Lecture 4: Recent developments in the study of regular maps

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Preamble/Reminder

A map M is 2-cell embedding of a connected graph or multi-graph into a surface. An automorphism of a map M is an incidence-preserving permutation of each of the (three) sets of vertices, edges and faces of M, and is uniquely determined by its effect on any flag (v, e, f), so $|\mathrm{Aut}\, M| \leq 4|E|$.

If this bound is attained, then M is a (fully) regular map. Such M may be orientable (and reflexible), or non-orientable.

Similarly, if the carrier surface is orientable, and $\mathrm{Aut}^{\mathrm{o}} M$ is the group of all orientation-preserving automorphisms of M, then $|\mathrm{Aut}^{\mathrm{o}} M| \leq 2|E|$, and when this bound is attained, M is orientably-regular (or just regular).

Three kinds of regular maps:

- Orientable and flag-transitive (and reflexible)
- Non-orientable and flag-transitive
- Orientable and arc-transitive but chiral (irreflexible).

Properties:

- Type $\{m, k\}$: $k =$ valence, $m =$ covalence (face-size)
- Orientability: orientability of carrier surface
- Characteristic: Euler characteristic of carrier surface

$$\chi = |V| - |E| + |F|$$
$$= \begin{cases} |\mathrm{Aut}^{\circ} M| \left(\frac{1}{k} - \frac{1}{2} + \frac{1}{m}\right) & \text{if } M \text{ orientable} \\ |\mathrm{Aut}\, M| \left(\frac{1}{2k} - \frac{1}{4} + \frac{1}{2m}\right) & \text{if } M \text{ flag-transitive} \end{cases}$$

- Genus: Genus of carrier surface.

Connection with Triangle Groups

Every regular map M of type $\{m, k\}$ has an arc-transitive group of automorphisms, generated by two automorphisms R and S that act like rotations about a face and incident vertex, and satisfy the defining relations for the ordinary $(m, k, 2)$ triangle group: $R^m = S^k = (RS)^2 = 1$.

Similarly, in the flag-transitive case, Aut M is generated by three automorphisms a, b, c that act like reflections (about axes through an incident face, edge and vertex, respectively) and satisfy the defining relations for the full $(m, k, 2)$ triangle group: $a^2 = b^2 = c^2 = (ab)^m = (bc)^k = (ac)^2 = 1$.

Conversely, a regular map can be constructed from any smooth quotient of one of these triangle groups:

- Vertices correspond to cosets of the image of $\langle S \rangle$ or $\langle b, c \rangle$
- Edges correspond to cosets of the image of $\langle RS \rangle$ or $\langle a, c \rangle$
- Faces correspond to cosets of the image of $\langle R \rangle$ or $\langle a, b \rangle$

and

- Incidence corresponds to non-empty intersection (of cosets).

Group-theoretic tests for map properties:

Reflexibility:	Automorphism $(R, S) \leftrightarrow (R^{-1}, S^{-1})$
Orientability:	$\langle R, S \rangle = \langle ab, bc \rangle$ has index 2 in $\langle a, b, c \rangle$
Duality:	Automorphism $(R, S) \leftrightarrow (S, R)$

Classification of regular maps

By type: Regular maps of given type $\{m, k\}$

By surface: Regular maps of given genus

By graph: Regular embeddings of a given graph

By group: What regular maps M with $\text{Aut}\, M = G$?

Within these, various properties are worth considering, e.g.

- Orientability, reflexibility, duality, Petrie duality, etc.
- Simple underlying graph (in the map and/or its dual)
- Underlying graph is a Cayley graph [see later]
- Quotients and covers
- Existence of semi-regular automorphisms.

Classification by type of the map

In this case, relatively little is known:

- Regular maps of types $\{2, m\}$, $\{3, 3\}$, $\{3, 4\}$ and $\{3, 5\}$ (on the sphere) and types $\{3, 6\}$ and $\{4, 4\}$ (on the torus) and their duals all known and well understood

- It is known that there are infinitely many rotary maps of any given hyperbolic type $\{k, m\}$ ... this follows easily from the residual finiteness of the triangle group $\Delta^{\mathrm{o}}(2, k, m)$

- Special case of Hurwitz maps (of type $\{3, 7\}$ ... for which the Hurwitz bound $|\mathrm{Aut}^{\mathrm{o}} M| \leq 84(g-1)$ is attained)

- Construction of infinite families of rotary but chiral maps of type $\{3, m\}$ for all $m \geq 7$ [various authors].

Hurwitz maps (of type $\{3,7\}$)

Map groups meeting the Hurwitz bound $|G^{o}| \leq 84(g-1)$ are quotients of the ordinary $(2,3,7)$ triangle group, including:

- Groups PSL$(2,q)$ for certain q [Macbeath (1969)]
- Alternating groups A_n for all but a few n [Conder (1980)]
- 12 of the 26 sporadic simple groups [one is the Monster]
- The Chevalley groups $G_2(q)$ for all $q \geq 5$ [Malle (1990)]
- The Ree groups $^2G_2(3^{2m+1})$ for all $m \geq 1$ [Malle, Jones]
- PSL(n,q) and SL(n,q) for all $n \geq 287$ [Lucchini et al]
- Various other families of groups of Lie type [Lucchini/Tamburini/Wilson, Vsemirnov, Zalesskii, et al]

But this Hurwitz bound (and its analogue when reflections are included) is rarely achieved ...

Hurwitz maps of 'small' genus [MC (1980s)]

Genus	Rfl	Ch
3	1	0
7	1	0
14	3	0
17	0	2
118	1	0
129	1	2
146	3	0
385	1	0
411	3	0
474	3	0
687	1	0

Genus	Rfl	Ch
769	3	0
1009	1	0
1025	0	8
1459	1	0
1537	1	0
2091	1	6
2131	3	0
2185	3	0
2663	0	2
3404	3	0
4369	3	0

Genus	Rfl	Ch
4375	1	0
5433	3	0
5489	0	2
6553	3	0
7201	1	4
8065	0	2
8193	1	12
8589	3	0
11626	1	0
11665	0	2
Total	50	42

Rfl = Reflexible   Ch = Chiral

Classification by group

Again, relatively little has been achieved:

- Cyclic groups and dihedral groups [easy cases]

- Orientably-regular M with $\text{Aut}^{\text{o}} M \cong \text{PSL}(2, q)$ or $\text{PGL}(2, q)$ … essentially by Macbeath (1967) and Sah (1969)

- Regular maps and hypermaps M with $\text{Aut}\, M \cong \text{PSL}(2, q)$ or $\text{PGL}(2, q)$ … Conder, Potočnik and Širáň (2009)

- The smallest genus of all rotary/regular maps on which a given group (from certain families) acts as $\text{Aut}^{\text{o}} M$ or $\text{Aut}\, M$ [in the study of the (strong) symmetric genus or cross-cap number, e.g. for A_n and S_n and sporadic simple groups]

• Nonorientable regular maps M with Aut $M \cong$ PSL$(3, p)$ for p prime ... Du & Kwak (2009) ... actually none at all!

• Study of finite groups generated by three involutions, two of which commute: relations $a^2 = b^2 = c^2 = (ac)^2 = 1$

– Alternating groups A_n, symmetric groups S_n [MC, 1980s]

– $PSL_2(q)$ [Sjerve & Cherkassoff (1994)]

– Other simple groups of Lie type [Nuzhin (1996–)]

– Other linear groups [Tamburini & Zucca (1997)]

– Sporadic simple groups [Timofeenko, then Mazurov (2003)]

Theorem [Mazurov (2003)]

Every sporadic finite simple group except the Mathieu groups M_{11}, M_{22}, M_{23} and the McLaughlin group McL can be generated by three involutions, two of which commute [so is the automorphism group of a non-orientable regular map].

Classification by underlying graph

- Take a graph X or family of graphs X_n
- What are the embeddings of X or X_n as a regular map?
- Necessary condition: the graph has to be arc-transitive
- One approach: find an arc-transitive subgroup G of the automorphism group of the graph that is generated by two elements R and S such that $R^m = S^k = (RS)^2 = 1$, with S inducing a k-cycle on the neighbourhood of some vertex v, and RS reversing an edge incident with v
- Enumeration and properties of the resulting maps (e.g. orientability, reflexibility, genus, etc.)

Classification by underlying graph (cont.)

Better progress here. Embeddings of the following families of graphs as orientably regular maps are now known:

- Cycle graphs C_n [on sphere, trivial]
- Complete graphs K_n [James & Jones (1985)]
- Cocktail party graphs [Nedela & Škoviera (1996)]
- Merged Johnson graphs [Jones (2005)]
- Some complete multipartite graphs [Du et al (2005)]
- Complete bipartite graphs $K_{n,n}$ [various authors]
- Graphs of order p or pq (where p, q prime) [various]
- Hypercube graphs Q_n for n odd [Du et al (2007)]
- Hypercube graphs Q_n for n even [CCDKNW (2010)]
- Hamming graphs $H_n(q)$ for $q > 2$ [Jones (preprint)]

Regular embeddings of Q_n [CCDKNW (2010)]

The automorphism group of the n-dimensional cube Q_n is the wreath product $\mathbb{Z}_2 \wr S_n$ (isomorphic to $(Z_2)^n \rtimes S_n$).
Let y be the n-cycle $(1, 2, 3, \ldots, n)$ in S_n, and let e_n be the nth standard basis vector $(0, 0, \ldots, 0, 1)$ of $(Z_2)^n$.

It is known by a theorem of Young Soo Kwon (2004) that embeddings of Q_n as an orientably regular map are in one-to-one correspondence with involutions $\sigma \in S_n$ fixing n such that $e_n\sigma$ and y generate a subgroup of order $2^n n$ in $\mathbb{Z}_2 \wr S_n$.

All such σ were known when n is odd [Du, Kwak & Nedela (2007)] and when $n = 2m$ for odd m [Jing Xu (2007)].

Theorem [Catalano/Conder/Du/Kwon/Nedela/Wilson]

For $n = 2m$ (even), the involution $\sigma \in S_n$ fixing n gives an orientably-regular embedding of Q_n (or equivalently, $e_n\sigma$ and $y = (1, 2, \ldots, n)$ generate a subgroup of order $2^n n$ in $\mathbb{Z}_2 \wr S_n$) if and only if

(a) σ commutes with y^m, so that the m orbits of $\langle y^m \rangle$ form a system of imprimitivity for $H = \langle \sigma, y \rangle$ on $\{1, 2, \ldots, n\}$, and

(b) σ is additive mod m, or equivalently, σ induces the same permutation on the blocks $B_i = \{i, i+m\}$ for $1 \leq i \leq m$ as multiplication on $\{1, 2, ..., m\}$ by some square root of 1 modulo m.

Also for n even, the proportion of orientably-regular embeddings of Q_n that are chiral tends to 1 as $n \to \infty$.

Classification by surface (genus/characteristic)

This is perhaps the most illuminating perspective:

- Characteristic 0, 1 & 2: Brahana [1927], Coxeter [1957]
- Orientably-regular, genus 3: Sherk [1959]
- Orientably-regular, genus 4, 5 and 6: Garbe [1969]
- Orientably-regular, genus 2 to 15, and non-orientable, genus 3 to 30: MC & Dobcsányi [2001]
- Regular non-orientable, genus $p+2$ for p odd prime: Breda, Nedela and Siráň (2005)
- Orientably-regular, genus 2 to 100, and non-orientable, genus 3 to 200: MC [2006]
- Orientably-regular, genus 2 to 300, and non-orientable, genus 3 to 300: MC [2011]

How was the latest census obtained?

Small homomorphic images of a finitely-presented group Σ can be obtained by computational methods ... most recently the Low IndexNormal Subgroups algorithm developed by Derek Holt and his student David Firth, to systematically enumerate the possibilities for a composition series of Σ/K, where K is a normal subgroup of small index in Σ.

Applying this to triangle groups $\Delta(2, k, m)$ and $\Delta^{o}(2, k, m)$ for suitable choices of k and m ... one of which may sometimes be taken as arbitrary ... and some post-computation analysis then gives the classification(s).

We even have beautiful pictures of many of these, thanks to Jarke van Wijk (a computer scientist at Eindhoven) ...

Questions about the genus spectrum

• Is there an orientably-regular map of every genus?
Answer: Yes, for every $g > 1$ there exists a regular map of type $\{4g, 4g\}$ with dihedral automorphism group of order $8g$ — but with only one vertex and one face, and multiple edges

• What are the genera of orientably-regular maps that have simple underlying graphs (with no multiple edges)?

• Are there non-orientable regular maps of all but finitely many genera?
Answer: No, since Breda, Nedela and Siráň (2005) proved that there's only one such map of genus $p+2$ when p is a prime congruent to 1 mod 12 (viz. one map of genus 15)

• What are the genera of orientably-regular but chiral maps?

Summary of results for small genus

Orientably-regular maps (up to isomorphism & duality)

Genus 2: 6 reflexible, 0 chiral
Genus 3: 12 reflexible, 0 chiral
Genus 4: 12 reflexible, 0 chiral
Genus 5: 16 reflexible, 0 chiral
Genus 6: 13 reflexible, 0 chiral
Genus 7: 12 reflexible, 4 chiral

Genus 2 to 100: 5972 reflexible, 1916 chiral (24% chiral)
Genus 101 to 200: 9847 reflexible, 4438 chiral (31%)
Genus 201 to 300: 10600 reflexible, 5556 chiral (34%)

Question: How prevalent is chirality?

Observations

- There is no orientably-regular but chiral map of genus 2, 3, 4, 5, 6, 9, 13, 23, 24, 30, 36, 47, 48, 54, 60, 66, 84, 95, 108, 116, 120, 139, 150, 167, 168, 174, 180, 186 or 198

- There is no regular orientable map of genus 20, 32, 38, 44, 62, 68, 74, 80 or 98 with simple underlying graph

- A lot of these exceptional genera are of the form $p+1$ where p is prime.

Theorems [MC, Jozef Siráň & Tom Tucker (2010)]

• If M is an irreflexible (chiral) orientably-regular map of genus $p+1$ where p is prime, then
either $p \equiv 1 \bmod 3$ and M has type $\{6, 6\}$,
or $p \equiv 1 \bmod 5$ and M has type $\{5, 10\}$,
or $p \equiv 1 \bmod 8$ and M has type $\{8, 8\}$.
In particular, there are no such maps of genus $p+1$ whenever p is a prime such that $p-1$ is not divisible by 3, 5 or 8.

• There is no regular map M with simple underlying graph on an orientable surface of genus $p+1$ where p is a prime congruent to 1 mod 6, for $p > 13$.

In fact, even more ...

We also have a complete classification of all rotary maps M for which $|\langle R, S\rangle|$ is coprime to the Euler characteristic χ (when χ is odd) or to $\chi/2$ (when χ is even).

This has all three main results to date as corollaries:

- No chiral orientably-regular maps of genus $p+1$ for primes p not congruent to 1 mod 3, 5 or 8,
- No regular orientable maps with simple underlying graph and genus $p+1$ for primes $p > 13$ congruent to 1 mod 6,
- No non-orientable regular maps of genus $p+2$ for primes $p > 13$ congruent to 1 mod 12.

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How were these things proved?

Let M be a rotary map on an orientable surface of genus g, and suppose $|\langle R, S\rangle|$ is coprime to $g-1$. Then:

- By the Riemann-Hurwitz formula, the type $\{k, m\}$ of M is restricted to one of five different families

- $\langle R, S\rangle$ is almost Sylow-cyclic (ASC) — i.e. every Sylow subgroup of odd order is cyclic, etc.

- Use Suzuki-Wong classification of nonsolvable ASC groups to deduce that $\langle R, S\rangle$ is solvable, except in one family

- Can then easily classify cases where the vertex-stabilizer and face-stabilizer intersect trivially, and use Ito's theorem and Schur's transfer theory to deal with the general case.

Regular Cayley maps (RCMs)

If the underlying graph of the rotary/regular map M is a Cayley graph Cay(G, S) for some group G, such that G acts regularly on the vertices of M (as a group of map automorphisms), then M is a *regular Cayley map* for G

— e.g. every rotary embedding of the n-cube Q_n gives a regular Cayley map for some group of order 2^n complementary to the vertex-stabilizer of order n.

With either definition, the embedding prescribes an order on the generating set S. Certain kinds of orderings give RCMs that are balanced, anti-balanced, or t-balanced for some t.

Recent research on regular Cayley maps

- General theory [Jajcay, Siráň, et al]
- Balanced regular Cayley maps for cyclic, dihedral and generalized quaternion groups [Wang & Feng]
- RCMs for abelian groups [MC, Jajcay & Tucker]
- RCMs of prime valency for abelian, dihedral and dicyclic groups [Kim, Kwon & Lee]
- t-balanced RCMs for cyclic groups [Kwon]
- t-balanced RCMs for semi-dihedral groups [Oh]
- RCMs for dihedral groups [Kovács].

Also the new census of orientably regular maps provides a well-spring of examples for analysis.

Curious theorem:

Let M be a regular Cayley map for a finite cyclic group A. Then M is reflexible if and only if M is anti-balanced (that is, if and only if the ordering of the generating set for A can be written in the form $\ldots x_3{}^{-1}, x_2{}^{-1}, x_1{}^{-1}, x_1, x_2, x_3, \ldots$).
Thus most regular Cayley maps for cyclic groups are chiral!
[Young Soo Kwon, Jozef Siráň & MC (2009)]

This is not as easy to prove as it looks, but is easier with this recent theorem [Tom Tucker & MC]:

If M is a regular Cayley map for a finite cyclic group A of order n, then the generating set S for A can be assumed to contain an element of order n. Moreover ...

Regular Cayley maps for finite cyclic groups

Using the previous theorem and a partial classification of RCMs for finite abelian groups, we can now classify all RCMs for finite cyclic groups (or equivalently, of all 'skew morphisms' of C_n with a generating orbit closed under inverses):

Case (1) applies to $n = 2$ only

Case (2) — the RCM is balanced, but can be represented also as a non-balanced Cayley map for $A \cong C_n$ if and only if n is divisible by the square of an odd prime.

Case (3) — here n is even, and the map is also a balanced RCM for $D_{n/2}$ with various properties determined by the prime factorisation of n.

[Details in a preprint by MC & Tom Tucker]

External symmetries of regular maps

If M is fully regular of type $\{m, k\}$ then its geometric dual M^D (obtainable by interchanging the roles of vertices and faces) is also regular, of type $\{k, m\}$, with Aut $M^D \cong$ Aut M.
This duality can be achieved by taking (a, b, c) to (c, b, a).

Take a walk on a regular map M, and at the each vertex, turn immediately left or immediately right, in an alternating fashion. This is called a Petrie polygon. The Petrie polygons of M are the faces of a new map M^P, the Petrie dual of M.

If M is regular of type $\{m, k\}$ with Petrie polygons of length q, then its Petrie dual M^P is also regular, of type $\{q, k\}$, with Petrie polygons of length m, and Aut $M^P \cong$ Aut M.
This duality can be achieved by taking (a, b, c) to (ac, b, c).

Wilson 'hole' operators

Given a k-valent map M, and any integer e coprime to k, we may define the power map M^e by taking M and replacing the cyclic rotation of edges at each vertex on the surface with the eth power of that rotation.

This stems from the concept of 'holes' for a regular map, introduced by Coxeter (1937) and extended by Wilson (1979) and also by Nedela and Škoviera (1997) in their work on exponents of orientable maps.

If M is regular, then so is M^e, with the same underlying graph, and the same automorphism group. Taking $M \mapsto M^e$ takes the vertex-stabilizing automorphism bc to $(bc)^e$, and the canonical generating triple (a, b, c) to $(a, (bc)^e c, c)$.

The duality operator D and Petrie duality operator P generate a group of order 6 (giving all permutations of the edge-preserving elements a, c and ac). If the map M is invariant under all six of these operators (in the group generated by D and P), then we say M has trinity symmetry.

If the map M is isomorphic to all of its power maps M^e (for e coprime to the valence), then M is invariant under all of the Wilson 'hole' operators, and we say M is kaleidoscopic.

Maps that are regular, kaleidoscopic and have trinity symmetry are in a sense the most highly symmetric of all.

Some very recent research (by Dan Archdeacon, Jozef Širáň, Young Soo Kwon and MC) has investigated these questions:

Do such maps exist? [Yes] And for what valencies?
How large is the group generated by all the map operators?