Lecture 5: Regular and Chiral Abstract Polytopes

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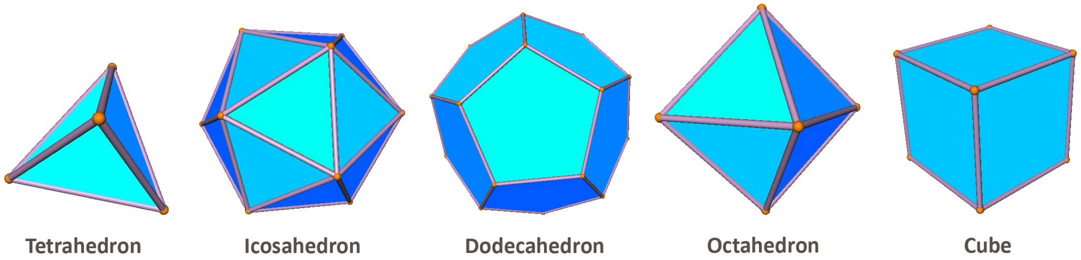
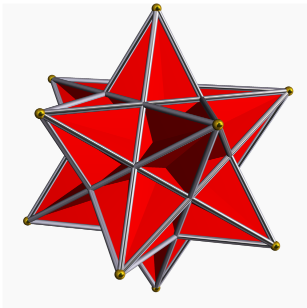
Polytopes

A polytope is a geometric structure with vertices, edges, and (usually) other elements of higher rank, and with some degree of uniformity and symmetry.

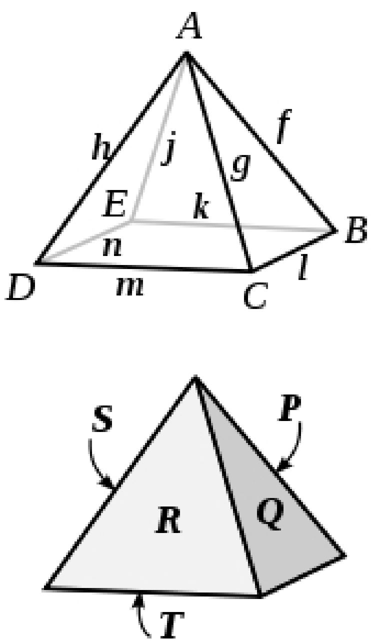
There are many different kinds of polytope, including both convex polytopes like the Platonic solids, and non-convex 'star' polytopes:

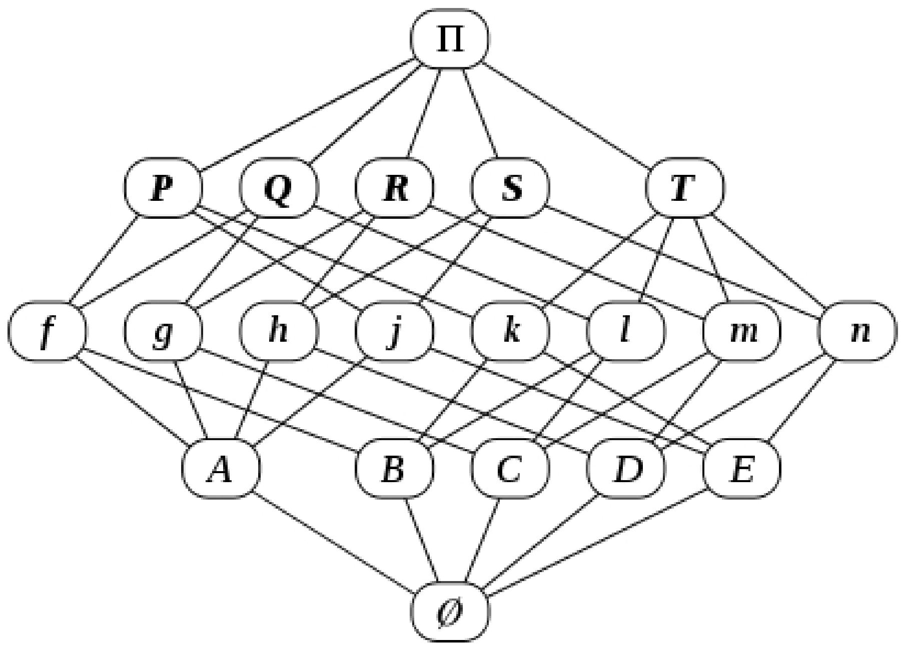




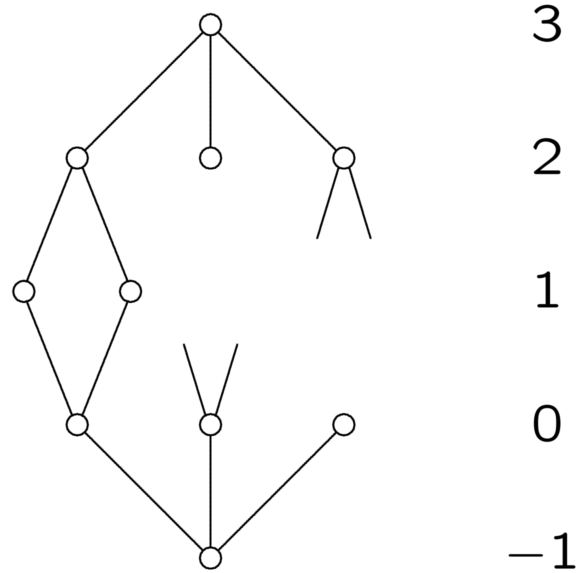
Abstract polytopes

An abstract polytopes is a generalised form of polytope, considered as a partially ordered set:





Definition



An abstract polytope of rank n is a partially ordered set $\mathcal{P}$ endowed with a strictly monotone rank function having range $\{-1, \ldots, n\}$. For $-1 \leq j \leq n$, elements of $\mathcal{P}$ of rank j are called the j-faces, and a typical j-face is denoted by F_j.

This poset must satisfy certain combinatorial conditions which generalise the properties of geometric polytopes.

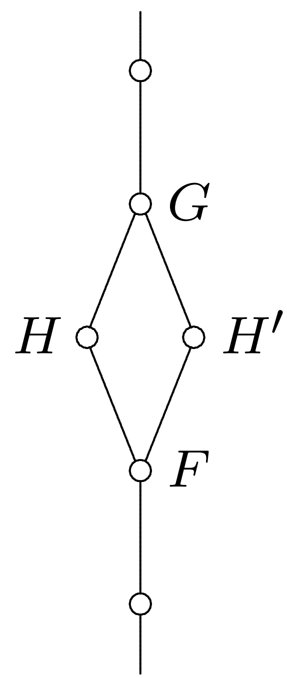
We require that $\mathcal{P}$ has a smallest (-1)-face F_{-1}, and a greatest n-face F_n, and that each maximal chain (or flag) of $\mathcal{P}$ has length $n+2$, e.g. $F_{-1} - F_0 - F_1 - F_2 - ... - F_{n-1} - F_n$.

The faces of rank $0, 1$ and $n-1$ are called the vertices, edges and facets of the polytope, respectively.

Two flags are called adjacent if they differ by just one face.

We require that $\mathcal{P}$ is strongly flag-connected, that is, any two flags Φ and Ψ of $\mathcal{P}$ can be joined by a sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$ such that each two successive faces Φ_{i-1} and Φ_i are adjacent, and $\Phi \cap \Psi \subseteq \Phi_i$ for all i.

Finally, we require the following homogeneity property, which is often called the diamond condition:



Whenever $F \leq G$, with rank$(F) = j-1$ and rank$(G) = j+1$, there are exactly two faces H of rank j such that $F \leq H \leq G$.

A little history

Regular maps

Brahana (1927), Coxeter (1948), ...

Convex geometric polytopes

Various (e.g. Coxeter, Grünbaum, et al)

'Non-spherical' polytopes

Grünbaum (1970s)

Incidence polytopes

Danzer & Schulte (1983)

Regular & chiral polytopes

Weber & Seifert (1933), Coxeter, Schulte, Weiss, McMullen, Monson, Leemans, Hubard, Pellicer et al

Relationship with maps

Every abstract 3-polytope is a map, with vertices, edges and faces of the map being 0-, 1- and 2-faces of the polytope.

But the converse is not always true. For a map to be a 3-polytope, the diamond condition must hold, and therefore

- every edge has two vertices (so there are no loops), and
- every edge lies on two faces (so there can be no ‘bridge’).
- given any face f and any vertex v on the boundary of f, there are exactly two edges incident with v and f.

Maps that satisfy these two conditions are called polytopal.

Note that flags of a 3-polytope are essentially the same as flags of a map: incident vertex-edge-face triples (v, e, f).

The same goes for automorphisms ...

Symmetries of abstract polytopes

An automorphism of an abstract polytope $\mathcal{P}$ is an order-preserving bijection $\mathcal{P} \to \mathcal{P}$.

Just as for maps, every automorphism is uniquely determined by its effect on any given flag. Why?

Suppose Φ is any flag $F_{-1} - F_0 - F_1 - F_2 - ... - F_{n-1} - F_n$, and α is any the automorphism of $\mathcal{P}$. Then for $0 \le i \le n{-}1$, the diamond condition tells us there are unique flags Φ^i and $(\Phi^\alpha)^i$ adjacent to Φ and Φ^α respectively, and differing in only the i-face, and it follows that α takes Φ^i to $(\Phi^\alpha)^i$. Then by strong flag connectedness, we know how α acts on every flag, and hence on every element of $\mathcal{P}$.

Regular polytopes

The number of automorphisms of an abstract polytope $\mathcal{P}$ is bounded above by the number of flags of $\mathcal{P}$.

When the upper bound is attained, we say that $\mathcal{P}$ is regular:

An abstract polytope $\mathcal{P}$ is regular if its automorphism group $\mathrm{Aut}\,\mathcal{P}$ is transitive (and hence regular) on the flags of $\mathcal{P}$.

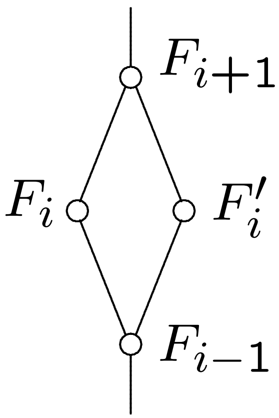
This is analogous to the definition of regular maps (although the latter is often weakened to include the case of orientable but irreflexible maps with the largest possible number of orientation-preserving automorphisms ... the chiral case).

Involutory 'swap' automorphisms

Let $\mathcal{P}$ be a regular abstract polytope, and let Φ be any flag $F_{-1} - F_0 - F_1 - F_2 - ... - F_{n-1} - F_n$. Call this the base flag.

For $0 \leq i \leq n-1$, there is an automorphism ρ_i that maps Φ to the adjacent flag Φ^i (differing from Φ only in its i-face).

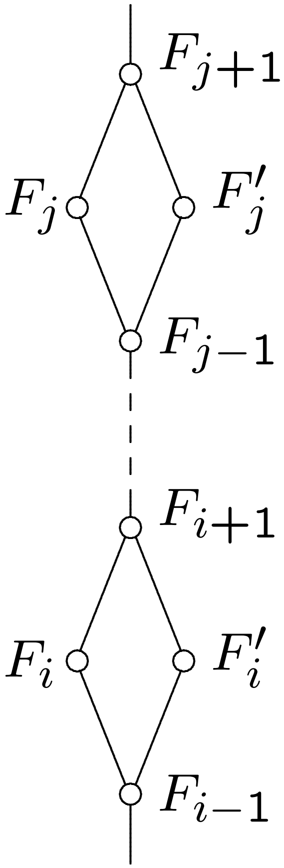
Then also ρ_i also takes Φ^i to Φ (by the diamond condition), so ρ_i swaps Φ with Φ^i, hence ${\rho_i}^2$ fixes Φ, so ρ_i has order 2:



. . . ρ_i swaps F_i with F_i'
and fixes every other F_j

Properties of the swap automorphisms

First, ρ_i commutes with ρ_j whenever $|j-i| \geq 2$:



. . . ρ_j swaps F_j with F'_j
and fixes all other F_k

. . . ρ_i swaps F_i with F'_i
and fixes all other F_k

Second, for any two i and j, the conjugate $\rho_i\rho_j\rho_i$ of ρ_j by ρ_i takes Φ^i to the flag $(\Phi^i)^j$ that is adjacent to Φ^i and differs from Φ^i in the j-face, and therefore differs from Φ in the i- and j-faces. By induction (and strong flag connectedness), the group generated by the automorphisms $\rho_0, \rho_1, \ldots, \rho_{n-1}$ is transitive on flags, and hence equals $\mathrm{Aut}\,\mathcal{P}$.

Third, consider the product $\rho_{i-1}\rho_i$, and let k_i be its order.

This element fixes the $(i-2)$-face F_{i-2} and the $(i+1)$-face F_{i+1} of Φ, and induces a cycle of length k_i on i-faces and a similar cycle of length k_i on $(i-1)$-faces in the 2-section

$$[F_{i-2}, F_{i+1}] = \{\, F \in \mathcal{P} \mid F_{i-2} \leq F \leq F_{i+1} \,\},$$

which is like a regular polygon with k_i vertices and k_i edges.

Connection with Coxeter groups

We have seen that the automorphism group Aut $\mathcal{P}$ of our regular polytope of rank n (or 'n-polytope') $\mathcal{P}$ is generated by the 'swap' automorphisms $\rho_0, \rho_1, \ldots, \rho_{n-1}$, which satisfy the following relations

- $\rho_i^2 = 1$ for $0 \le i \le n-1$,
- $(\rho_{i-1}\rho_i)^{k_i} = 1$ for $1 \le i \le n-1$,
- $(\rho_i\rho_j)^2 = 1$ for $0 \le i < i+1 < j \le n-1$.

These are precisely the defining relations for the Coxeter group $[\,k_1, k_2, ..., k_{n-1}]$ (with Schläfli symbol $\{\,k_1\,|\,k_2\,|\,..\,|\,k_{n-1}\}$). In particular, Aut $\mathcal{P}$ is a quotient of this Coxeter group.

Note: $[k, m]$ Coxeter group $\equiv$ full $(2, k, m)$ triangle group

Stabilizers and cosets

$$\begin{aligned}
\mathrm{Stab}_{\mathrm{Aut}\,\mathcal{P}}(F_0) &= \langle \rho_1, \rho_2, \rho_3, \ldots, \rho_{n-2}, \rho_{n-1} \rangle \\
\mathrm{Stab}_{\mathrm{Aut}\,\mathcal{P}}(F_1) &= \langle \rho_0, \rho_2, \rho_3, \ldots, \rho_{n-2}, \rho_{n-1} \rangle \\
\mathrm{Stab}_{\mathrm{Aut}\,\mathcal{P}}(F_2) &= \langle \rho_0, \rho_1, \rho_3, \ldots, \rho_{n-2}, \rho_{n-1} \rangle \\
&\vdots \\
\mathrm{Stab}_{\mathrm{Aut}\,\mathcal{P}}(F_{n-2}) &= \langle \rho_0, \rho_1, \rho_2, \ldots, \rho_{n-3}, \rho_{n-1} \rangle \\
\mathrm{Stab}_{\mathrm{Aut}\,\mathcal{P}}(F_{n-1}) &= \langle \rho_0, \rho_1, \rho_2, \ldots, \rho_{n-3}, \rho_{n-2} \rangle
\end{aligned}$$

As $\mathcal{P}$ is flag-transitive, Aut $\mathcal{P}$ acts transitively on i-faces for all i, so i-faces can be labelled with cosets of $\mathrm{Stab}_{\mathrm{Aut}\,\mathcal{P}}(F_i)$, for all i, and incidence is given by non-empty intersection.

Also this can be reversed, giving a construction for regular polytopes from smooth quotients of (string) Coxeter groups, as for regular maps, but under certain extra assumptions ...

The Intersection Condition

When $\mathcal{P}$ is regular, the generators ρ_i for Aut $\mathcal{P}$ satisfy an extra condition known as the intersection condition, namely

$$\langle \rho_i : i \in I \rangle \cap \langle \rho_i : i \in J \rangle = \langle \rho_i : i \in I \cap J \rangle$$

for every two subsets I and J of the index set $\{0, 1, \ldots, n-1\}$.

Conversely, this condition on generators $\rho_0, \rho_1, \ldots, \rho_{n-1}$ of a quotient of a Coxeter group $[k_1, k_2, .., k_{n-1}]$ ensures the diamond condition and strong flag connectedness. Hence:

If G is a finite group generated by n elements $\rho_0, \rho_1, \ldots, \rho_{n-1}$ which satisfy the defining relations for a string Coxeter group of rank n, with orders of the ρ_i and products $\rho_i\rho_j$ preserved, and these generators ρ_i satisfy the intersection condition, then there exists a regular polytope $\mathcal{P}$ with Aut $\mathcal{P} \cong G$.

Infinite families of regular polytopes

There are many families of regular polytopes, including these:

- Regular n-simplex, type $[3, \overset{n-1}{\ldots}, 3]$, autom group S_{n+1}
- Cross polytope (or n-orthoplex), type $[3, \overset{n-2}{\ldots}, 3, 4]$
- n-dimensional cubic honeycomb, type $[4, 3, \overset{n-2}{\ldots}, 3, 4]$

Other examples and families (including regular maps) can be constructed from smooth quotients of Coxeter groups, as described earlier.

The 'rotation subgroup' of a regular polytope

In the group $\mathrm{Aut}\,\mathcal{P} = \langle \rho_0, \rho_1, \ldots, \rho_{n-2}, \rho_{n-1} \rangle$, we may define

$$\sigma_j = \rho_{j-1}\rho_j \quad \text{for} \quad 1 \leq j \leq n-1.$$

These generate a subgroup of index 1 or 2 in $\mathrm{Aut}\,\mathcal{P}$, containing all all words of even length in $\rho_0, \rho_1, \ldots, \rho_{n-2}, \rho_{n-1}$. This subgroup may be denoted by $\mathrm{Aut}^+\mathcal{P}$, or $\mathrm{Aut}^o\mathcal{P}$.

If the index is 1, then $\mathrm{Aut}^+\mathcal{P} = \mathrm{Aut}\,\mathcal{P}$ has a single orbit on flags of $\mathcal{P}$, but if the index is 2, then $\mathrm{Aut}^+\mathcal{P}$ has two orbits on flags, with adjacent flags in different orbits.

Note also that ${\sigma_1}^{\rho_0} = \rho_0^{-1}(\rho_0\rho_1)\rho_0 = \rho_1\rho_0 = {\sigma_1}^{-1}$,

and similarly ${\sigma_2}^{\rho_0} = \rho_0(\rho_1\rho_2)\rho_0 = \rho_0\rho_1\rho_0\rho_2 = {\sigma_1}^2\sigma_2$,

while ${\sigma_i}^{\rho_0} = (\rho_{i-1}\rho_i)^{\rho_0} = \rho_{i-1}\rho_i = \sigma_i$ for $3 \leq i \leq n-1$.

Chirality

If the map M is orientable and has maximum rotational symmetry but admits no reflections, then its automorphism group has at least two orbits on flags (with adjacent flags being in different orbits), and the map is chiral.

This can be generalised: an abstract n-polytope $\mathcal{P}$ is said to be chiral if its automorphism group has two orbits on flags, with adjacent flags being in distinct orbits.

In this case, for each flag $\Phi = \{F_{-1}, F_0, \ldots, F_n\}$, there exist automorphisms $\sigma_1, \ldots, \sigma_{n-1}$ such that each σ_j fixes all faces in $\Phi \setminus \{F_{j-1}, F_j\}$, and cyclically permutes j-faces of $\mathcal{P}$ in the rank 2 section $[F_{j-2}, F_{j+1}] \; = \; \{\, F \in \mathcal{P} \mid F_{j-2} \leq F \leq F_{j+1}\}$.

Now given any base flag $\Phi = \{F_{-1}, F_0, \ldots, F_n\}$ of $\mathcal{P}$, the automorphisms $\sigma_1, \ldots, \sigma_{n-1}$ described above may be chosen such that σ_i takes Φ to the flag $\Phi^{i,i-1}$ (which differs from Φ in only its $(i-1)$- and i-faces), for $1 \leq i < n$.

Whenever $i < j$, the automorphism $\sigma_i \sigma_{i+1} \ldots \sigma_j$ fixes all of Φ except its $(i-1)$- and j-faces, and so has order 2.

Hence for a chiral n-polytope $\mathcal{P}$, these automorphisms σ_i generate $\mathrm{Aut}\,\mathcal{P}$, and satisfy (among others) the relations

$$(\sigma_i \sigma_{i+1} \ldots \sigma_j)^2 = 1 \quad \text{for} \quad 1 \leq i < j < n,$$

which are defining relations for the orientation-preserving subgroup of the Coxeter group $[k_1, \ldots, k_{n-1}]$, namely the subgroup generated by the elements $\sigma_i = \rho_{i-1}\rho_i$ for $1 \leq i < n$.

Conversely, if G is any finite group generated by elements $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ satisfying these relations, with orders of the σ_i and products $\sigma_i \sigma_{i+1} \ldots \sigma_j$ preserved, and these σ_i satisfy a modified version of the intersection condition, then there exists an abstract n-polytope $\mathcal{P}$ of type $[k_1, \ldots, k_{n-1}]$ which is regular or chiral, with $G \cong \mathrm{Aut}\,\mathcal{P}$ if $\mathcal{P}$ is chiral, or $G \cong \mathrm{Aut}^+\mathcal{P}$ of index 2 in $\mathrm{Aut}\,\mathcal{P}$ if $\mathcal{P}$ is regular.

Moreover, the polytope $\mathcal{P}$ is *regular* if and only if there exists an involutory group automorphism $\rho\colon \mathrm{Aut}\,\mathcal{P} \to \mathrm{Aut}\,\mathcal{P}$ such that $\rho(\sigma_1) = {\sigma_1}^{-1}$, $\rho(\sigma_2) = {\sigma_1}^2\sigma_2$, and $\rho(\sigma_i) = \sigma_i$ for $3 \le i \le n-1$ (or in other words, acting like conjugation by the generator ρ_0 in the regular case).

Chiral polytopes (for which no such ρ exists) occur in pairs, with one being the 'mirror image' of the other.

Duality

The dual of an n-polytope $\mathcal{P}$ is the n-polytope $\mathcal{P}^*$ obtained from $\mathcal{P}$ by reversing the partial order. The polytope $\mathcal{P}$ is called self-dual if $\mathcal{P} \cong \mathcal{P}^*$. In that case an incidence-reversing bijection $\delta\colon \mathcal{P} \to \mathcal{P}$ is called a duality.

If $\mathcal{P}$ is a chiral n-polytope, the reverse of a flag can lie in either one of two flag orbits. We say that $\mathcal{P}$ is properly self-dual if there exists a duality of $\mathcal{P}$ mapping a flag Φ to a flag Φ^δ in the same orbit as Φ (under $\mathrm{Aut}\,\mathcal{P}$), or improperly self-dual if $\mathcal{P}$ has a duality mapping the flag Φ to a flag in the other orbit of $\mathrm{Aut}\,\mathcal{P}$.

For 3-polytopes considered as maps, the polytope dual is a mirror image of the map dual. Hence the map is self-dual (as a map) iff it is improperly self-dual as a 3-polytope.

Finding chiral polytopes

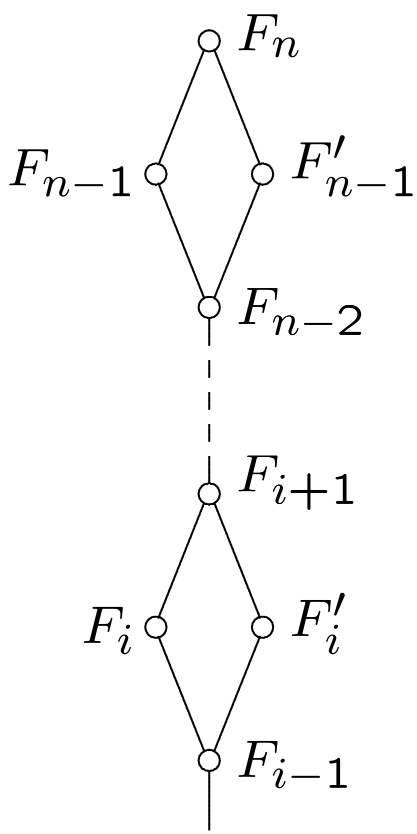
Chiral polytopes appear to be much more rare than regular polytopes, which is surprising since they have a smaller degree of symmetry. This may just hold for small examples, or for small ranks, or of course it could be simply that we don't know enough examples!

Chiral polytopes can be constructed from string Coxeter groups, or by using other algebraic/combinatorial/geometric methods (e.g. building 'new' ones from old).

More on that in tomorrow's lecture, after just one more observation, about the apparent impossibility of building chiral polytopes from one rank to the next ...

Drawback to inductive construction(s)

If $\mathcal{P}$ is a chiral n-polytope, then the stabilizer in Aut$\,\mathcal{P}$ of each $(n-2)$-face F_{n-2} of $\mathcal{P}$ is transitive on the flags of F_{n-2}, and therefore every $(n-2)$-face of $\mathcal{P}$ is regular!



... σ swaps F_{n-1} with F'_{n-1}

... σ swaps F_i with F'_i