Condition of convex optimization and spherical intrinsic volumes

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(joint work with Dennis Amelunxen)

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Motivation

Regular convex cones

- Fix a regular cone $C \subset \mathbb{R}^n$, i.e., a closed convex cone with nonempty interior that does not contain a nontrivial linear subspace.
- The dual cone of C is defined as $\breve{C} := \{z \in \mathbb{R}^n \mid \forall x \in C : z^T x \leq 0\}$. We call C *self-dual* if $\breve{C} = -C$.
- The positive orthant $\mathbb{R}^n_+$ and products $\mathcal{L}^{n_1} \times \ldots \times \mathcal{L}^{n_r}$ of Lorentz cones $\mathcal{L}^n := \{x \in \mathbb{R}^n \mid x_n \geq (x_1^2 + \cdots + x_{n-1}^2)^{1/2}\}$ are self-dual.
- The cone of positive semidefinite matrices Sym^k_+ is self-dual as well.

Renegars condition number

- The homogeneous convex feasibility problem is to decide for a given matrix $A \in \mathbb{R}^{m\times n}$, $1 \leq m < n$, the alternative

$$\exists x \in \mathbb{R}^n \setminus 0 \text{ s.t. } \quad Ax = 0\,,\ x \in \breve{C}\,, \qquad \text{(P)}$$

$$\exists y \in \mathbb{R}^m \setminus 0 \text{ s.t. } \quad A^T y \in C\,. \qquad \text{(D)}$$

- The set of ill-posed inputs Σ_{R} is defined as the set of matrices A, for which (P) and (D) are both feasible. The feasibility problem has no unique solution if $A \in \Sigma_{\mathrm{R}}$.
- Renegar's condition number $\mathcal{R}_C(A)$ of A is defined as inverse distance to ill-posedness with respect to spectral norm:

$$\mathcal{R}_C(A) := \frac{\|A\|}{d(A, \Sigma_{\mathrm{R}})}\,,$$

where $d(A, \Sigma_{\mathrm{R}}) = \min\{\|A - A'\| \mid A' \in \Sigma_{\mathrm{R}}\}$.

Relevance for complexity

- Jim Renegar realized that the complexity of solving linear—and more generally convex optimization problems—can be bounded in terms of the condition number $\mathcal{R}_C(A)$.
- For simplicity, we only focus here on the homogeneous convex feasibility problem.
- Vera, Rivera, Peña, Hui: There is an interior-point algorithm that solves the homogeneous convex feasibility problem, for $C \subseteq \mathbb{R}^n$ a self-scaled cone with a self-scaled barrier function, in $O(\sqrt{\nu_C} \cdot \log(\nu_C \cdot \mathcal{R}_C(A)))$ interior-point iterations.
- $\nu_C \leq n$ for the cones C of (LP), (SOCP), (SDP).

Average probabilistic analysis for $C = \mathbb{R}^n_+$

- To understand the complexity of convex optimization, we want to analyze the probabilistic behaviour of $\mathcal{R}_C(A)$.
- First step: average analysis. Assume that entries of $A \in \mathbb{R}^{m\times n}$ are iid standard Gaussian, i.e., $A \sim N(0, I)$.
- For $C = \mathbb{R}^n_+$ several papers on average analysis: B, Cheung, Cucker, Hauser, Lotz, Müller, Wschebor (also for condition numbers closely related to $\mathcal{R}(A)$).
- As a result:

We have $\mathbb{E}\log\mathcal{R}(A) = \mathcal{O}(\log m)$ for $C = \mathbb{R}^n_+$.

Smoothed probabilistic analysis for $C = \mathbb{R}^n_+$

- More realistic viewpoint: Smoothed analysis.
- Fix $\sigma > 0$. Let $\bar{A} \in \mathbb{R}^{m\times n}$ st $\|\bar{A}\| \leq 1$ and assume $A \sim N(\bar{A}, \sigma I)$.

Dunagan, Spielman, Teng (2011). For $C = \mathbb{R}^n_+$,

$$\sup_{\|\bar{A}\|\leq 1} \mathbb{E}_{A\sim N(\bar{A},\sigma I)} \log \mathcal{R}(A) = \mathcal{O}\big(\log \frac{n}{\sigma}\big).$$

- Extension to more general distributions by Amelunxen & B.

Future goal: Smoothed analysis for any regular cone!

So far achieved for average analysis: this talk.

A coordinate-free condition number

The Grassmann manifolds $\mathrm{Gr}_{n,m}$

- The known probabilistic analyses of $\mathcal{R}_C(A)$ rely on the product structure of the cone $C = \mathbb{R}_+ \times \cdots \times \mathbb{R}_+$ and cannot be extended.
- Working with a coordinate-free, geometric notion of condition allows to overcome this difficulty, at the price of working in the intrinsic geometric setting of Grassmann manifolds.
- The Grassmann manifold $\mathrm{Gr}_{n,m}$ is the set of m-dimensional linear subspaces W of $\mathbb{R}^n$.
- $\mathrm{Gr}_{n,m}$ is a compact manifold on which the orthogonal group $O(n)$ acts transitively.
- $\mathrm{Gr}_{n,m}$ is a Riemannian manifold with orthogonal invariant metric.
- The corresponding volume form defines an orthogonal invariant probability measure on $\mathrm{Gr}_{n,m}$.

The homogeneous convex feasibility problem

- Let $C \subset \mathbb{R}^n$ be a regular cone and $1 \leq m < n$. We define the sets of dual feasible and primal feasible subspaces, resp., as

$$\mathcal{D}_m(C) := \left\{ W \in \mathrm{Gr}_{n,m} \mid W \cap C \neq \{0\} \right\}$$
$$\mathcal{P}_m(C) := \left\{ W \in \mathrm{Gr}_{n,m} \mid W^\perp \cap \breve{C} \neq \{0\} \right\} .$$

- Farkas Lemma: $W \cap \mathrm{int}(C) \neq \emptyset \iff W^\perp \cap \breve{C} = \{0\}$, hence $\mathcal{D}_m(C) \cup \mathcal{P}_m(C) = \mathrm{Gr}_{n,m}$.
- The boundaries of $\mathcal{D}_m(C)$ and $\mathcal{P}_m(C)$ coincide with

$$\Sigma_m(C) := \mathcal{D}_m(C) \cap \mathcal{P}_m(C) .$$

  $\Sigma_m(C)$ is called the set of ill-posed subspaces and consists of the subspaces W touching the cone C.
- Duality: $W \mapsto W^\perp$ maps $\mathcal{D}_m(C)$ to $\mathcal{P}_{n-m}(\breve{C})$ and maps $\mathcal{P}_m(C)$ to $\mathcal{D}_{n-m}(\breve{C})$.

Grassmann condition number

- Let Π_{W_i} denote the orthogonal projection onto $W_i \in \mathrm{Gr}_{n,m}$. The spectral norm $d_p(W_1, W_2) := \|\Pi_{W_1} - \Pi_{W_2}\|$ is called the projection distance of $W_1, W_2 \in \mathrm{Gr}_{n,m}$.
- We define the Grassmann condition as the function

$$\mathscr{C}_C\colon \mathrm{Gr}_{n,m} \to [1, \infty] , \quad \mathscr{C}_C(W) := \frac{1}{d_p(W, \Sigma_m(C))} ,$$

where $d_p(W, \Sigma_m(C)) := \min\{d_p(W, W') \mid W' \in \Sigma_m(C)\}$.

- We may characterize $\mathscr{C}_C$ also in term of the geodesic distance d_g of the Riemannian manifold $\mathrm{Gr}_{n,m}$.
- Prop. $d_p(W, \Sigma_m(C)) = \sin d_g(W, \Sigma_m(C))$.

Comparison with Renegar's condition number

- Let $A \in \mathbb{R}^{m \times n}$ with $\mathrm{rk}(A) = m$ and put $W := \operatorname{im} A^T$. Belloni & Freund essentially showed:

$$\mathscr{C}_C(W) \;\leq\; \mathcal{R}_C(A) \;\leq\; \kappa(A) \cdot \mathscr{C}_C(W)\,,$$

  where $\kappa(A)$ denotes the usual matrix condition number, i.e., the ratio between the largest and the smallest singular value of A.
- In particular, $\mathscr{C}_C(W) = \mathcal{R}_C(A)$ if $\kappa(A) = 1$.
- Can break up the probabilistic study of Renegar's condition number $\mathcal{R}_C(A)$ into the study of $\mathscr{C}_C$ and κ. In particular, for random A,

$$\mathbb{E} \log \mathcal{R}_C(A) \leq \mathbb{E} \log \kappa(A) + \mathbb{E} \log \mathscr{C}_C(A).$$

Main results

Average analysis of Grassmann CN: I

If $A \in \mathbb{R}^{m\times n}$ is a Gaussian random matrix, then $W := \operatorname{im} A^T$ is uniformly distributed in $\mathrm{Gr}_{n.m}$.

Theorem I (Amelunxen, B)

Let $C \subset \mathbb{R}^n$ be a regular cone. For $W \in \mathrm{Gr}_{n,m}$ uniformly distributed,

$$\mathrm{Prob}[\mathscr{C}_C(W) > t] \ < \ 6 \cdot \sqrt{m(n-m)} \cdot \frac{1}{t}\,, \quad \text{if } t > n^{\frac{3}{2}}\,,$$
$$\mathbb{E}\,[\ln \mathscr{C}_C(W)] \ < \ 1.5 \cdot \ln(n) + 1.5\,.$$

- Recall: $\mathscr{C}_C(W) > t$ iff $d_p(W, \Sigma_m(C)) < 1/t$
- $\mathrm{Prob}[\mathscr{C}_C(W) > t]$ equals the relative volume of the tube of radius $1/t$ around $\Sigma_m(C)$, relative to the volume of $\mathrm{Gr}_{n,m}$.

Average analysis of Grassmann CN: II

Theorem II (Amelunxen, B)

Let $C \subset \mathbb{R}^n$ be a regular self-dual cone. For $W \in \mathrm{Gr}_{n,m}$ uniformly distributed,

$$\mathrm{Prob}[\mathscr{C}_C(W) > t] \;<\; 20 \cdot v(C) \cdot \sqrt{m} \cdot \frac{1}{t}\,, \quad \text{if } t > m\,,$$

$$\mathbb{E}\left[\ln \mathscr{C}_C(AW)\right] \;<\; \ln(m) + \max\{\ln(v(C)), 0\} + 3\,,$$

with the excess over the Lorentz cone $v(C)$ bounded as follows:

C	$\mathbb{R}^n_+$	$\mathcal{L}^n$	$\mathcal{L}^{n_1} \times \ldots \times \mathcal{L}^{n_r}$ (assuming some conjecture)	Sym^k_+ (assuming Conjecture SDP)
$v(C) \leq$	$\sqrt{2}$	1	2^{r-1}	2

.

Intrinsic volumes

Spherical intrinsic volumes

- A set $K \subseteq S^{n-1}$ is called spherical convex iff $C := \mathrm{cone}(K)$ is a convex cone. Then $K = S^{n-1} \cap C$.
- The α-tube $\mathcal{T}(K,\alpha)$ around K is defined as the α-neighborhood of K in S^{n-1} with respect to angular distance d.
- Put $\mathcal{O}_{n-1} := \mathrm{vol}_{n-1}(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$.
- H. Weyl's tube formula:

$$\mathrm{vol}_{n-1}\, \mathcal{T}(K,\alpha) = V_n(C) \cdot \mathcal{O}_{n-1} + \sum_{j=1}^{n-1} V_j(C) \cdot \mathrm{vol}_{n-1}\, \mathcal{T}(S^{i-1},\alpha)\ .$$

- The uniquely determined coefficients $V_1(C), \ldots, V_n(C)$ are called intrinsic volumes of C (or K).
- Note $V_n(C) = \frac{\mathrm{vol}_{n-1} K}{\mathcal{O}_{n-1}}$. We further set $V_0(C) := \frac{\mathrm{vol}_{n-1}(S^{n-1} \cap \breve{C})}{\mathcal{O}_{n-1}}$.
- The intrinsic volumes are orthogonal invariant.

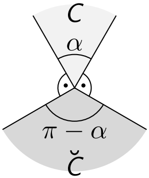
Probabilistic interpretation

- Let C be a polyhedral cone and $\Pi_C\colon \mathbb{R}^n \to C$ denote the projection map onto C.
- $\Pi_C(x)$ lies in the interior of a unique face of C. Let $d_C(x)$ denote the dimension of this face.
- The proof of Weyl's formula reveals: for $0 \le j \le n$,

$$V_j(C) = \underset{p\in S^{n-1}}{\text{Prob}}\,[d_C(p) = j] = \underset{x\in\mathcal{N}(0,I_n)}{\text{Prob}}\,[d_C(x) = j]$$

- Ex. $C \subseteq \mathbb{R}^2$ with angle $\alpha \le \pi$. Then $\breve{C}$ has angle $\pi - \alpha$.

$$V_0(C) = \tfrac{\pi-\alpha}{2\pi},\; V_1 = \tfrac{1}{2},\; V_2(C) = \tfrac{\alpha}{2\pi}.$$



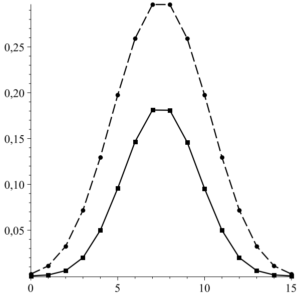
Properties of intrinsic volumes

- Conclusions from $V_j(C) = \underset{x \in \mathcal{N}(0,I_n)}{\text{Prob}} [d_C(x) = j]$:
- The $V_0(C), \dots, V_n(C)$ form a probability distribution on $\{0, 1, \dots, n\}$, i.e., $\sum_{j=0}^{n} V_j(C) = 1,\ V_j(C) \geq 0$.
- Duality implies $V_j(\check{C}) = V_{n-j}(C)$.
- The vector $V_j(C_1 \times C_2)$ is obtained from $V_j(C_1)$ and $V_j(C_2)$ by (cyclic) convolution.
- Ex. $\mathbb{R}_+^n = \mathbb{R}_+ \times \cdots \times \mathbb{R}_+$. The *n*-fold convolution of $V(\mathbb{R}_+) = (\frac{1}{2}, \frac{1}{2})$ (Bernoulli) yields the symmetric binomial distribution:

$$V_j(\mathbb{R}_+^n) = 2^{-n} \binom{n}{j}.$$

Example

- We have explicit formulas of the intrinsic volumes of $\mathcal{L}^n$ (easy) and for Sym_+^k (complicated), see talk by Dennis Amelunxen.
- The following graphics compares $V_j(\mathrm{Sym}_+^5)$ with $2V_j(\mathcal{L}^{15})$ (dashed); note $\mathrm{Sym}^5 \simeq \mathbb{R}^{15}$.



The logconcavity conjecture

Logconcavity conjecture

For any closed convex cone $C \subset \mathbb{R}^n$, the sequence of intrinsic volumes $V_0(C), \ldots, V_n(C)$ is logconcave, i.e., $V_j(C)^2 \geq V_{j-1}(C) \cdot V_{j+1}(C)$.

- We proved this conjecture for $\mathbb{R}^n_+$ and products of Lorentz cones.
- The conjecture is trivially true for $n = 1, 2$. For $n = 3$, $K \subseteq S^2$, it follows from the well known isoperimetric inequality

$$\mathrm{vol}_1(\partial K)^2 \geq vol_2(K)\big(4\pi - \mathrm{vol}_2(K)\big).$$

- For euclidean space, the logconcavity of the inner volumes is true, as a consequence of the Alexandrov-Fenchel inequalities.
- The euclidean case can obtained as a limit of the spherical case, but apparently, the spherical case seems more general.

Excess over Lorentz cones

- The Lorentz cone $\mathcal{L}^n = \{x \in \mathbb{R}^n \mid x_n \geq (x_1^2 + \cdots + x_{n-1})^{1/2}\}$ satisfies

$$f_j(n) := V_j(\mathcal{L}^n) = \frac{\binom{(n-2)/2}{(j-1)/2}}{2^{n/2}} .$$

- For a self-dual cone $C \subseteq \mathbb{R}^n$ we define the excess $v(C)$ over the Lorentz cone as

$$v(C) := \max_{0 \leq j \leq n} \frac{V_j(C)}{f_j(n)} .$$

- By definition, $v(\mathcal{L}^n) = 1$. We can show $v(\mathbb{R}^n_+) \leq \sqrt{2}$.

Conjecture SDP

The cone Sym^k_+ of positive semidefinite matrices satisfies $v(\mathrm{Sym}^k_+) \leq 2$.

- The conjecture is numerically checked for small values of k.

A tube formula for Grassmannians

The tube formula for $\mathrm{Gr}_{n,m}$

Let $C \subset \mathbb{R}^n$ be a regular cone and $\mathcal{T}(\Sigma_m(C), \alpha)$ denote the α-tube around $\Sigma_m(C)$ wrt geodesic distance.

Theorem

For $1 \leq m \leq n-1$ and $0 \leq \alpha \leq \frac{\pi}{2}$,

$$\frac{\mathrm{vol}\,\mathcal{T}(\Sigma_m(C), \alpha)}{\mathrm{vol}\,\mathrm{Gr}_{n,m}} \leq \frac{4m(n-m)}{n} \binom{n/2}{m/2} \sum_{j=0}^{n-2} V_{j+1}(C) \cdot F_j^{nm}(\alpha)$$

with the following functions (independent of C)

$$F_j^{nm}(\alpha) = \frac{\mathcal{O}_{n-2}}{\mathcal{O}_j \mathcal{O}_{n-2-j}} \cdot \sum_{i=0}^{n-2} d_{ij}^{nm} \cdot \int_0^\alpha (\cos\rho)^i (\sin\rho)^{n-2-i}\, d\rho \ ,$$

where $d_{ij}^{nm} := \binom{\frac{m-1}{\frac{i-j}{2}+\frac{m-1}{2}}} \cdot \binom{\frac{n-m-1}{\frac{i+j}{2}-\frac{m-1}{2}}} \cdot \binom{n-2}{j}^{-1}$ if $i+j+m \equiv 1 \pmod 2$ and $d_{ij}^{nm} := 0$ otherwise.

Discussion

$$\frac{\operatorname{vol} \mathcal{T}(\Sigma_m(C), \alpha)}{\operatorname{vol} \mathrm{Gr}_{n,m}} \leq \frac{4m(n-m)}{n} \binom{n/2}{m/2} \sum_{j=0}^{n-2} V_{j+1}(C) \cdot F_j^{nm}(\alpha)$$

- The result is an extension of Weyl's spherical tube formula.
- The only dependence on C is through the intrinsic volumes!
- For the proof we may assume wlog that C has a smooth boundary with positive curvature (by continuity)!

- Theorems I-II follow by estimations using $V_j(C) \leq 1$ or, more precisely, $V_j(C) \leq v(C) f_j(n)$, respectively.

Sharpness of the bound

$$\frac{\mathrm{vol}\,\mathcal{T}(\Sigma_m(C),\alpha)}{\mathrm{vol}\,\mathrm{Gr}_{n,m}} \leq \frac{4m(n-m)}{n}\binom{n/2}{m/2}\sum_{j=0}^{n-2} V_{j+1}(C)\cdot F_j^{nm}(\alpha)$$

- The bound is asymptotically sharp for $\alpha \to 0$.
- If the tube $\mathcal{T}(C \cap S^{n-1}, \alpha)$ is convex, we can even obtain get an exact formula, by using modified functions $F_j^{nm}(\alpha)$.
- If the cone C has smooth boundary with positive curvature, then $\mathcal{T}(C \cap S^{n-1}, \alpha)$ is convex for sufficiently small radius α.
- However, for our cones of interest, this convexity assumption is violated.

- Under the convexity assumption, the exact formula was already obtained by Glasauer 1995 (PhD thesis, University of Freiburg).
- However, Glasauer's works with measure theoretic techniques, which don't provide inequalities and thus results for our cones of interest.

The main geometric idea of proof

- $\mathrm{Gr}_{n,m}$ is a Riemannian manifold and thus has exponential maps $\exp_W\colon T_W \mathrm{Gr}_{n,m} \to \mathrm{Gr}_{n,m}$ at $W \in \mathrm{Gr}_{n,m}$.
- Let $C \subseteq \mathbb{R}^n$ be a regular cone such that $K := S^{n-1} \cap C$ has smooth boundary $M := \partial K$ with positive curvature.
- Then $\Sigma_m := \Sigma_m(C)$ is a smooth oriented hypersurface of $\mathrm{Gr}_{n,m}$ bounding $\mathcal{D}_m(C)$ and $\mathcal{P}_m(C)$. Let ν denote the unit normal vector field of Σ_m pointing inside $\mathcal{D}_m(C)$.
- The α-tube $\mathcal{T}(\Sigma_m, \alpha)$ around Σ_m is the image of

$$\Psi\colon \Sigma_m \times [-\alpha, \alpha] \to \mathrm{Gr}_{n,m},\ (W, \theta) \mapsto \exp_W\big(\theta\, \nu(W)\big).$$

- By the coarea formula

$$\mathrm{vol}\, \mathcal{T}(\Sigma_m, \alpha) \;=\; \int_{-\alpha}^{\alpha} d\theta \int_{\Sigma_m} \mathrm{NJ}\Psi \, d\Sigma_m \,.$$

- Need a parametrization of Σ_m and need to compute $\mathrm{NJ}\Psi$.

Geometry of ill-posed set Σ_m

- Recall that $\Sigma_m := \Sigma_m(C) \subseteq \mathrm{Gr}_{n,m}$ consists of the m-dimensional subspaces W touching C.
- Each W touches $K = S^{n-1} \cap C$ in a unique point p due to positive curvature of $M = \partial K$. Write $Y := p^\perp \cap W$. Then $Y \in \mathrm{Gr}(T_pM, m-1)$ and $W = \mathbb{R}p + Y$.
- The fiber over p of the map

$$\Pi_M \colon \Sigma_m \to M, W \mapsto p, \text{ where } W \cap K = \{p\}$$

basically equals $F_p := \mathrm{Gr}(T_pM, m-1)$. We can thus view Σ_m as an embedding of the $(m-1)$th Grassmann bundle over M.
- By the coarea formula:

$$\int_{\Sigma_m} \mathrm{NJ}\Psi \, d\Sigma_m = \int_{p \in M} dM(p) \int_{Y \in F_p} \mathrm{NJ}\Pi_M \cdot \mathrm{NJ}\Psi \, dF_p(Y) \, .$$

Thank you, and

All the Best for You, Mike!!!

Twisted characteristic polynomial

- Let $\mathcal{W}_p \colon T_pM \to T_pM$ denote the Weingarten map of M at p: the eigenvalues of $\mathcal{W}_p$ are the principle curvatures of the smooth hypersurface M of S^{n-1}.
- Let $\sigma_k(p)$ denote the kth elementary symmetric polynomial in the principal curvatures of M at p.
- Weyl: For $1 \leq j \leq n-1$

$$V_j(C) \; = \; \frac{1}{\mathcal{O}_{j-1} \cdot \mathcal{O}_{n-j-1}} \cdot \int_{p \in M} \sigma_{n-j-1}(p) \, dM \; ,$$

- Let $Y \in \mathrm{Gr}(T_pM, m-1)$ and $\Pi_Y \colon V \to Y$ denote the orthogonal projection onto Y.
- We define the twisted characteristic polynomial of $\mathcal{W}_p$ with respect to Y as

$$\mathrm{ch}_Y(\mathcal{W}_p, t) := \det \left(\mathcal{W}_p - \left(t \cdot \Pi_Y - \tfrac{1}{t} \cdot \Pi_{Y^\perp} \right) \right) \cdot t^{n-m-1} \; .$$

Normal Jacobians

Recall $\Psi(W,\theta) = \exp_W\big(\theta\,\nu(W)\big)$ and $F_p := \mathrm{Gr}(T_pM, m-1)$

Theorem (technical!)

$$(\mathrm{NJ}\Pi_M \cdot \mathrm{NJ}\Psi)(W,\theta) \;=\; (\cos\theta)^{n-2}\,\mathrm{ch}_Y(\mathcal{W}_p, \tan\theta)$$

$$\mathop{\mathbb{E}}_{Y\in F_p}\Big[\big|\,\mathrm{ch}_Y(\mathcal{W}_p, t)\big|\Big] \;\le\; \sum_{i,j=0}^{n-2} d_{ij}^{nm}\cdot\sigma_{n-2-j}(p)\cdot t^{n-i-2}\,.$$

Wrapping up:

$$\begin{aligned}
\mathrm{vol}\,\mathcal{T}(\Sigma_m,\alpha) \;&=\; \int_{-\alpha}^{\alpha} d\theta \int_{p\in M} dM(p) \int_{Y\in F_p} (\cos\theta)^{n-2}\big|\,\mathrm{ch}_Y(\mathcal{W}_p,\tan\theta)\big|\, dF_p(Y) \\
&=\; \int_{-\alpha}^{\alpha} (\cos\theta)^{n-2}\, d\theta \int_{p\in M} \mathrm{vol}(F_p) \mathop{\mathbb{E}}_{Y\in F_p}\Big[\big|\,\mathrm{ch}_Y(\mathcal{W}_p,\tan\theta)\big|\Big]\, dM(p) \\
&\overset{Thm.}{\le}\; \mathrm{vol}(F_p)\sum_{i,j} d_{ij}^{nm} \int_{p\in M} \sigma_{n-2-j}(p)\, dM(p) \int_{-\alpha}^{\alpha} (\cos\theta)^{n-2}(\tan\theta)^{n-i-2}\, d\theta
\end{aligned}$$