Cubical Cohomology Ring: Algorithmic Approach

Tomasz Kaczynski*
Université de Sherbrooke
and
Marian Mrozek†
Jagiellonian University in Kraków and WSB-NLU in Nowy Sącz

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Abstract

A cohomology ring algorithm in a dimension-independent framework of combinatorial cubical complexes is developed with the aim of applying it to the topological analysis of high-dimensional data. This approach is convenient in the cup-product computation and motivated, among others, by interpreting pixels or voxels in digital images as cubes. The S-complex theory and so called co-reductions are adopted to build a cohomology ring algorithm speeding up the algebraic computations.

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1 Introduction

In the past two decades, the homology and cohomology theories gained a vivid attention outside of the mathematics community prompted by its modern applications in sciences and engineering. The development of computational approach to these theories is motivated, among others, by problems in dynamical systems [24], material science [6, 10, 14], electromagnetism [12, 11, 19, 31], geometric modeling [13, 20], image understanding and digital image processing [1, 2, 3, 18, 23, 29], and point-cloud data analysis [5, 8]. Conversely, that development is enabled by the progress in computer science. Although algebraic topology has raised from applications and has been thought of as a computable tool at its early stage, its practical implementation had to wait until the modern generation of powerful computers due to a very high complexity of operations, involved especially in high-dimensional problems.

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Until recently, the main progress has been done in computation of homology groups of finitely representable structures. The software libraries CHomP [4] and RedHom [32] are examples of a systematic approach to computing homology in arbitrary dimensions and for a variety of data sets. Examples of programs with more specific aims are GAP package HAP [15], PLEX [7], and Dionysus [25]. Many applied problems are related to the topological analysis of data which express well in cubical grids. For example, in [2, 29] pixels and voxels are represented by unit squares or cubes in digital images. This naturally leads to the framework of cubical sets and cubical homology presented in [21]. In this paper, we shall work in the same framework but extending the material to dual cochain groups.

Cohomology theory, not less important than homology from the point of view of applications but intrinsically more difficult, had to wait longer for computer implementations. Wherever a mathematical model was making it possible as, for example, in the case of orientable manifolds, the duality has been used to avoid explicitly working with cohomology. This approach is observable in works on computational electromagnetism such as [19, 31, 12]. However, among features distinguishing cohomology from homology is the cup product, which renders a ring structure on the cohomology. The cup product is a difficult concept which has been more challenging to make it explicit enough for computer programs than homology or cohomology groups. Some of significant application-oriented work on computing the cohomology ring of simplicial complexes is done by Real *at. al.* [17].

The notion of the singular cubical homology and cohomology and a cubical cup product formula were first introduced in 1951 by Serre [33, Chapitre 2]. However, we wish to emphasize that this pioneer approach is far from the combinatorial and application–driven spirit of our work for three primary reasons: First, because singular cubes are defined as equivalence classes of all continuous functions from the standard cube $[0,1]^d$ to a given topological space. Secondly, because Serre, in his original work, does not directly derive algebraic properties of the singular cubical cohomology ring by arguments within his theory: he only refers to the isomorphism between the singular and simplicial cohomologies. Finally, because the Serre's result on this topic is hidden as a part of a highly theoretical work addressed to lecturers with a deep pure mathematics background and beyond the reach of most of the computer engineering community for instance. For such reasons, authors working on applications of cohomology to 3D digital images e.g. [17, 16] in the framework of $3D$ cellular cubical complexes tend to derive the needed cubical formulas from the simplicial theory rather than from Serre's work. Our philosophy is based on observation, that the combinatorial cubical complexes presented in [21] are a more friendly framework to directly derive explicit formulas such as the cup product formula for instance and to implement them in dimension–independent algorithms than the simplicial or singular setting.

Let us recall the general definition of the cup product used in standard literature on homological algebra [30].

Definition 1.1 The *cup product* $\smile : H^p(X) \times H^q(X) \to H^{p+q}(X)$ is defined on cohomology classes of cocycles z^p and z^q as follows.

$$[z^p] \smile [z^q] := \mathrm{diag}^*([z^p] \times [z^q]),$$

where $[z^p] \times [z^q]$ is the *cohomology cross product* and diag^* is the homomorphism induced by the *diagonal map* $\mathrm{diag} : X \to X \times X$ given by $\mathrm{diag}(x) := (x, x)$.

The main goal of this paper is to provide an explicit formula for computing the cup product, when X is a cubical set. We will see that, in the context of cubical sets, the cochain cross product

$$\times : C^p(X) \times C^q(Y) \to C^{p+q}(X \times Y)$$

is simply the dual of the cubical product

$$\diamond\ : C_p(\mathbb{R}^{d_1}) \times C_q(\mathbb{R}^{d_2}) \to C_{p+q}(\mathbb{R}^{d_1+d_2})$$

introduced in [21], restricted to cubical sets $X \in \mathbb{R}^{d_1}$ and $Y \in \mathbb{R}^{d_2}$. The concept of cross product is much easier and more natural in the context of cubical sets than for simplical or singular complexes, because the cartesian product of generating cubes is again a generating cube. This is not true for simplices. These considerations lead us to Definition 2.16 of the cubical cup product in Section 2.3.

In order to make the formula for the cup product explicit, we need to derive an explicit formula for a chain map $\mathrm{diag}_\# : \mathcal{C}(X) \to \mathcal{C}(X \times X)$ induced by the diagonal map. Actually, this task is more complex than it may seem at the first glance and Section 2.2 is devoted mainly to the related constructions.

Note that the choice of a chain map is not unique, thus the correctness of the definition and the properties of the cup product are achieved at the cohomology level. These properties are discussed in Section 2.3. The explicit formula for computing the cubical cup product on generating elementary cubes of a cubical set $X \in \mathbb{R}^d$ is developed by induction with respect to the embedding dimension d (Theorem 2.22 and Theorem 2.24). The coordinate-wise formula is presented in Corollary 2.26. We end the section with an example illustrating the use of the formula.

Cubical sets arising from large data sets which are present in applications are built of a huge number of generating elementary cubes. In order to benefit from the established formula in this context, one needs reduction algorithms which render the computation efficient. The goal of Section 3 is to show that the techniques of S-reductions of S-complexes successfully developed in [26, 28] with the purpose of computing homology of large cubical complexes may be adapted in computing the cohomology ring of a cubical set. The terminology of S-complexes, S-reduction pairs, the coreduction algorithm and the concept of homology models are reviewed and adapted for cohomology. We finish the paper with Section 3.5, where computations via S-reductions are carried through and compared on two explicit examples.

The implementation of the methods presented in this paper and experimentation is a joint work with P. Dłotko, currently in progress.

2 Cubical Cohomology

2.1 Cubical cohomology groups

Recall from [21, Chapter 2] that $X \subset \mathbb{R}^d$ is a *cubical set* if it is a finite union of *elementary cubes*

$$Q = I_1 \times I_2 \times \cdots \times I_d \subset \mathbb{R}^d$$

where I_i is an interval of the form $I = [k, k+1]$ (non-degenerate) or $I = [k, k]$ (degenerate) for some $k \in \mathbb{Z}$. For short, $[k] := [k, k]$. The dimension $\dim Q$ of Q is the number of non-degenerate intervals in its product expression and the embedding number emb Q is d. The set of all elementary cubes in $\mathbb{R}^d$ is denoted by $\mathcal{K}(\mathbb{R}^d)$ and those of dimension k by $\mathcal{K}_k(\mathbb{R}^d)$. Those which are contained in X are denoted by $\mathcal{K}(X)$, respectively, $\mathcal{K}_k(X)$.

The group $C_k(\mathbb{R}^d)$ of *cubical k-chains* is the free abelian group generated by $\mathcal{K}_k(\mathbb{R}^d)$, its *canonical basis*. For $k < 0$ and $k > d$, we set $C_k(\mathbb{R}^d) := 0$. We abandon here the distinction made in [21] between the geometric cubes $Q \in \mathcal{K}(\mathbb{R}^d)$ and their algebraic dual chains $\widehat{Q}$ so to focus on the duality in the sense of cochains. We recall that the *cubical cross product*

$$\times : C_p(\mathbb{R}^n) \times C_q(\mathbb{R}^m) \to C_{p+q}(\mathbb{R}^{n+m})$$

is defined on the canonical basis elements $P \in \mathcal{K}_p^n$ and $Q \in \mathcal{K}_q^m$ as the cartesian product $P \times Q$ and extended on all pairs of chains (c, c') by bilinearity. In [21], this operation is called *cubical product* and denoted by $c \diamond c'$ in order to distinguish it from the cartesian product but we abandon this notation here to emphasize its equivalence to the cross product in homological algebra.

Given $k \in \mathbb{Z}$, the *cubical boundary map* $\partial_k : C_k(\mathbb{R}^d) \to C_{k-1}(\mathbb{R}^d)$ is a homomorphism defined on the elements Q of the basis $\mathcal{K}_k(\mathbb{R}^d)$ by induction on d as follows.

For $d = 1$ put $\partial_0 Q = 0$ if $Q = [a] \in \mathcal{K}_0(\mathbb{R}^d)$, and $\partial_1 Q := [a+1] - [a]$ if $Q = [a, a+1] \in \mathcal{K}_1(\mathbb{R}^d)$ for some $a \in \mathbb{Z}$.

For $d > 1$ decompose Q as $Q = I \times P$, where emb $I = 1$ and emb $P = d - 1$, and put

$$\partial_k Q := \partial_p I \times P + (-1)^p I \times \partial_q P, \tag{1}$$

where $p = \dim I$ and $q = \dim P$. The pair $(\mathcal{C}(\mathbb{R}^d), \partial) := \{(C_k(\mathbb{R}^d), \partial_k)\}_{k \in \mathbb{Z}}$ is called the *cubical chain complex of* $\mathbb{R}^d$. We refer to [21, Chapter 2] for the properties of the cubical cross product and cubical boundary maps, in particular for this one:

Proposition 2.1 [21, Proposition 2.34] *For any $c \in C_p(\mathbb{R}^n)$, and $c' \in C_q(\mathbb{R}^m)$*

$$\partial_{p+q}(c \times c') = \partial_p c \times c' + (-1)^p c \times \partial_q c'.$$

In the virtue of the above formula, the cubical cross product induces the isomorphism

$$C_p(\mathbb{R}^n) \otimes C_q(\mathbb{R}^m) \cong C_{p+q}(\mathbb{R}^{n+m})$$

(for the definition of the tensor product of chain complexes see e. g. [30, Chapter 7]).

For any $c, d \in C_p(\mathbb{R}^d)$, the notation $\langle c, d\rangle$ is used for the scalar product defined on the elements P, Q of the canonical basis $\mathcal{K}_p$ by

$$\langle P, Q\rangle := \begin{cases} 1 & \text{if } P = Q, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The *support* $|c|$ of c is the union of all $Q \in \mathcal{K}_p$ such that $\langle c, Q\rangle \neq 0$. Given a cubical set $X \in \mathbb{R}^d$, the *cubical chain complex of* X denoted by $\mathcal{C}(X)$ is the restriction of $\mathcal{C}(\mathbb{R}^d)$ to the chains c whose *support* is contained in X. We refer to [21] for the properties of cubical chain complexes and for the computation of their homology.

Definition 2.2 Let $X \in \mathbb{R}^d$ be a cubical set. The *cubical cochain complex* $(\mathcal{C}^*(X), \delta)$ is defined as follows. For any $k \in \mathbb{Z}$, the k–dimensional cochain group

$$C^k(X) = \operatorname{Hom}(C_k(X), \mathbb{Z}),$$

where Hom is the functor assigning to any abelian group G the group of all homomomorphisms from G to $\mathbb{Z}$, called the *dual* of G. Elements of $C^k(X)$ are called *cochains* and denoted either by c^k, d^k or by $c^\star, d^\star$, if we don't need to specify their dimension k. The value of a cochain c^k on a chain d_k is denoted by $\langle c^k, d_k\rangle$. Note that this notation is also used for the scalar product in chain complexes introduced in (2) and in S-complexes in Section 3 but it is easy to figure out from the context which scalar product we mean.

The k'th coboundary map $\delta^k : C^k(X) \to C^{k+1}(X)$ is the dual homomorphism of ∂_{k+1} defined by

$$\langle \delta^k c^k, d_{k+1}\rangle := \langle c^k, \partial_{k+1} d_{k+1}\rangle.$$

Note that $C^k(X)$ is the free abelian group generated by the dual canonical basis $\{Q^\star | Q \in \mathcal{K}_k(X)\}$ where

$$\langle Q^\star, P\rangle := \begin{cases} 1 & \text{if } P = Q, \\ 0 & \text{otherwise.} \end{cases}$$

The notation $Q^\star$, for the dual of Q, is aimed to be distinct from H^* for the cohomology functor.

Definition 2.3 Given a cubical set $X \in \mathbb{R}^d$, the group of k–dimensional *cocycles* of X is $Z^k(X) := \ker \delta^k$, and the group of k–dimensional *coboundaries* of X is $B^k := \operatorname{im} \delta^{k-1}$. The kth *cohomology group* of X is the quotient group

$$H^k(X) := Z^k(X)/B^k(X).$$

Definition 2.4 The *cubical cross product* of cochains $c^p \in C^p(X)$ and $c^q \in C^q(Y)$ is a cochain in $C^{p+q}(X \times Y)$ defined on any elementary cube $R \times S \in \mathcal{K}_{p+q}(X \times Y)$, where $R \in \mathcal{K}(X)$ and $S \in \mathcal{K}(Y)$, as follows.

$$\langle c^p \times c^q, R \times S\rangle := \begin{cases} \langle c^p, R\rangle \cdot \langle c^q, S\rangle & \text{if } \dim R = p \text{ and } \dim S = q, \\ 0 & \text{otherwise.} \end{cases}$$

We easily check the following.

Proposition 2.5 *The cross product of cochains is a bilinear map. Moreover, for $P \in \mathcal{K}_p(X)$, $Q \in \mathcal{K}_q(Y)$,*

$$P^\star \times Q^\star = (P \times Q)^\star.$$

Algebraic properties of the cubical product on cubical chains derived in [21, Section 2.2] readily extend to the cross product on cochains, in particular this one:

Proposition 2.6 *If $c^p \times c^q \in C^{p+q}(X)$, then*

$$\delta(c^p \times c^q) = \delta c^p \times c^q + (-1)^p c^p \times \delta c^q.$$

2.2 Constructing chain maps

The most important step towards an explicit formula for the cup product is the construction of a homology chain map $\operatorname{diag}_\# : \mathcal{C}(X) \to \mathcal{C}(X \times X)$ induced by the diagonal map $\operatorname{diag} : X \to X \times X$ given by $\operatorname{diag}(x) := (x, x)$. This is done using the construction presented in [21, Chapter 6]. We briefly outline that construction. We use the notation $F : X \rightrightarrows Y$ for *cubical set-valued maps* from X to Y introduced there. The Acyclic Selector Theorem [21, Theorem 6.22] affirms that, if such a map has acyclic values, then it admits a *chain selector*, that is, a chain map $\varphi : \mathcal{C}(X) \to \mathcal{C}(Y)$ with the property that, for any vertex $V \in \mathcal{K}_0(X)$, $\varphi(V)$ is a vertex in $F(V)$, and for any elementary cube $Q \in \mathcal{K}(X)$, the support $|\varphi(Q)|$ of $\varphi(Q)$ is contained in $F(Q)$. Any two such chain selectors are chain homotopic. If a continuous map $f : X \to Y$ admits an acyclic-valued *representation*, that is, a cubical map F with the property $f(x) \in F(x)$, the homology map of any chain selector φ of F is the homology map $H_*(f)$ of f. In general, computing homology map may be hard, because an acyclic-valued representation may not always exist and one has to apply a process of *rescaling*. However, our diagonal map does admit an acyclic-valued representation $\operatorname{Diag} : X \rightrightarrows (X \times X)$ given by $\operatorname{Diag}(x) := Q \times Q$, where $Q \in \mathcal{K}(X)$ is the *cubical enclosure* of x, that is, the smallest cubical set containing x, denoted by $\operatorname{ch}(x)$.

Given any maps $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$ on cubical sets, we define $f \times g : X_1 \times X_2 \to Y_1 \times Y_2$ by $(f \times g)(x_1, x_2) := (f(x_1), g(x_2))$. Given $F_i : X_i \rightrightarrows Y_i$ for $i = 1, 2$, we define $F_1 \times F_2 : X_1 \times X_2 \rightrightarrows Y_1 \times Y_2$ by $(F_1 \times F_2)(x, y) := F_1(x) \times F_2(y)$.

Given chain maps $\varphi : \mathcal{C}(X_1) \to \mathcal{C}(Y_1)$ and $\psi : \mathcal{C}(X_2) \to \mathcal{C}(Y_2)$, their *cubical tensor product* is defined on generators $Q = Q_1 \times Q_2 \in \mathcal{K}_k(X_1 \times X_2)$ as follows. Set $p := \dim Q_1$ and $q := \dim Q_2$. Note that $p + q = k$. Put

$$(\varphi \otimes \psi)_k(Q) := \varphi_p(Q_1) \times \psi_q(Q_2). \tag{3}$$

Lemma 2.7 *Let X_1, X_2, Y_1, Y_2 be cubical sets and let $f : X_1 \to Y_1$, $g : X_2 \to Y_2$ be continuous maps which admit acyclic-valued representations F, G. Let φ and ψ be the chain selectors of F and, respectively, G.*

(a) *The set-valued map $F \times G$ is an acyclic-valued cubical representation of $f \times g$.*

(b) *The chain map $\varphi \otimes \psi$ is a chain selector of $F \times G$.*

Proof: The statement in (a) is straightforward because the product of acyclic sets is acyclic. To prove (b), we start form showing that $\varphi \otimes \psi$ is a chain map. Given any $Q = Q_1 \times Q_2 \in \mathcal{K}_k(X_1 \times X_2)$, we have

$$\begin{aligned}
\partial(\varphi \otimes \psi)_k(Q) &= \partial\varphi_p(Q_1)\times\psi_q(Q_2) + (-1)^p\varphi_p(Q_1)\times\partial\psi_p(Q_2) \\
&= \varphi_{p-1}(\partial Q_1)\times\psi_q(Q_2) + (-1)^p\varphi_p(Q_1)\times\psi_{q-1}(\partial Q_2) \\
&= (\varphi \otimes \psi)_{k-1}\left(\partial Q_1\times Q_2 + (-1)^{\dim Q_1} Q_1\times\partial Q_2\right) \\
&= (\varphi \otimes \psi)_{k-1}(\partial Q).
\end{aligned}$$

It remains to check the chain selector conditions (6.12),(6.13) in [21, Theorem 6.22]. First, we have

$$|(\varphi \otimes \psi)_k(Q)| = |\varphi_p(Q_1)| \times |\psi_q(Q_2)| \subset F(Q_1) \times G(Q_2).$$

Finally, for any vertex $V = V_1 \times V_2 \in \mathcal{K}_0(X_1 \times X_2)$, $W_1 := \varphi(V_1)$ and $W_2 := \psi(V_2)$ are vertices, hence

$$(\varphi \otimes \psi)_k(V) = W_1 \times W_2 \in \mathcal{K}_0(X).$$

■

The following statement is easily verified.

Proposition 2.8 *Let $X_1 \subset R^n$ and $X' \subset \mathbb{R}^{d-n}$ be the images of a cubical set $X \subset \mathbb{R}^d$ under its projections onto, respectively, the first n and the complementary $d - n$ coordinates in $\mathbb{R}^d$. Consider the inclusion map $j : X \hookrightarrow X_1 \times X_2$. Then*

(a) *The map $J : X \rightrightarrows X_1 \times X_2$ given by $J(x) := \mathrm{ch}\,(x)$ is an acyclic-valued cubical representation of j;*

(b) *The inclusion of chain complexes $\iota : \mathcal{C}(X) \hookrightarrow \mathcal{C}(X_1 \times X_2)$ is a chain selector of J.*

Theorem 2.9 *Let X, Y be cubical sets and let $\lambda : X \times Y \to Y \times X$ be the transpose given by $\lambda(x, y) := (y, x)$.*

(a) *The map $\Lambda : X \times Y \rightrightarrows Y \times X$ given by $\Lambda(x, y) := Q_2 \times Q_1$, where $Q_1 := \mathrm{ch}\,(x)$ and $Q_2 := \mathrm{ch}\,(y)$, is an acyclic-valued cubical representation of λ;*

(b) *Let $\lambda_\# : \mathcal{C}(X \times Y) \to C(Y \times X)$ be the map defined on generators $Q = Q_1 \times Q_2 \in \mathcal{K}_k(X \times Y)$ by*

$$\lambda_k(Q) := (-1)^{\dim Q_1 \dim Q_2} Q_2 \times Q_1.$$

Then $\lambda_\#$ is a chain selector of Λ.

Proof: The statement (a) is obvious. For (b), the conditions (6.12),(6.13) in [21, Theorem 6.22] follow instantly from the definitions, so it remains to check that $\lambda_\#$ is a chain map, that is, it commutes with the boundary map. Let $Q = Q_1 \times Q_2 \in \mathcal{K}_k(X \times Y)$. On one hand, we have

$$\begin{aligned}\partial\lambda_k(Q) &= (-1)^{\dim Q_1 \dim Q_2}\partial(Q_2\times Q_1)\\ &= (-1)^{\dim Q_1 \dim Q_2}\left(\partial Q_2\times Q_1 + (-1)^{\dim Q_2}Q_2\times\partial Q_1\right)\\ &= (-1)^{\dim Q_1 \dim Q_2}\partial Q_2\times Q_1 + (-1)^{(\dim Q_1+1)\dim Q_2}Q_2\times\partial Q_1.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\lambda_{k-1}(\partial Q) &= \lambda_{k-1}\left(\partial Q_1\times Q_2 + (-1)^{\dim Q_1}Q_1\times\partial Q_2\right)\\ &= (-1)^{\dim \partial Q_1 \dim Q_2}Q_2\times\partial Q_1 + (-1)^{\dim Q_1+\dim Q_1\dim\partial Q_2}\partial Q_2\times Q_1\\ &= (-1)^{(\dim Q_1-1)\dim Q_2}Q_2\times\partial Q_1 + (-1)^{\dim Q_1+\dim Q_1(\dim Q_2-1)}\partial Q_2\times Q_1\\ &= (-1)^{(\dim Q_1+1)\dim Q_2}Q_2\times\partial Q_1 + (-1)^{\dim Q_1\dim Q_2}\partial Q_2\times Q_1.\end{aligned}$$

Hence $\partial\lambda_k(Q) = \lambda_{k-1}(\partial Q)$. ■

From Proposition 2.8 and Theorem 2.9 we derive the following corollary.

Corollary 2.10 *Let X_1, X_2, Y_1, Y_2 be cubical sets and consider the permutation*

$$\tau : (X_1 \times Y_1) \times (X_2 \times Y_2) \to (X_1 \times X_2) \times (Y_1 \times Y_2)$$

given by $\tau(x_1, y_1, x_2, y_2) = (x_1, x_2, y_1, y_2)$. Then the map given by $T(x) := \tau(\mathrm{ch}\,(x))$ is an acyclic-valued representation of τ and the map

$$\tau_\# : \mathcal{C}(X_1 \times Y_1 \times X_2 \times Y_2) \to \mathcal{C}(X_1 \times X_2 \times Y_1 \times Y_2)$$

defined on products of $P_1 \in \mathcal{K}(X_1)$, $P_2 \in \mathcal{K}(Y_1)$, $Q_1 \in \mathcal{K}(X_2)$, and $Q_2 \in \mathcal{K}(Y_1)$ by

$$\tau_\#\left((P_1\times P_2)\times(Q_1\times Q_2)\right) := (-1)^{\dim P_2 \dim Q_1}\left(P_1\times Q_1\right)\times\left(P_2\times Q_2\right)$$

is its chain selector.

Here are dual statements of Theorem 2.9 and Corollary 2.10 for cochains.

Corollary 2.11

Let $\lambda : X \times Y \to Y \times X$ be the transpose defined in Theorem 2.9. Then the induced cochain map $\lambda^\# := \mathrm{Hom}\,(\lambda_\#) : \mathcal{C}^*(Y \times X) \to \mathcal{C}^*(X \times Y)$ satisfies

$$\lambda^{p+q}(c^q\times c^p) = (-1)^{pq}c^p\times c^q.$$

Proof: (a). Let $P \in \mathcal{K}(X)$ and $Q \in \mathcal{K}(Y)$. If $\dim P = p$ and $\dim Q = q$, then

$$\begin{aligned}\langle\lambda^{p+q}(c^q\times c^p), P\times Q\rangle &= \langle c^q\times c^p, \lambda_{p+q}(P\times Q)\rangle = \langle c^q\times c^p, (-1)^{pq}Q\times P\rangle\\ &= (-1)^{pq}\langle c^q, Q\rangle\cdot\langle c^p, P\rangle = (-1)^{pq}\langle c^p, P\rangle\cdot\langle c^q, Q\rangle\\ &= (-1)^{pq}\langle c^p\times c^q, P\times Q\rangle.\end{aligned}$$

If $\dim P \neq p$ or $\dim Q \neq q$, then both sides are null. ■

Corollary 2.12 *Let $\tau : (X_1 \times Y_1) \times (X_2 \times Y_2) \to (X_1 \times X_2) \times (Y_1 \times Y_2)$ be the permutation discussed in Corollary 2.10. The map*

$$\tau^{\#} := \mathrm{Hom}\,(\tau_{\#}) : \mathcal{C}^*(X_1 \times X_2 \times Y_1 \times Y_2) \to \mathcal{C}^*(X_1 \times Y_1 \times X_2 \times Y_2)$$

induced by $\tau_{\#}$ is given on products of duals of $P_1 \in \mathcal{K}(X_1)$, $P_2 \in \mathcal{K}(Y_1)$, $Q_1 \in \mathcal{K}(X_2)$, and $Q_2 \in \mathcal{K}(Y_1)$ by the formula

$$\tau^{\#}\left((P_1^\star \times P_2^\star) \times (Q_1^\star \times Q_2^\star)\right) = (-1)^{\dim P_2 \dim Q_1} \left(P_1^\star \times Q_1^\star\right) \times \left(P_2^\star \times Q_2^\star\right).$$

Let $X \subset \mathbb{R}^d$ be a cubical set. We proceed to the construction of a homology-representative chain map $\mathrm{diag}_{\#} : \mathcal{C}(X) \to \mathcal{C}(X \times X)$ induced by the diagonal map. It is straightforward to see that the map diag admits an acyclic-valued representation $\mathrm{Diag} : X \rightrightarrows (X \times X)$ given by $\mathrm{Diag}(x) := Q \times Q$ where $Q = \mathrm{ch}\,(x) \in \mathcal{K}(X)$. The construction of its chain selector goes by induction on $d = \mathrm{emb}\ X$.

Case $d = 1$: If $Q = [v] \in \mathcal{K}_0(X)$, we put

$$\mathrm{diag}_0([v]) := [v] \times [v]. \tag{4}$$

If $Q = [v_0, v_1] \in \mathcal{K}_1(X)$, we put

$$\mathrm{diag}_1([v_0, v_1]) := [v_0] \times [v_0, v_1] + [v_0, v_1] \times [v_1]. \tag{5}$$

In order to show that our formula defines a chain map that induces a map in homology, we need to make two observations.

Lemma 2.13 *Let* $\mathrm{emb}\ X = 1$. *The map* $\mathrm{diag}_{\#}$ *defined by (4) and (5) is a chain selector for* Diag.

Proof: The conditions [21, Theorem 6.22, (6.12),(6.13)] follow instantly from the definitions, so it remains to check that $\mathrm{diag}_{\#}$ is a chain map. Since $\partial_0 = 0$ and $d = 1$, it is enough to check that $\partial_1 \mathrm{diag}_1 = \mathrm{diag}_0 \partial_1$. Let $Q = [v_0, v_1] \in \mathcal{K}_1(X)$. On one hand,

$$\begin{aligned} \partial\, \mathrm{diag}_1([v_0, v_1]) &= 0 + [v_0] \times ([v_1] + [v_0]) - ([v_1] - [v_0]) \times [v_1] + 0 \\ &= [v_1] \times [v_1] - [v_0] \times [v_0]. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathrm{diag}_0(\partial [v_0, v_1]) &= \mathrm{diag}_0([v_1] - [v_0]) = \mathrm{diag}_0([v_1]) - \mathrm{diag}_0([v_o]) \\ &= [v_1] \times [v_1] - [v_0] \times [v_0], \end{aligned}$$

hence the conclusion follows. ■

Remark 2.14 The formula

$$\mathrm{diag}_1^-([v_0, v_1]) := [v_0, v_1] \times [v_0] + [v_1] \times [v_0, v_1]. \tag{6}$$

leads to a different chain selector of Diag and a different definition of a cup product on cochains. Note that the two choices are homologous since

$$\mathrm{diag}_1([v_0, v_1]) - \mathrm{diag}_1^-([v_0, v_1]) = \partial([v_0, v_1] \times [v_0, v_1]).$$

Induction step. Suppose that $\mathrm{diag}_{\#}$ is defined for cubical sets of embedding numbers $n = 1, \ldots, d-1$ and let us construct it for a cubical set X of the embedding number d.

Note that

$$\begin{aligned} \mathrm{diag}_{\mathbb{R}^d}(x_1, \ldots, x_d) &= (x_1, \ldots, x_d, x_1, x_2, \ldots, x_d) \\ &= \tau(x_1, x_1, x_2, \ldots, x_d, x_2, \ldots, x_d) \\ &= \tau(\mathrm{diag}_{\mathbb{R}}(x_1), \mathrm{diag}_{\mathbb{R}^{d-1}}(x_2, \ldots, x_d)), \end{aligned}$$

where $\tau : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is the permutation of coordinates which transposes the $(d+1)$st coordinate with the precessing $d-1$ coordinates. The formula for the chain maps induced by τ is provided by Corollary 2.10.

Consider the images $X_1 \subset \mathbb{R}$ and $X' \subset \mathbb{R}^{d-1}$ of X under its projections of, respectively, the first and the complementary $d-1$ coordinates in $\mathbb{R}^d$. Let diag^{X_1} and $\mathrm{diag}^{X'}$ be the diagonal maps defined, respectively, on X_1 and X'. Then the diagonal map on X is the composition

$$\mathrm{diag} = \tau \circ (\mathrm{diag}^{X_1} \times \mathrm{diag}^{X'}) \circ j. \tag{7}$$

where j is the inclusion map discussed in Proposition 2.8. Note that τ takes values in $X_1 \times X' \times X_1 \times X' \supset X \times X$ but the composition of maps on the left-hand side of (7) takes values in $X \times X$.

Lemma 2.15 *Let* $\mathrm{emb}\ X > 1$. *Define the map* $\mathrm{diag}_{\#} : \mathcal{C}(X) \to \mathcal{C}(X \times X)$ *by the formula*

$$\mathrm{diag}_{\#} := \pi \circ \tau_{\#} \circ (\mathrm{diag}_{\#}^{X_1} \otimes \mathrm{diag}_{\#}^{X'}) \circ \iota, \tag{8}$$

where

1. $\iota : \mathcal{C}(X) \to \mathcal{C}(X_1 \times X')$ *is the inclusion of chain complexes,*
2. $\mathrm{diag}_{\#}^{X_1} : \mathcal{C}(X_1) \to \mathcal{C}(X_1 \times X_1)$ *and* $\mathrm{diag}_{\#}^{X'} \to \mathcal{C}(X' \times X')$ *are defined by the induction hypothesis,*
3. $\mathrm{diag}_{\#}^{X_1} \otimes \mathrm{diag}_{\#}^{X'} : \mathcal{C}(X_1 \times X') \to \mathcal{C}(X_1 \times X_1 \times X' \times X')$ *is defined by the formula (3),*
4. $\tau_{\#} : \mathcal{C}(X_1 \times X_1 \times X' \times X') \to \mathcal{C}(X_1 \times X' \times X_1 \times X')$ *is defined in Corollary 2.10, and*
5. $\pi : \mathcal{C}(X_1 \times X' \times X_1 \times X') \to \mathcal{C}(X \times X)$ *is the projection defined on elementary cubes* $Q \in \mathcal{K}(X_1 \times X' \times X_1 \times X')$ *by*

$$\pi(Q) := \begin{cases} Q & \text{if } Q \in \mathcal{K}(X \times X), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathrm{diag}_{\#}$ *is a chain selector for* Diag.

Proof: We leave to the reader the verification that the composition of the acyclic-valued representations of the maps involved in the formula (7) described in the Lemmas has acyclic values which, moreover, are contained in $X \times X$. It follows from [21, Corollary 6.31]), that the composition of corresponding chain selectors is an induced chain map

$$\left(\tau \circ (\mathrm{diag}^{X_1} \times \mathrm{diag}^{X'}) \circ j\right)_{\#} : \mathcal{C}(X) \to \mathcal{C}(X_1 \times X' \times X_1 \times X').$$

Since, in addition, this chain map has values supported in $X \times X$, the composition with the projection π restricts its range to $\mathcal{C}(X \times X)$ without affecting the selection property. ∎

2.3 Cubical cup product

Definition 2.16 Let X be a cubical set. The *cubical cup product*

$$\smile : C^p(X) \times C^q(X) \to C^{p+q}(X)$$

of cochains c^p and c^q is defined by the formula

$$c^p \smile c^q = diag^{p+q}(c^p \times c^q).$$

In particular, for $Q \in \mathcal{K}_{p+q}(X)$

$$\langle c^p \smile c^q, Q\rangle = \langle c^p \times c^q, \mathrm{diag}_{p+q}(Q)\rangle \tag{9}$$

The following proposition is an easy consequence of Definition 2.16 and the respective properties of the cross product.

Proposition 2.17 [Algebraic properties of the cup product on cochains]

(a) $(c^p \smile c^q) \smile c^r = c^p \smile (c^q \smile c^r)$ *(associative law);*

(b) $(c^p + d^p) \smile c^q = (c^p \smile c^q) + (d^p \smile c^q)$ *(distributive law);*

(c) $1_{C_0(X)} \smile c^p = c^p \smile 1_{C_0(X)} = c^p$ *where* $1_{C_0(X)} := \sum_{V \in \mathcal{K}_0(X)} V^\star$ *(unit element);*

Theorem 2.18 [Bounding properties of the cup product]

(a) $\delta(c^p \smile c^q) = \delta c^p \smile c^q + (-1)^p c^p \smile \delta c^q$.

(b) *If* $z^p \in Z^p(X)$ *and* $z^q \in Z^q(X)$, *then* $z^p \smile z^q \in Z^{p+q}(X)$;

(c) *If* $x^p, y^p \in Z^p(X)$, $x^p - y^p \in B^p(X)$, *and* $z^q \in Z^q(X)$, *then*

$$x^p \smile z^q = y^p \smile z^q \text{ and } z^q \smile x^p = z^q \smile y^p.$$

Proof: (a) Let $Q \in \mathcal{K}_k(X)$, where $k = p + q + 1$.

$$\begin{aligned}
\langle \delta(c^p \smile c^q), Q\rangle &= \langle c^p \smile c^q, \partial Q\rangle = \langle c^p \times c^q, \operatorname{diag}_k \partial Q\rangle \\
&= \langle c^p \times c^q, \partial \operatorname{diag}_{k+1} Q\rangle = \langle \delta(c^p \times c^q), \operatorname{diag}_{k+1} Q\rangle \\
&= \langle \delta c^p \times c^q + (-1)^p c^p \times \delta c^q, \operatorname{diag}_{k+1} Q\rangle \\
&= \langle \delta c^p \smile c^q + (-1)^p c^p \smile \delta c^q, Q\rangle.
\end{aligned}$$

(b) is an immediate consequence of (a).

(c) Let $x^p - y^p = \delta w$. It follows from (a) that

$$(x^p - y^p) \smile z^q = \delta w \smile z^q = \delta(w \smile z^q) = 0$$

because $\delta z^q = 0$. Hence $x^p \smile z^q = y^p \smile z^q$. The second equation follows by the same argument. ■

The property (a) in Theorem 2.18 is the analogy of the bounding property of cubical cross product defined in [21, Section 2.2]. The property (b) asserts that the cup product sends cocycles to cocycles, and by (c) that it doesn't depend on a representative of a cohomology class. Thus, we get the definition:

Definition 2.19 The *cup product* $\smile\, : H^p(X) \times H^q(X) \to H^{p+q}(X)$ is defined on cohomology classes of cocycles as follows.

$$[z^p] \smile [z^q] := [z^p \smile z^q]. \tag{10}$$

The properties of the cup product listed in Proposition 2.17 easily extend to cohomology classes. The following property holds only for cohomology classes.

Theorem 2.20 [Graded-cummutative law]
If z^p and z^q are cocycles, then

$$[z^q] \smile [z^p] = (-1)^{pq}[z^p \smile z^q].$$

Proof: Let λ be the transpose defined in Theorem 2.9, applied for $X = Y$. Let $Q \in \mathcal{K}(X)$. On one hand, we have

$$\langle z^p \smile z^q, Q\rangle = \langle z^p \times z^q, \operatorname{diag}_{p+q}(Q)\rangle.$$

On the other hand, from Corollary 2.11,

$$\begin{aligned}
\langle z^q \smile z^p, Q\rangle &= \langle z^q \times z^p, \operatorname{diag}_{p+q} Q\rangle = (-1)^{pq}\langle \lambda^{p+q}(z^p \times z^q), \operatorname{diag}_{p+q}(Q)\rangle \\
&= (-1)^{pq}\langle z^p \times z^q, \left(\lambda_{p+q} \circ \operatorname{diag}_{p+q}\right)(Q)\rangle.
\end{aligned}$$

Note that $\lambda \circ \operatorname{diag} = \operatorname{diag}$ on X. The corresponding chain maps $\psi_\# = \lambda_\# \circ \operatorname{diag}_\#$ and $\varphi_\# = \operatorname{diag}_\#$ are both chain selectors for Diag. By [21, Theorem 6.25], they are chain homotopic. This implies that the associated cochain maps $\psi^\#$ and $\varphi^\#$ are cochain homotopic (see the definition preceeding Theorem 44.1 in [30]), hence they induce the same homomorphism in cohomology. Thus, we get

$$[z^q] \smile [z^p] = (-1)^{pq} H^*(\psi)[z^p \times z^q] = (-1)^{pq} H^*(\varphi)[z^p \times z^q] = (-1)^{pq}[z^p] \smile [z^q].$$

■

The algebraic properties listed in Theorem 2.17 and in Theorem 2.20 are referred to as the graded ring properties. Thus we arrived at the key definition:

Definition 2.21 Let X be a cubical set. The *cubical cohomology ring* of X is the graded abelian group $H^\star(X)$ with the graded multiplication given by the cup product.

We now derive an explicit formula, suitable for computations, for the cubical cup product (9) on the generating cochains $P^\star \in \mathcal{K}^p(X)$ and $Q^\star \in \mathcal{K}^q(X)$. The formula is developed recursively with respect to $d = \text{emb } X$ and presented in three stages, the first one for the case $d = 1$, the second one for the recursion step, and the last one is the final recursion-free coordinate-wise formula.

Theorem 2.22 *Let X be a cubical set in $\mathbb{R}$ and let $P, Q \in \mathcal{K}(X)$, $P = [a, b]$, $Q = [c, d]$ be elementary intervals, possibly degenerated. Then*

$$P^\star \smile Q^\star = \begin{cases} [a]^\star & \text{if } a = b = c = d, \\ [c,d]^\star & \text{if } a = b = c = d - 1, \\ [a,b]^\star & \text{if } b = c = d = a + 1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $P^\star \smile Q^\star$ is either zero, or a dual od an elementary interval.

Proof: Let $p = \dim P$ and $q = \dim Q$. Then $p, q \in \{0, 1\}$. Let $k := p + q$. Consider first the case $k = 0$. Let $R = [v] \in \mathcal{K}_0(X)$. Then

$$\begin{aligned} \langle P^\star \smile Q^\star, R\rangle &= \langle P^\star \times Q^\star, \text{diag}_0(R)\rangle = \langle P^\star \times Q^\star, [v] \times [v]\rangle \\ &= \langle P^\star, [v]\rangle \cdot \langle Q^\star, [v]\rangle \\ &= \begin{cases} 1 & \text{if } P = Q = [v], \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

When $k = p + q = 1$, let $R = [v_0, v_1] \in \mathcal{K}_1(X)$. Then

$$\begin{aligned} \langle P^\star \smile Q^\star, R\rangle &= \langle P^\star \times Q^\star, \text{diag}_1(R)\rangle = \langle P^\star \times Q^\star, [v_0] \times [v_0, v_1] + [v_0, v_1] \times [v_1]\rangle \\ &= \langle P^\star, [v_0]\rangle \cdot \langle Q^\star, [v_0, v_1]\rangle + \langle P^\star, [v_0, v_1]\rangle \cdot \langle Q^\star, [v_1]\rangle \\ &= \begin{cases} 1 & \text{if } P = [v_0] \text{ and } Q = [v_0, v_1], \\ 1 & \text{if } P = [v_0, v_1] \text{ and } Q = [v_1], \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We reach the conclusion by expressing the values in terms of intervals $[a, b]$, $[c, d]$.

■

Example 2.23 Since $[a]^\star \smile [a,a+1]^\star = [a,a+1]^\star$ and $[a,a+1]^\star \smile [a]^\star = 0$, we see that the graded commutative law in Theorem 2.20 does not hold for the cup product on chain complexes.

Theorem 2.24 *Let* $\operatorname{emb} X = d > 1$*, and suppose that the formula for* $\smile$ *is given for cochains on cubical sets of embedding numbers* $n = 1, \ldots, d-1$*. Consider the decomposition of elementary cubes* $P = P_1 \times P_2 \in \mathcal{K}_p(X)$ *and* $Q = Q_1 \times Q_2 \in \mathcal{K}_q(X)$ *with* $\operatorname{emb}\ P_1 = \operatorname{emb}\ Q_1 = 1$ *and* $\operatorname{emb}\ P_2 = \operatorname{emb}\ Q_2 = d-1$*. Let* $x = P_1^\star \smile Q_1^\star$ *and* $y = P_2^\star \smile Q_2^\star$ *be computed using the induction hypothesis. Then*

$$P^\star \smile Q^\star = \begin{cases} (-1)^{\dim P_2 \dim Q_1}\ x{\times}y & \text{if } |x{\times}y| \in \mathcal{K}(X), \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Consider a generator $R = R_1 \times R_2 \in \mathcal{K}_k(X)$, where $\operatorname{emb}\ R_1 = 1$ and $\operatorname{emb}\ R_2 = d-1$. Let $p = \dim\ R_1$ and $q = \dim R_2$. Note that $p+q = k$. By Lemma 2.15 we get

$$\begin{aligned} \langle P^\star \smile Q^\star, R\rangle &= \langle P^\star{\times}Q^\star, \operatorname{diag}_k(R)\rangle \\ &= \langle P^\star{\times}Q^\star, \left(\pi_k \circ \tau_k \circ (\operatorname{diag}^{X_1} \otimes \operatorname{diag}^{X'})_k \circ j_k\right)(R)\rangle \\ &= \langle \tau^k(\pi^k(P^\star{\times}Q^\star)), \left((\operatorname{diag}^{X_1} \otimes \operatorname{diag}^{X'})_k \circ j_k\right)(R)\rangle. \end{aligned}$$

Since π_k is a projection, its dual map π^k is an injection which extends any element of $\mathcal{C}^\star(X \times X)$ as zero on elementary cubes in $\mathcal{K}(X_1 \times X' \times X_1 \times X') \setminus \mathcal{K}(X \times X)$. Set $\sigma = (-1)^{\dim P_2 \dim Q_1}$. By Proposition 2.8, Corollary 2.12 and Lemma 2.7, we get

$$\begin{aligned} \langle P^\star \smile Q^\star, R\rangle &= \sigma \cdot \langle (P_1^\star{\times}Q_1^\star) \times (P_2^\star{\times}Q_2^\star), (\operatorname{diag}^{X_1} \otimes \operatorname{diag}^{X'})_k(R)\rangle \\ &= \sigma \cdot \langle P_1^\star{\times}Q_1^\star, \operatorname{diag}_p^{X_1}(R_1)\rangle \cdot \langle P_2^\star{\times}Q_2^\star, \operatorname{diag}_q^{X'}(R_2)\rangle \\ &= \sigma \cdot \langle P_1^\star \smile Q_1^\star, R_1\rangle \cdot \langle P_2^\star \smile Q_2^\star, R_2\rangle \\ &= \sigma \cdot \langle x, R_1\rangle \cdot \langle y, R_2\rangle \\ &= \sigma \cdot \langle x{\times}y, R\rangle, \end{aligned}$$

with a non trivial value assumed if and only if $|x| = R_1$ and $|y| = R_2$. However, $R \in \mathcal{K}(X)$ and the conclusion follows. ∎

Example 2.25 Let $X = [0,1]^2$, $P = [0] \times [0,1]$, and $Q = [0,1] \times [0]$. We get

$$P^\star \smile Q^\star = (-1)^{1 \cdot 1}\,([0]^\star \smile [0,1]^\star) \times ([0,1]^\star \smile [1]^\star) = -[0,1]^{2\star}.$$

However, if $X = P \cup Q$, we get $P^\star \smile Q^\star = 0$.

We now pass to the coordinate-wise formula. Let P and Q be as in Theorem 2.24. Consider their decompositions to products of intervals in $\mathbb{R}$:

$$P = I_1 \times I_2 \times \cdots \times I_d \text{ and } Q = J_1 \times J_2 \times \cdots \times J_d.$$

Let $P'_j := I_{j+1} \times I_{j+2} \times \cdots \times I_d$. Put

$$\mathrm{sgn}\,(P,Q) := (-1)^{\sum_{j=1}^{d} \dim P'_j \dim J_j} = (-1)^{\sum_{j=1}^{d} \left(\dim J_j \sum_{i=j+1}^{d} \dim I_i\right)}.$$

From Theorem 2.24 we easily derive the following.

Corollary 2.26 *With the above notation,*

$$P^\star \smile Q^\star = \mathrm{sgn}\,(P,Q)(I_1^\star \smile J_1^\star) \times (I_2^\star \smile J_2^\star) \times \cdots \times (I_d^\star \smile J_d^\star), \tag{11}$$

provided the right-hand side is supported in X, *and* $P^\star \smile Q^\star = 0$ *otherwise.*

Example 2.27 We illustrate the cup-product formula in the cubical torus $T := \Gamma^1 \times \Gamma^1 \subset \mathbb{R}^4$, where $\Gamma^1 = \partial[0,1]^2$ is the boundary of the square. Since it is hard to draw pictures in R^4, we parameterize Γ^1 by the interval $[0,4]$ with identified endpoints $0 \sim 4$, which permits visualizing T as the square $[0,4]^2$ with pairs of identified facing edges, as shown in the figure below.

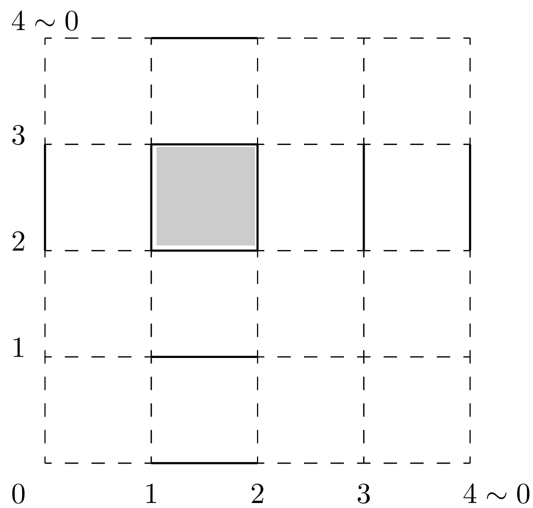

Figure 1: The graphical representation of the cubical torus discussed in Example 2.27. The solid line vertical edges carry the cocycle x^1 and the horizontal ones the cocycle y^1. The gray square carries $x^1 \smile y^1$.

Consider the cocycle x^1 generated by the sum of four solid line vertical edges with $[2,3]$ at the second coordinate, and y^1 by the sum of solid line horizonal edges with $[1,2]$ at the first coordinate. Only the edges of the parametric square $[1,2] \times [2,3]$ may contribute to non-zero terms of $x^1 \smile y^1$. Thus,

$$\begin{aligned} x^1 \smile y^1 &= \{([1] \times [2,3])^* + ([2] \times [2,3])^*\} \smile \{([1,2] \times [2])^* + ([1,2] \times [3])^*\} \\ &= 0 - ([1,2] \times [2,3])^* + 0 + 0 = -([1,2] \times [2,3])^*. \end{aligned}$$

The cohomology classes of cochains x^1 and y^1 generate $H^1(T)$, and $[Q^*]$, where $Q := ([1,2] \times [2,3])$, generates $H^2(T)$.

It is known that the ring structure introduced in Definition 2.21 may be used to distinguish non homeomorphic spaces even if their homology and cohomology groups are isomorphic. This is shown in the example presented in Section 3.5.

Remark 2.28 All what we have done until now can be extended to chain complexes $\mathcal{C}(X;R) := \mathcal{C}(X) \otimes R$ with coefficients in a ring with unity R, which are graded modules over R. This gives rise to the cohomology ring $H^*(X;R)$. We have initially chosen coefficients in $\mathbb{Z}$ for the sake of clarity and, in particular, to avoid confusion of two rings, R and the graded cohomology ring. In addition, the case of integer coefficients is the most general, because, due to the universal coefficient theorem [30, Theorem 53.1], other cases can be derived from it. However we introduce the ring coefficients in the next section because, for computational purposes, it is often convenient to choose coefficients in the finite field $R = \mathbb{Z}_p$, p a prime number. Field coefficients are sufficient in many practical applications.

3 Computing cohomology

The aim of this section is to show that the techniques of S-reductions of S-complexes developed in [26, 28] in order to construct efficient algorithms computing homology of cubical complexes may be easily adapted to provide algorithms computing the cohomology ring of a cubical set.

3.1 S-Complexes

Let R be a ring with unity and let S be a finite set. Denote by $R(S)$ the free module over R generated by S. Let $(S_q)_{q\in\mathbb{Z}}$ be a gradation of S such that $S_q = \emptyset$ for all $q < 0$. Then $(R(S_q))_{q\in\mathbb{Z}}$ is a gradation of the module $R(S)$ in the category of moduli over the ring R. For every element $s \in S$ the unique number q such that $s \in S_q$ is called the *dimension* of s and denoted by $\dim s$. We use the notation $\langle\cdot,\cdot\rangle : R(S) \times R(S) \to R$ for the scalar product which is defined on generators by

$$\langle t, s\rangle = \begin{cases} 1 & \text{for } t = s, \\ 0 & \text{otherwise,} \end{cases}$$

and extended it bilinearly to $R(S) \times R(S)$.

We recall (see [26, 28]) that the pair (S, κ), where $\kappa : S \times S \to R$ is a map such that $\kappa(s,t) = 0$ unless $\dim s = \dim t + 1$, is called an *S-complex*, if the pair $(R(S), \partial^\kappa)$ is a free chain complex with base S and the *boundary map* $\partial^\kappa : R(S) \to R(S)$ defined on generators $s \in S$ by

$$\partial^\kappa(s) := \sum_{t\in S} \kappa(s,t)t.$$

The *homology of an S-complex* (S,κ) is the homology of the associated chain complex $(R(S), \partial^\kappa)$, denoted $H(S,\kappa)$ or simply $H(S)$. The elements of $R(S)$ are called *chains*.

The cubical complex $\mathcal{C}(X)$ discussed in Section 2 is a particular case of an S-complex. Namely, let X be a cubical set, $S = \mathcal{K}(X)$ the set of all elementary cubes of X and let $\kappa(Q, P) := \langle \partial Q, P \rangle$, where ∂ is the cubical boundary map. Then (S, κ) is an S-complex and its chain complex $R(S, \kappa)$ is equal to $\mathcal{C}(X; R) = \mathcal{C}(X) \otimes R$, the cubical chain complex with coefficients in R.

Let $R^\star(S) := \mathrm{Hom}\,(R(S), R)$ be the group of cochains. The *coboundary map* defined as the dual $\delta^\kappa := (\partial^\kappa)^\star$ satisfies

$$\delta^\kappa(t^\star) := \sum_{s \in S} \kappa(s, t) s^\star \ .$$

for duals of generators $t \in S$. Moreover, for any pair of a chain $c \in R(S_q)$ and a cochain $d \in R^\star(S_{q-1})$ we have

$$\langle \partial^\kappa c, d \rangle = \langle c, \delta^\kappa d \rangle.$$

The cohomology of the cochain complex $(\mathrm{Hom}\,(R(S), R), \delta^\kappa)$ is called the *cohomology of the S-complex* and denoted $H^*(S)$.

In the following, we will drop the superscript κ in ∂^κ and δ^κ whenever κ is clear from the context.

The technique of S-reductions consists in replacing the original set of generators S by a subset $S' \subset S$, and the original coincidence index κ by the restriction $\kappa' := \kappa|_{S' \times S'}$. This has to be done in such a way that (S', κ') is still an S-complex, and the (co)homology does not change. A subset $\mathcal{K}' \subset \mathcal{K}$ is an *S-subcomplex* of the S-complex $\mathcal{K}$ if $(\mathcal{K}', \kappa')$, with $\kappa' := \kappa|_{\mathcal{K}' \times \mathcal{K}'}$, the restriction of κ to $\mathcal{K}' \times \mathcal{K}'$, is itself an S-complex, i.e. if $(R[\mathcal{K}'], \partial^{\kappa'})$ is a chain complex. Note that the concept of an S-subcomplex is not the same as the chain subcomplex (see [9, Example 1]).

Two important special cases of S-subcomplexes are the closed and open subset of an S-complex. In order to define these concepts we introduce the following notation for any subset $A \subset S$

$$\begin{aligned} \mathrm{bd}_S A &:= \{t \in S \mid \ \kappa(s, t) \neq 0 \text{ for some } s \in A\}, \\ \mathrm{cbd}_S A &:= \{s \in S \mid \ \kappa(s, t) \neq 0 \text{ for some } t \in A\}. \end{aligned}$$

We say that $\mathcal{K}' \subset \mathcal{K}$ is *closed* in $\mathcal{K}$ if $\mathrm{bd}_{\mathcal{K}} \mathcal{K}' \subset \mathcal{K}'$. We say that $\mathcal{K}' \subset \mathcal{K}$ is *open* in $\mathcal{K}$ if the complement $\mathcal{K} \setminus \mathcal{K}'$ is closed. Note that if $\mathcal{K}'$ is closed in $\mathcal{K}$, then $\partial^\kappa(R[\mathcal{K}']) \subset R[\mathcal{K}']$. Therefore, there is a well defined restriction

$$\partial^\kappa|_{R[\mathcal{K}']} : R[\mathcal{K}'] \to R[\mathcal{K}'],$$

which gives rise to a chain subcomplex $(R[\mathcal{K}'], \partial^\kappa|_{R[\mathcal{K}']})$ of the chain complex $(R[\mathcal{K}], \partial^\kappa)$.

Similarly, if $\mathcal{K}'$ is open in $\mathcal{K}$, then there is a well defined quotient complex $(R[\mathcal{K}]/R[\mathcal{K} \setminus \mathcal{K}'], \partial')$ with the boundary map ∂' taken as the respective quotient map of ∂^κ.

The following theorem is a straightforward extension of Theorem 3.4 in [26] to cohomology.

Theorem 3.1 *Let (S, κ) be an S-complex over the ring R, $S' \subset S$ a closed subset and $S'' := S \setminus S'$ the associated open subset. Then we have the following long exact sequence of homology modules*

$$\ldots \stackrel{\partial_{q+1}}{\to} H_q(S') \stackrel{\iota_q}{\to} H_q(S) \stackrel{\pi_q}{\to} H_q(S'') \stackrel{\partial_q}{\to} H_{q-1}(S') \stackrel{\iota_{q-1}}{\to} \ldots$$

and the following long exact sequence of cohomology modules

$$\ldots \stackrel{\iota^{q-1}}{\to} H^{q-1}(S') \stackrel{\delta^q}{\to} H^q(S'') \stackrel{\pi^q}{\to} H^q(S) \stackrel{\iota^q}{\to} H^q(S') \stackrel{\delta^{q+1}}{\to} \ldots$$

in which $\iota_* : H_q(S') \to H_q(S)$ and $\iota^* : H^q(S) \to H^q(S')$ are induced by the inclusion $\iota : R(S') \to R(S)$, whereas $\pi_* : H_q(S) \to H_q(S'')$ and $\pi^* : H^q(S'') \to H^q(S)$ are induced by the projection $\pi : R(S) \to R(S'')$.

3.2 S-Reduction Pairs and the Coreduction Algorithm

Let (S, κ) be an S-complex. A pair (a, b) of elements of S is called an *S-reduction pair* if $\kappa(a, b)$ is invertible and either $\text{cbd}_S a = \{b\}$ or $\text{bd}_S b = \{a\}$. In the first case the S-reduction pair is referred to as an *elementary reduction pair* and in the other case as an *elementary coreduction pair*.

Arguing analogously to the proof of [26, Theorem 4.1] we obtain the following theorem.

Theorem 3.2 *Assume S is an S-complex and (a, b) is an S-reduction pair in S. If (a, b) is an elementary reduction pair then $\{a, b\}$ is open in S. If (a, b) is an elementary coreduction pair then $\{a, b\}$ is closed in S. Moreover, in both cases $\{a, b\}$ is an S-subcomplex of S and $H(\{a, b\}) = H^*(\{a, b\}) = 0$.*

Theorems 3.1 and 3.2 result in the following corollary.

Corollary 3.3 *If (a, b) is an S-reduction pair in an S-complex S, then the homology modules $H(S)$ and $H(S \setminus \{a, b\})$ as well as the cohomology modules $H^*(S)$ and $H^*(S \setminus \{a, b\})$ are isomorphic.*

Corollary 3.3 lies at the heart of the coreduction homology algorithm presented in [26, Algorithm 6.1]. The same algorithm without any changes may be used to speed up computation of cohomology modules. The algorithm consists in performing as many S-reductions as possible before applying the general Smith diagonalization algorithm to the reduced S-complex in order to compute the homology or cohomology module. To make it useful one needs to find as many S-reduction pairs as feasible. In the case of simplicial complexes and cubical complexes it is straightforward to provide examples which admit elementary reduction pairs, but elementary coreduction pairs are not possible. However, it is easy to observe that by removing a vertex one obtains an open subcomplex which admits elementary coreduction pairs. Moreover, the homology of this subcomplex coincides with the reduced homology of the original complex and the cohomology of this complex coincides with the reduced cohomology of the original complex. Therefore, not only elementary reduction pairs, but also elementary coreduction pairs are useful when computing the homology or cohomology of simplicial or cubical complexes.

If the reduced S-complex is small when compared with the original S-complex then the coreduction algorithm is fast, because the reduction process is linear whereas the Smith diagonalization algorithm is supercubical. In fact, numerical experiments indicate that elementary coreduction pairs provide essentially deeper reductions than the elementary reduction pairs and the speed up is essential. For details we refer the reader to [26, Section 5].

3.3 Homology Models

The Smith diagonalization algorithm applied to the reduced S-complex enables computing the cohomology module up to an isomorphism. In order to compute the cohomology ring of a cubical set it is not sufficient to have the cohomology generators in a reduced S-complex. It is necessary to construct the cohomology generators in the original cubical set.

In order to achieve this we need the following theorem which is an immediate consequence of [28, Theorem 2.8 and 2.9].

Theorem 3.4 *Assume (S,κ) is an S-complex, (a,b) is an S-reduction pair in S and $\bar{S} := S \setminus \{a,b\}$. Then $(\bar{S},\bar{\kappa})$ with $\bar{\kappa} := \kappa_{|\bar{S}\times\bar{S}}$ is an S-subcomplex of (S,κ) and the maps $\psi = \psi^{(a,b)} : R(S) \to R(\bar{S})$, respectively, $\iota = \iota^{(a,b)} : R(\bar{S}) \to R(S)$ given by*

$$\psi(c) = c - \frac{\langle c,a\rangle}{\langle \partial b,a\rangle}\partial b - \langle c,b\rangle b, \tag{12}$$

$$\iota(c) = c - \frac{\langle \partial c,a\rangle}{\langle \partial b,a\rangle} b \tag{13}$$

are mutually inverse chain equivalences. In particular, the chain complexes $(R(S),\partial^{\kappa})$ and $(R(\bar{S}),\partial^{\bar{\kappa}})$ are chain homotopic.

Since mutually inverse chain equivalences induce isomorphisms in cohomology, we get the following corollary.

Corollary 3.5 *If (S,κ), is an S-complex over the ring R and (a,b) is an S-reduction pair in S, then homologies $H(S)$ and $H(S\setminus\{a,b\})$ and cohomoloogies $H^*(S)$ and $H^*(S\setminus\{a,b\})$ are isomorphic. Moreover, the isomorphisms are induced by the chain maps defined in* (12) *and* (13) *for homology and their duals for cohomology.*

As we already mentioned, the coreduction algorithm consists in performing a sequence of reductions. A *reduction sequence* of an S-complex (S,κ) is a sequence of pairs $\omega = \{(a_i,b_i)\}_{i=1,2,\ldots n}$ in S such that (a_i,b_i) is a reduction pair in (S^{i-1},κ^{i-1}), where the S-complexes (S^i,κ^i) are defined recursively by taking

$$\begin{aligned} S^0 &:= S, \\ \kappa^0 &:= \kappa, \\ S^i &:= S^{i-1}\setminus\{a_i,b_i\}, \\ \kappa^i &:= \kappa^{i-1}_{|S^i}. \end{aligned}$$

We then use the notation S^ω for the last chain complex in the sequence of S-complexes $\{S^i\}_{i=1,2,\ldots n}$ and call this S-complex the *ω-reduction* of S. A *homology model of S* is an ω reduction S^ω together with the chain equivalences.

$$\begin{aligned}\iota^\omega &= \iota^{(a_1,b_1)} \circ \iota^{(a_2,b_2)} \circ \cdots \circ \iota^{(a_n,b_n)} : R(S^\omega) \to R(S),\\ \psi^\omega &= \psi^{(a_n,b_n)} \circ \psi^{(a_{n-1},b_{n-1})} \circ \cdots \circ \psi^{(a_1,b_1)} : R(S) \to R(S^\omega).\end{aligned}$$

In order to discuss the benefits from constructing a homology model of an S-complex S let us define first the *weight of S* by

$$w(S) := \max\left\{\max(\operatorname{card}\operatorname{bd} s, \operatorname{card}\operatorname{cbd} s) \mid s \in S\right\}.$$

The construction of a homology model of an S-complex S may be performed in time $O(w(S)\operatorname{card} S)$ (see [26, Theorem 6.2]). In particular, in the case of cubical sets of fixed embedding dimension the homology model construction takes linear time. When the *ω-reduction* of S is small relative to S, one can profit from the homology model whenever homology generators and/or a decomposition of a homology class on the generators are needed. To construct the generators of $H(S)$ one constructs the generators of $H(S^\omega)$ and transports them to $H(S)$ via the map ι^ω. The cost of this transport is $O(w(S)\operatorname{card} S)$ (see [28, Theorem 3.1]). To decompose a homology class in $H(S)$ one transports the class via the map ψ^ω to $H(S^\omega)$ and finds the decomposition there. See [28, Section 3.1]) for details.

3.4 Homology models for cohomology

Precisely the same method may be used to speed up the construction of the cohomology generators in $H^*(S)$. One only uses the dual $(\iota^\omega)^\star$ of ι^ω to transport the cochains in the S-complex to its ω reduction and the dual $(\psi^\omega)^\star$ of ψ^ω to transport the cochains in the ω-reduction back to the original S-complex. For this, one needs the following proposition, which may be proved by an elementary computation.

Proposition 3.6 *The duals of the chain maps ψ and ι are given by*

$$\psi^\star : R^\star(\bar{S}) \ni c \mapsto c - \frac{\langle b^\star, \delta c\rangle}{\langle \partial b, a\rangle} a^\star \in R^\star(S), \tag{14}$$

$$\iota^\star : R^\star(S) \ni c \mapsto c - \frac{\langle b^\star, c\rangle}{\langle \partial b, a\rangle} \delta a^\star \in R^\star(\bar{S}). \tag{15}$$

■

Surprisingly, there are even more benefits from the homology model for cohomology computations than for homology computations. This is because of the following theorem.

Theorem 3.7 *Assume ω is a reduction sequence of an S-complex consisting only of elementary coreduction pairs. Then $(\psi^\omega)^\star$ is an inclusion*

$$R^\star(S^\omega) \hookrightarrow R^\star(S),$$

that is, $(\psi^\omega)^\star(c) = c$ for any $c \in R^\star(S^\omega)$.

Proof: If (a,b) is an elementary coreduction pair, then $\operatorname{bd} b = \{a\}$. Since $a \notin \bar{S} = S \setminus \{a,b\}$, we have $\langle b^\star, \delta c\rangle = \langle \partial b, c\rangle = 0$ for any $c \in R(\bar{S})$. Therefore, $(\psi^{(a,b)})^\star(c) = c$ for any $c \in R^\star(\bar{S})$. Since the reduction sequence ω consist only of elementary coreduction pairs, the conclusion follows. ■

The consequence of this lemma is that in the case of a reduction sequence ω consisting only of elementary coreduction pairs there is no need to transport the cohomology generators from the ω-reduction back to the original S-complex. The cohomology generators constructed in the ω-reduction are the cohomology generators in the original S-complex. This is particularly useful for computing the ring structure of a cubical set X, because we can apply formula (11) directly to the cohomology generators in the ω-reduction.

3.5 Example

`<<COMMENT>> The figures are clear when looking at the screen but not a gray-scale print. I changed the width command to 0.9, but then they take a lot of vertical space, so I have trimmed the blanc edges. TK.` !!

Observe that the set

$$\begin{aligned} X &:= \left\{(x,y,z)\in\mathbb{R}^3 \mid \max(|x-\frac{1}{2}|,|y-\frac{1}{2}|,|z-\frac{1}{2}|)=\frac{1}{2}\right\}\cup \\ &\quad \left\{(x,y,z)\in\mathbb{R}^3 \mid y=0,\ \max(|x-\frac{3}{2}|,|z-\frac{3}{2}|)=\frac{1}{2}\right\}\cup \\ &\quad \left\{(x,y,z)\in\mathbb{R}^3 \mid x=0,\ \max(|y-\frac{3}{2}|,|z-\frac{3}{2}|)=\frac{1}{2}\right\} \end{aligned}$$

is a cubical subset of $\mathbb{R}^3$ homeomorphic to $S^2 \vee S^1 \vee S^1$ (see Fig. 2 top left).

Coreduction algorithm applied to X outputs a model X^ω with some coreduction sequence ω. The model consists of the following cells.

$$\{[0]\times[0]\times[0], (1,2)\times[1]\times[0], (0,1)\times(0,1)\times[1], [0]\times(0,1)\times[2]\}.$$

It is straightforward to observe that the cohomology classes of the cocycles

$$\begin{aligned} \alpha &:= ([1,2]\times[1]\times[0])^\star \\ \beta = &:= ([0]\times[0,1]\times[2])^\star \end{aligned}$$

form the basis of the first cohomology module of X^ω and the cohomology class of the cocycle

$$\gamma := ([0,1]\times[0,1]\times[1])^\star$$

generates the second cohomology module of X^ω. By Theorem 3.7 and Theorem 3.4 $[\alpha]$, $[\beta]$ form the basis of $H^1(X)$ and $[\gamma]$ generates $H^2(X)$.

Moreover, a straightforward computation based on Corollary 2.26 shows that

$$[\alpha] \smile [\beta] = 0. \tag{16}$$

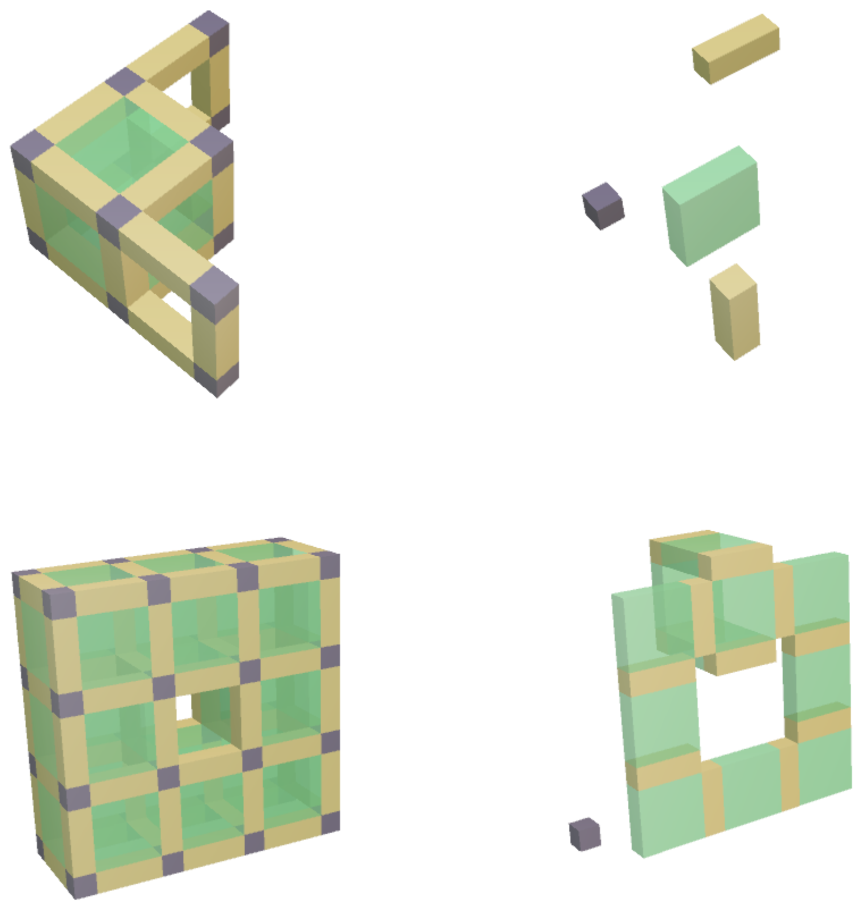

Figure 2: Wedge of S^2 and two S^1 (top left) and wedge of S^2 and two S^1 coreduced (top right). Cubical torus (bottom left) and cubical torus coreduced (bottom right). For visualization purposes, the vertices are displayed as small black cubes, edges as long square-based prisms and 2D faces as large square-based prisms. The 2D faces of cubical sets are partially transparent to enable viewing through.

$$T := \left\{(x,y,z)\in\mathbb{R}^3 \mid \frac{1}{2}\le \max(|x-\frac{3}{2}|,|y-\frac{3}{2}|)\le\frac{3}{2},\ z\in\{0,1\}\right\}\cup \left\{(x,y,z)\in\mathbb{R}^3 \mid \max(|x-\frac{3}{2}|,|y-\frac{3}{2}|)\in\{\frac{1}{2},\frac{3}{2}\},\ 0\le z\le 1\right\}$$

which is a cubical subset of $\mathbb{R}^3$ homeomorphic to a torus (see Fig. 2 bottom left).

Coreduction algorithm applied to T outputs a model T^ω with some coreduction sequence ω. The model consists of the following cells.

$$\{[0]\times[0]\times[0], (1,2)\times[2]\times[0], (1,2)\times(2,3)\times[0], (1,2)\times[3]\times[0],$$
$$(1,2)\times[2]\times(0,1), (1,2)\times[3]\times(0,1), (0,1)\times(0,1)\times[1], [1]\times(0,1)\times[1],$$
$$(1,2)\times(0,1)\times[1], [2]\times(0,1)\times[1], (2,3)\times(0,1)\times[1], (0,1)\times[1]\times[1],$$
$$(2,3)\times[1]\times[1], (0,1)\times(1,2)\times[1], (2,3)\times(1,2)\times[1], (0,1)\times[2]\times[1],$$
$$(1,2)\times[2]\times[1], (2,3)\times[2]\times[1], (0,1)\times(2,3)\times[1], [1]\times(2,3)\times[1],$$
$$(1,2)\times(2,3)\times[1], [2]\times(2,3)\times[1], (2,3)\times(2,3)\times[1], (1,2)\times[3]\times[1]\}$$

It is straightforward to observe that the cohomology classes of the cocycles

$$\alpha = ([1]\times[0,1]\times[1])^\star + ([2]\times[0,1]\times[1])^\star + ([0,1]\times[1]\times[1])^\star + ([2,3]\times[1]\times[1])^\star$$
$$+([0,1]\times[2]\times[1])^\star + ([2,3]\times[2]\times[1])^\star + ([1]\times[2,3]\times[1])^\star + ([2]\times[2,3]\times[1])^\star$$

and

$$\beta = ([1,2]\times[2]\times[0])^\star + ([1,2]\times[3]\times[0])^\star + ([1,2]\times[2]\times[1])^\star + ([1,2]\times[3]\times[1])^\star$$

form the basis of the first cohomology module of T^ω and the cohomology class of the cocycle

$$\gamma = ([1,2]\times[2,3]\times[1])^\star$$

generates the second cohomology module of T^ω. By Theorem 3.7 and Theorem 3.4 $[\alpha]$, $[\beta]$ form the basis of $H^1(T)$ and $[\gamma]$ generates $H^2(T)$.

Moreover, a straightforward computation based on Corollary 2.26 shows that

$$\alpha \smile \beta = \pm\gamma. \tag{17}$$

Equations (16) and (17) show that the cohomology rings of X and T are different. In this simple case it is possible to make the necessary computations by hand. However, one can have two cubical sets homeomorphic respectively to X and T whose representation consists of millions of cubes. Such cubical sets often result from rigorous numerics of dynamical systems, data or image analysis. Then benefit from computing the ring structure via the homology model is evident. This is visible even in the case of a simple rescaling (for the definition of rescaling see [21, Section 6.4.2]) of the cubical sets in our two examples (see Fig. 3).

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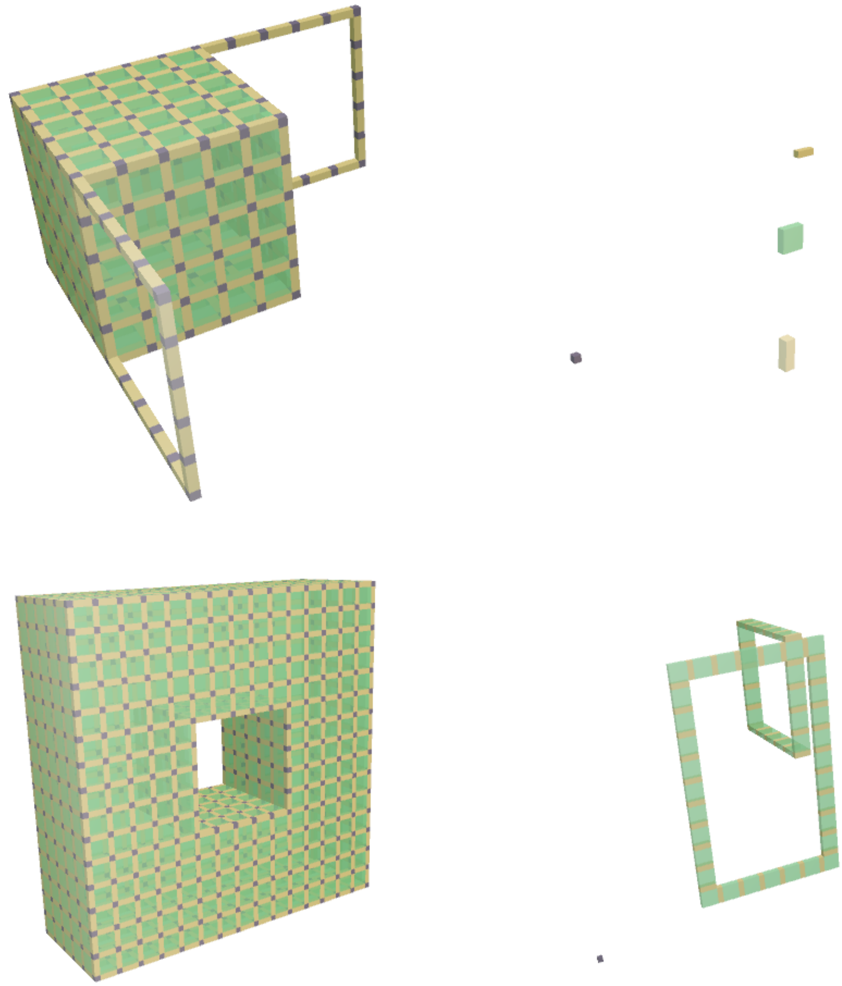

Figure 3: Rescaled wedge of the sphere and two circles from Figure 2 (top left) and the result of its correduction (top right). Rescaled cubical torus (bottom left) and the result of its correduction (bottom right).

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Département de mathématiques
Université de Sherbrooke
Sherbrooke, Qc, Canada J1K 2R1
t.kaczynski@usherbrooke.ca

Institute of Computer Science
Jagiellonian University,
ul. St. Łojasiewicza 6
30-348 Kraków, Poland
and Division of Computational Mathematics
WSB-NLU
ul. Zielona 27
33-300 Nowy Sącz, Poland
mrozek@ii.uj.edu.pl