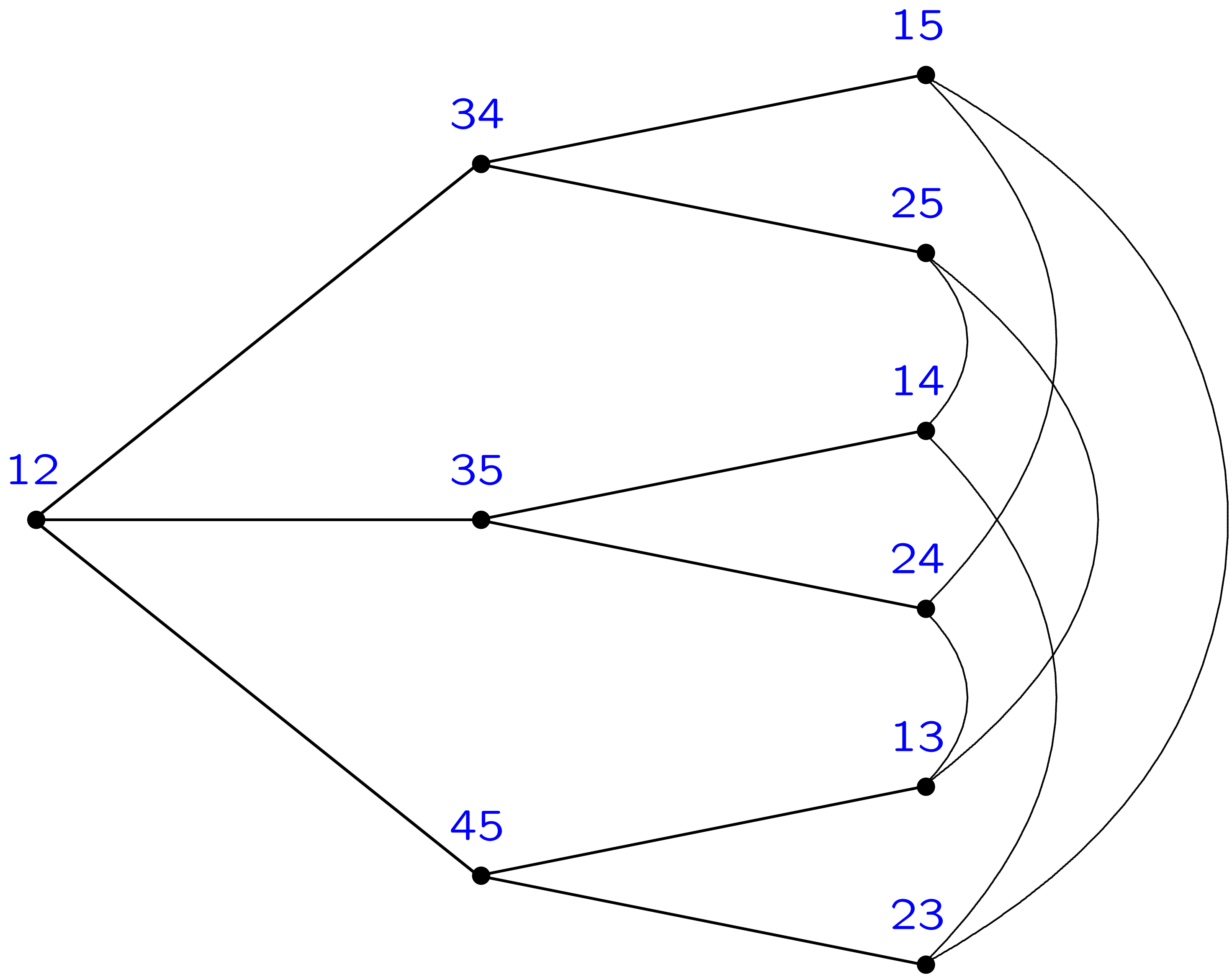
This is a graph of degree 3 and diameter 2

Question: What's the largest possible number of nodes?








15
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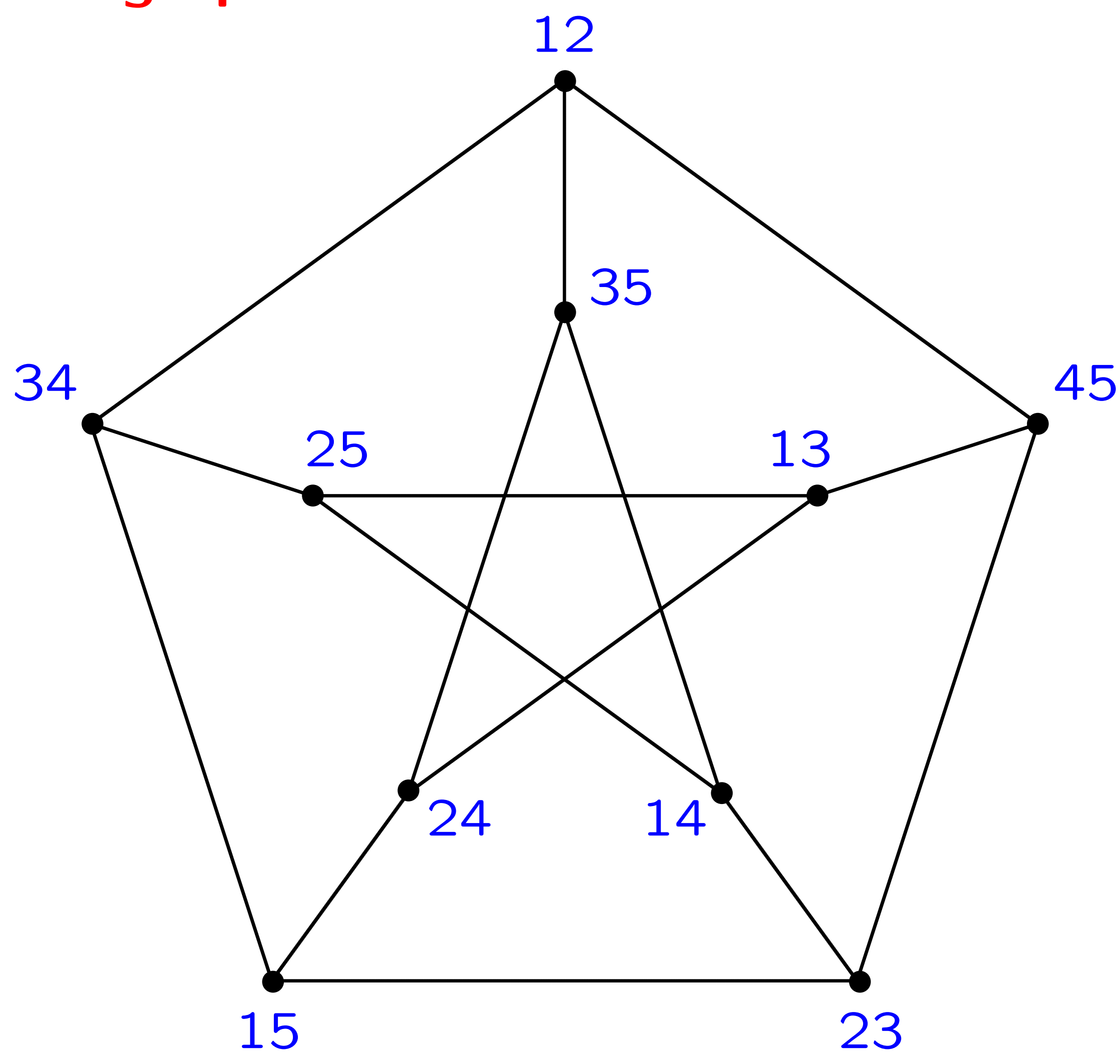
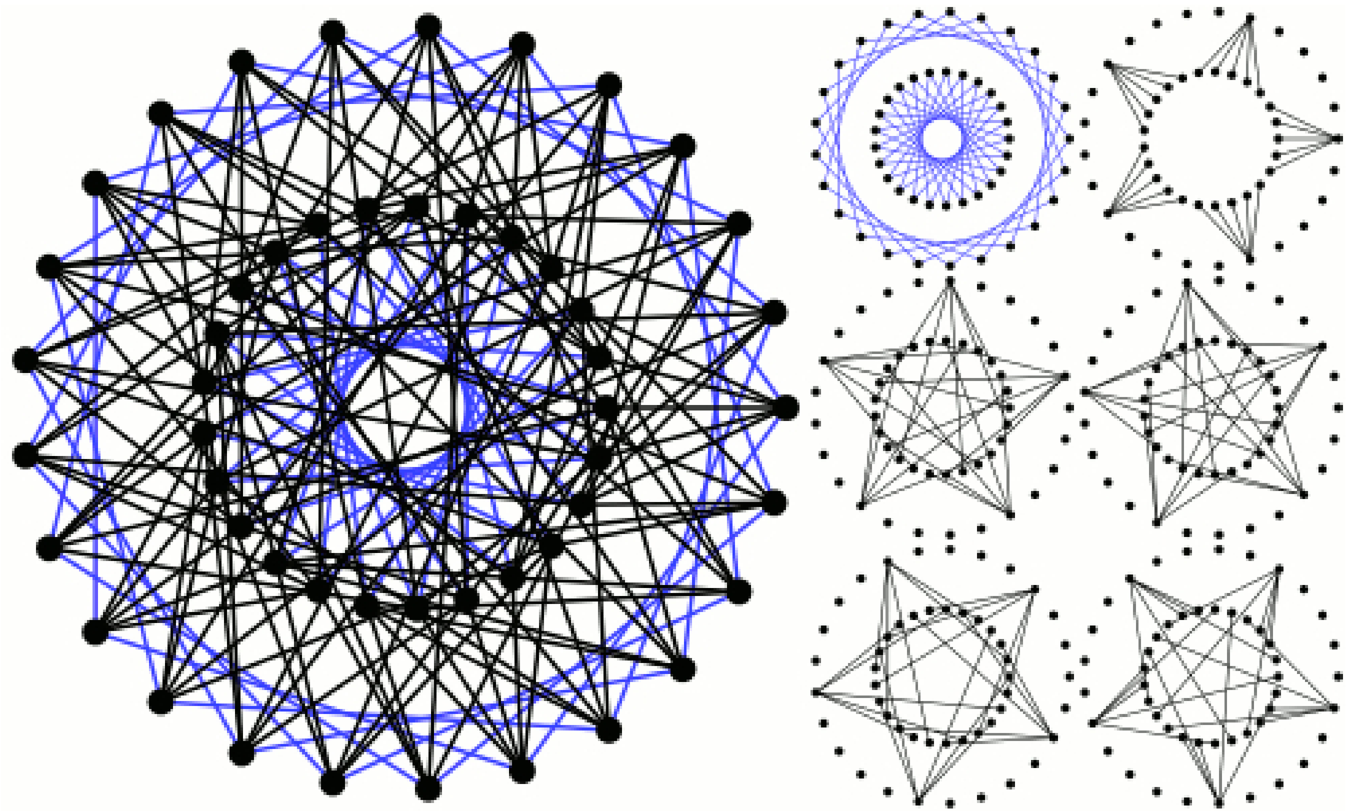
This is the Petersen graph:



The largest 7-valent graph of diameter 2



... is also highly symmetric: the Hoffman-Singleton graph

Question/Exercise

Suppose X is a d-regular simple graph of diameter D.

Counting the largest possible number of vertices at distance k from a given vertex, for $1 \le k \le D$, gives the Moore bound

$$|V(X)| \le 1 + d + d(d-1) + d(d-1)^2 + \cdots + d(d-1)^{D-1}.$$

Are there well-known families of graphs that give infinitely many examples of graphs for which this bound is attained?

[Hint: What happens when d or D is small?]

Symmetries of discrete structures

Suppose V is a finite or countable set endowed with some structure that can defined by subsets or ordered sequences of elements, e.g.

- a graph (V, E) with vertex-set V and edge-set E
- a map (V, E, F) with edge-set E and face-set F
- a design (V, B) with block-set B (a subset of $2^V = \mathcal{P}(V)$)
- a polytope $(\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_n)$ with $\mathcal{F}_i =$ set of all i-faces (and $V =$ set of all flags of the polytope).

The symmetry of any such discrete structure can be measured by its automorphisms: incidence-preserving bijections.

Under composition, these form a group, called the automorphism group (or symmetry group) of the structure.

Symmetric graphs

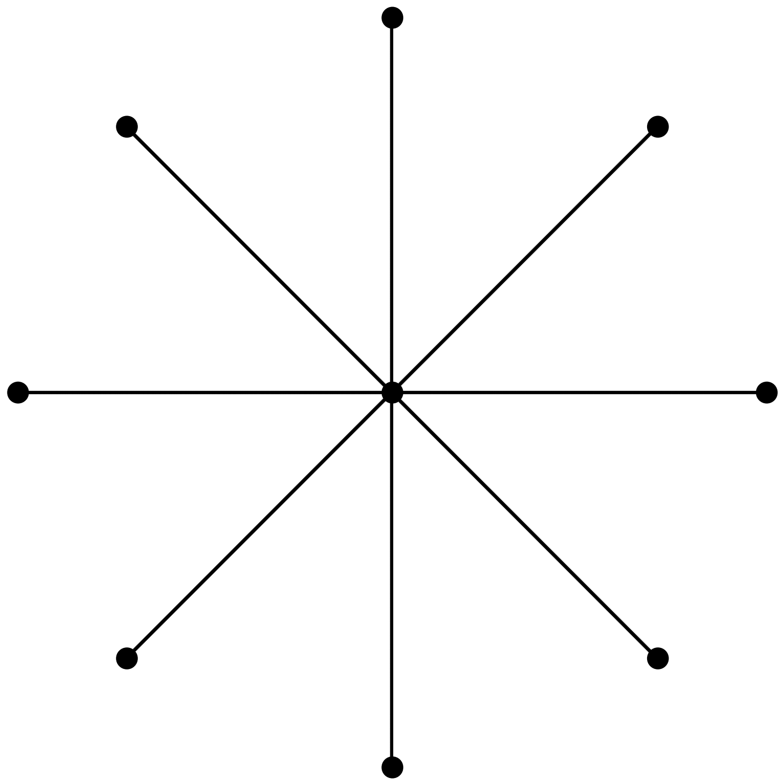
Let X be a simple graph — with no loops or multiple edges.

When the automorphism group has a single orbit on vertices, i.e. when the graph looks the same from every vertex/node, the graph is called vertex-transitive

When the automorphism group has a single orbit on edges, i.e. when the graph looks the same from every edge, the graph is called edge-transitive

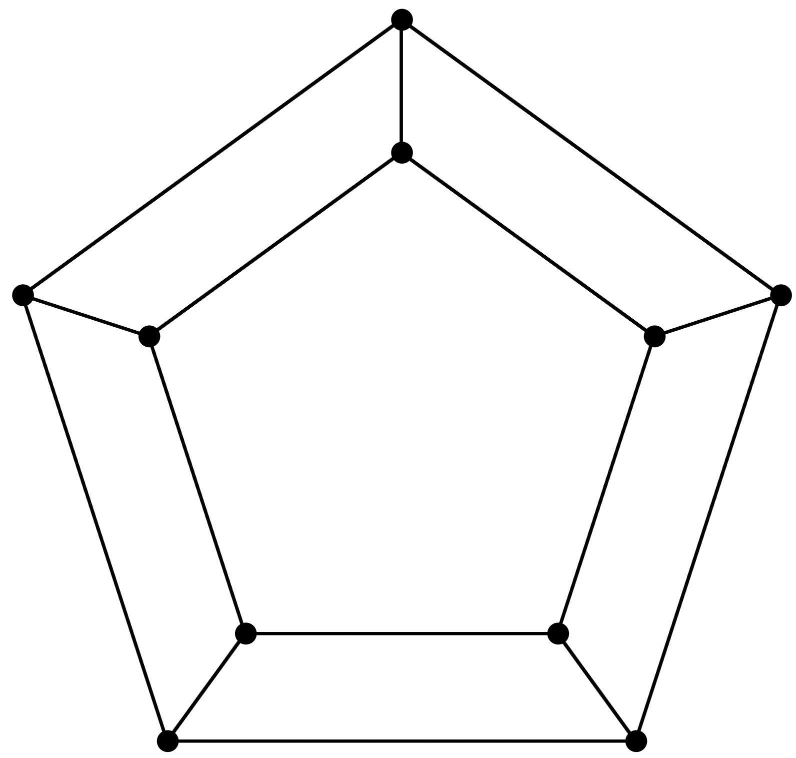
When the automorphism group has a single orbit on arcs, i.e. when the graph looks the same **along** every ordered edge, the graph is called arc-transitive, or symmetric.

Example:



This is edge-transitive but not vertex-transitive

Example:



This is vertex-transitive but not edge-transitive

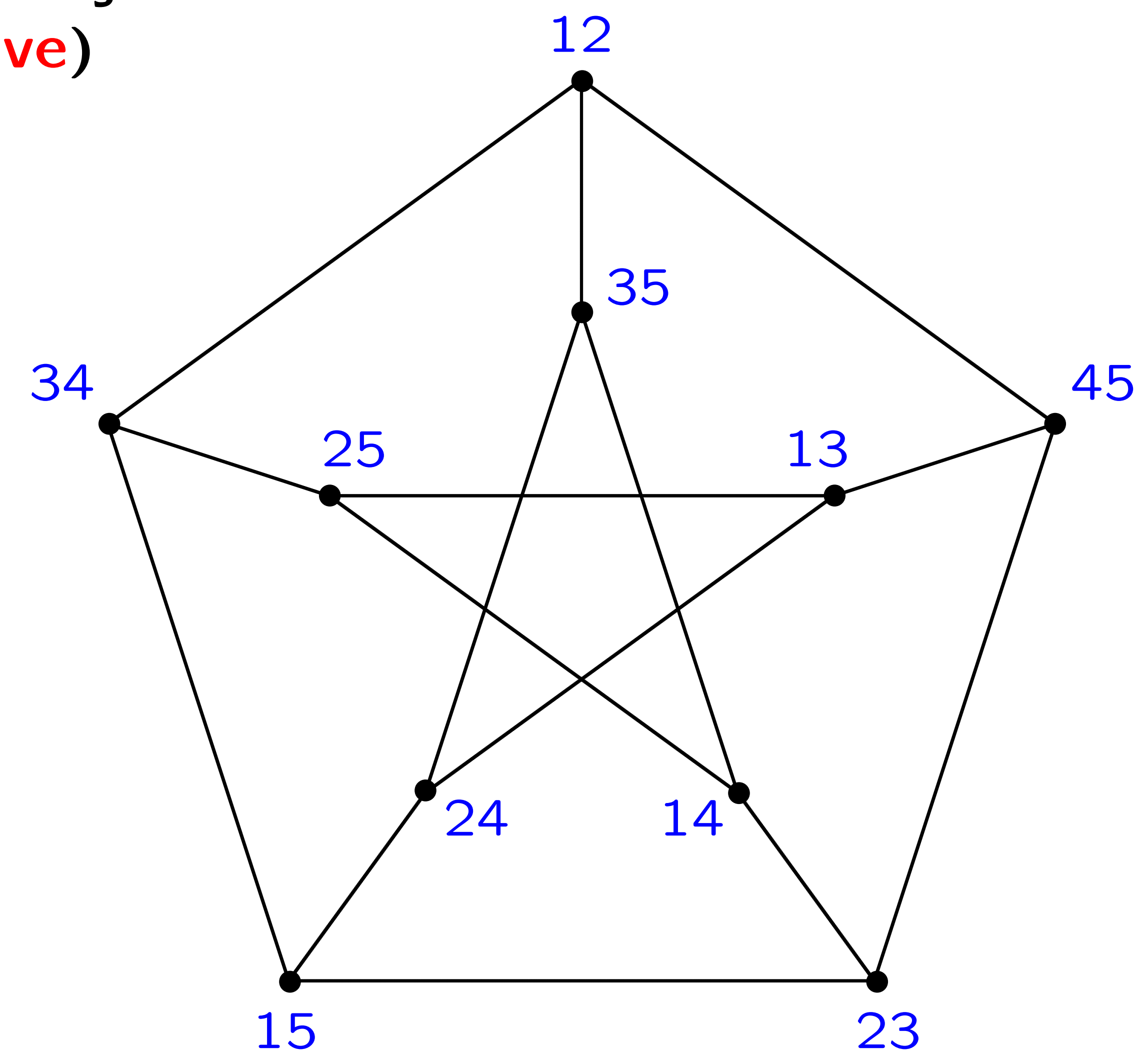
Higher levels of arc-transitivity

- An s-arc in a graph X is a sequence $(v_0, v_1, v_2, \ldots, v_s)$ of $s+1$ vertices of X such that $\{v_{i-1}, v_i\}$ is an edge of X for $0 < i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $0 < i < s$
– or in other words, such that any two consecutive v_i are adjacent and any three consecutive v_i are distinct.

- The graph X is called s-arc-transitive if its automorphism group $\mathrm{Aut}\, X$ is transitive on the set of all s-arcs of X

e.g. 0-arc-transitive means the same as vertex-transitive, while 1-arc-transitive means arc-transitive (or symmetric).

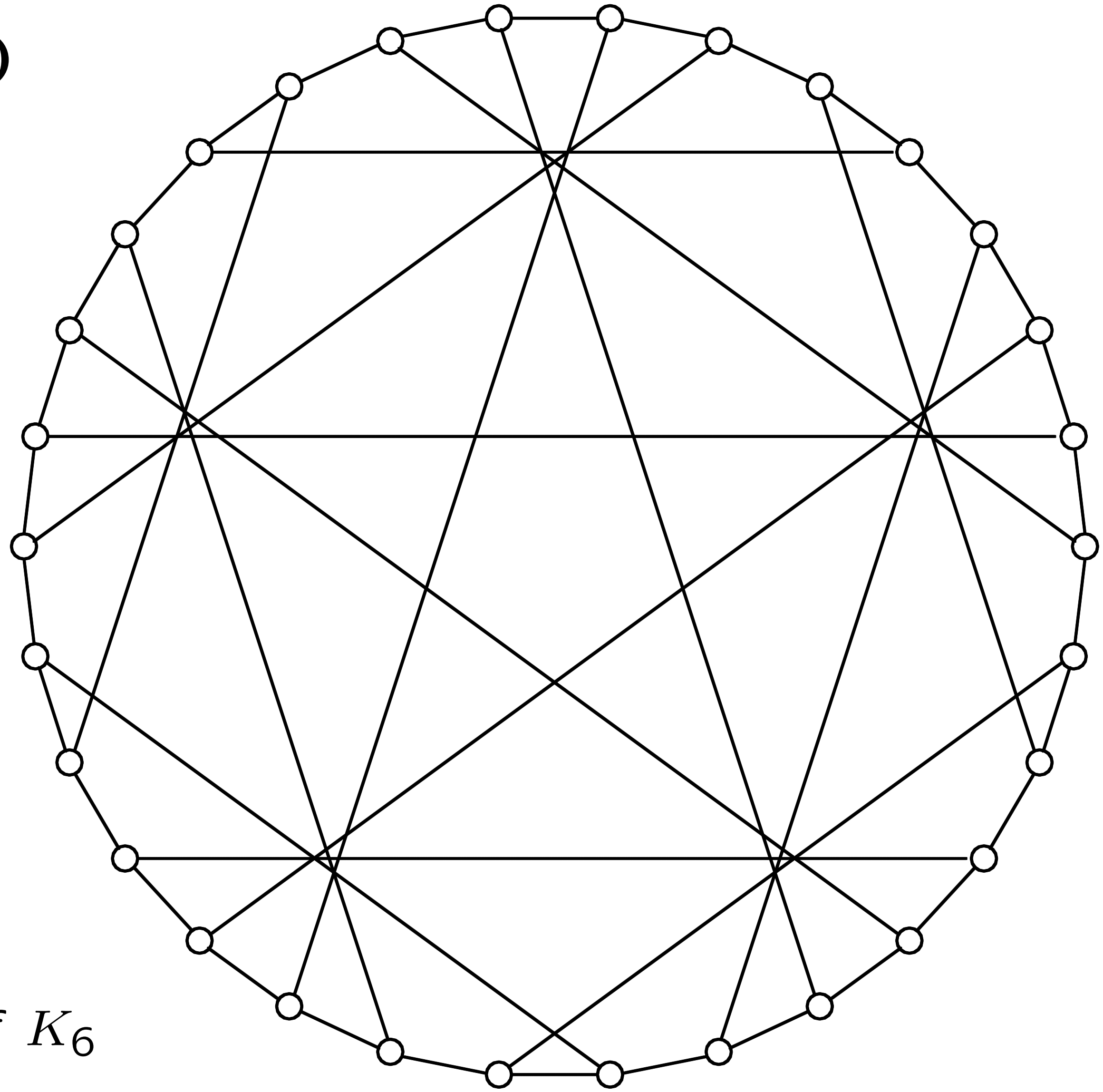
The Petersen graph is symmetric
(in fact 3-arc-transitive)



Every 3-arc has form

ab — cd — ae — bc

Tutte's 8-cage
(**5-arc-transitive**)



Associated with
1-factorisations of K_6

Construction of symmetric graphs

The automorphism group of a symmetric graph has particular properties that can be modelled. All examples of a given type can then often be constructed from a 'universal model'.

Various people including Tutte, Conway, Biggs, Weiss, Holt, Djoković, Lorimer and Praeger contributed greatly to the theoretical analysis and construction of symmetric graphs.

Subsequent use of group theory and discrete computation has led to the construction of many new examples, and infinite families of examples, and complete lists of small examples of various kinds. One of the graphs that arose in this way is the largest known 3-valent graph of diameter 10.

Recent work by Auckland student Eyal Loz

Tables of the largest known graphs of given degree d and diameter k have been built up and occasionally adjusted over the last 50 years (by computer scientists, engineers and mathematicians). Finding the largest possible is known as the degree-diameter problem.

In his PhD thesis project (2005-2008), Eyal used group-based voltage graph methods to construct new examples as 'covers' of old ones. Roughly speaking, this involves linking together a chain of copies of a suitably-chosen small graph, with a 'voltage group' determining the linkages.

The result?

Degree-Diameter Table (as at August 2011)

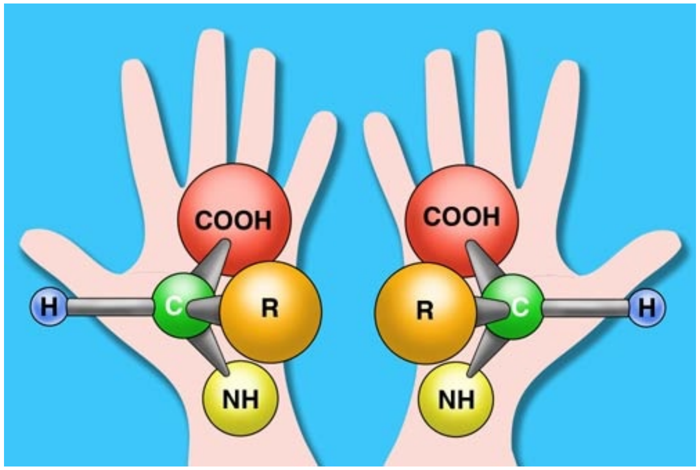
d\k	2	3	4	5	6	7	8	9	10
3	10	20	38	70	132	196	336	600	1 250
4	15	41	98	364	740	1 320	3 243	7 575	17 703
5	24	72	212	624	2 772	5 516	17 030	57 840	187 056
6	32	111	390	1 404	7 917	19 383	76 461	307 845	1 253 615
7	50	168	672	2 756	11 988	52 768	249 660	1 223 050	6 007 230
8	57	253	1 100	5 060	39 672	131 137	734 820	4 243 100	24 897 161
9	74	585	1 550	8 200	75 893	279 616	1 686 600	12 123 288	65 866 350
10	91	650	2 286	13 140	134 690	583 083	4 293 452	27 997 191	201 038 922
11	104	715	3 200	19 500	156 864	1 001 268	7 442 328	72 933 102	600 380 000
12	133	786	4 680	29 470	359 772	1 999 500	15 924 326	158 158 875	1 506 252 500
13	162	851	6 560	40 260	531 440	3 322 080	29 927 790	249 155 760	3 077 200 700
14	183	916	8 200	57 837	816 294	6 200 460	55 913 932	600 123 780	7 041 746 081
15	186	1 215	11 712	76 518	1 417 248	8 599 986	90 001 236	1 171 998 164	10 012 349 898
16	198	1 600	14 640	132 496	1 771 560	14 882 658	140 559 416	2 025 125 476	12 951 451 931
17	274	1 610	19 040	133 144	3 217 872	18 495 162	220 990 700	3 372 648 954	15 317 070 720
18	307	1 620	23 800	171 828	4 022 340	26 515 120	323 037 476	5 768 971 167	16 659 077 632
19	338	1 638	23 970	221 676	4 024 707	39 123 116	501 001 000	8 855 580 344	18 155 097 232
20	381	1 958	34 952	281 820	8 947 848	55 625 185	762 374 779	12 951 451 931	78 186 295 824

Chirality

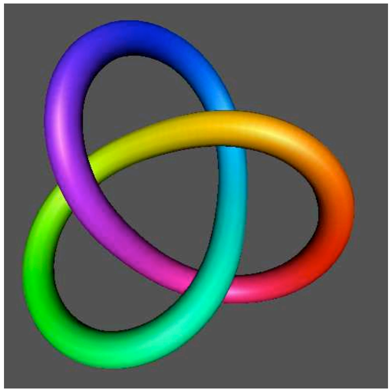
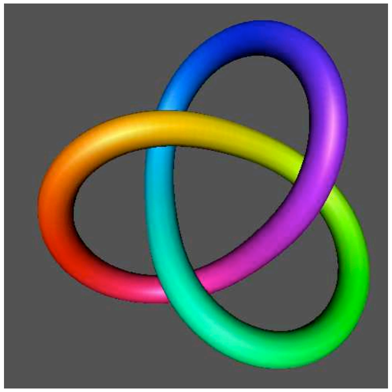
An object is called chiral if it differs from its mirror images.

History/terminology



The term 'chiral' means handedness, derived from the Greek word $\chi\epsilon\iota\rho$ (or ‘kheir’) for ‘hand’. It is usually attributed to the scientist William Thomson (Lord Kelvin) in 1884, although the philosopher Kant had earlier observed that left and right hands are inequivalent except under mirror image.

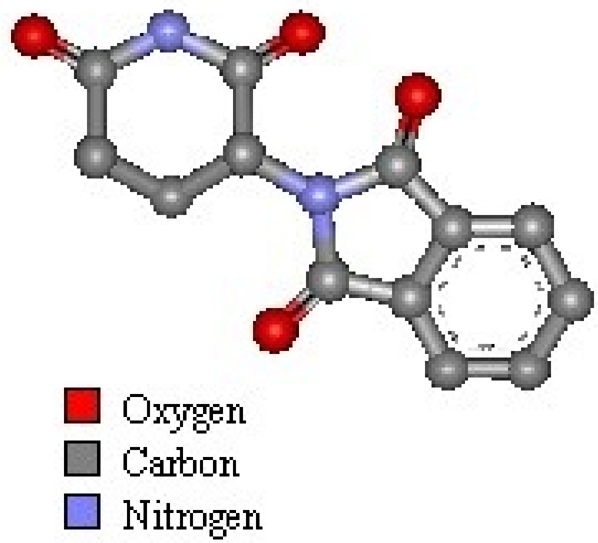
Chirality in mathematics

The right and left trefoil knots are inequivalent . . . with Jones polynomials $t + t^3 - t^4$ and $t^{-1} + t^{-3} - t^{-4}$ respectively

Many of the other invariants of these knots (including their Alexander polynomials) are exactly the same for both, some because they are mirror images of each other, and in purely mathematical terms they have equal importance, but . . .

Chirality in biology/chemistry/medicine



The two enantiomorphs of thalidomide have vastly different effects ... one is a sedative, but its mirror image causes birth defects ... making the context important

Other examples (in food science)

- DNA, proteins, amino acids and sugars are all chiral

- Aspartame is a sweetening agent (sweeter than sucrose) but its mirror image molecule is bitter

- (S)-carvone smells a lot like caraway while its mirror image (R)-carvone is like spearmint

- Mirror image amino acids are called L- and D-aminoacids; human proteins are exclusively built from L-aminoacids

... again in all these cases, the **context** is important.

Question: How prevalent is chirality?

If an object is equivalent to its mirror image (with respect to some axis/hyperplane) then it has reflectional symmetry.

In biological/chemical/medical/physical contexts, objects tend to be chiral — but the following is a remarkable phenomenon:

When a discrete object has a large degree of **rotational** symmetry, it often happens that it has also **reflectional** symmetry, so that chirality is not necessarily the norm!

e.g. the Platonic solids are all reflexible