"Metric Aspects in Algebraic Geometry, on the Average". (Celebrating the work of Mike Shub)

Luis M. Pardo[1]

May, 2012

[1] Univ. de Cantabria.

Mike's Influence

Mike's ideas have strongly influenced my work in the last decade.

Mike's Influence

Mike's ideas have strongly influenced my work in the last decade.

Specially, his work with Steve Smale on Numerical Solving of Polynomial Equations.

Mike's Influence

Mike's ideas have strongly influenced my work in the last decade.

Specially, his work with Steve Smale on Numerical Solving of Polynomial Equations.

His influence and their work was essential to deal with Smale's 17th Problem : [Beltrán-P., 2009]...

Mike's Influence

Mike's ideas have strongly influenced my work in the last decade.

Specially, his work with Steve Smale on Numerical Solving of Polynomial Equations.

His influence and their work was essential to deal with Smale's 17th Problem : [Beltrán-P., 2009]...

But these are already "old" mathematical results.

Mike's Influence

Mike's ideas have strongly influenced my work in the last decade.

Specially, his work with Steve Smale on Numerical Solving of Polynomial Equations.

His influence and their work was essential to deal with Smale's 17th Problem : [Beltrán-P., 2009]...

But these are already "old" mathematical results. I wanted something fresh, specifically oriented for this conference in Mike's honor.

Mike's Influence

Mike's ideas have strongly influenced my work in the last decade.

Specially, his work with Steve Smale on Numerical Solving of Polynomial Equations.

His influence and their work was essential to deal with Smale's 17th Problem : [Beltrán-P., 2009]...

But these are already "old" mathematical results. I wanted something fresh, specifically oriented for this conference in Mike's honor.

Thus, I tried to work on some (maybe modest and preliminary) results based on ideas from Mike's work.

Some Self-Constraints

Some Self-Constraints

* No Condition Number.

Some Self-Constraints

* No Condition Number.
* No Complexity.

Some Self-Constraints

* No Condition Number.
* No Complexity.
* No Homotopy/Path Continuation Methods.

Some Self-Constraints

* No Condition Number.
* No Complexity.
* No Homotopy/Path Continuation Methods.
* No Polynomial System Solving.

Mike's Inspiring Source

L. Blum, M. Shub, *Evaluating Rational Functions: Infinite Precision is finite cost and Tractable on average*, SIAM J. on Comput. **15** (1986) 384–398.

Mike's Inspiring Source

L. Blum, M. Shub, *Evaluating Rational Functions: Infinite Precision is finite cost and Tractable on average*, SIAM J. on Comput. **15** (1986) 384–398.

Main outcome of this manuscript:

Mike's Inspiring Source

L. Blum, M. Shub, *Evaluating Rational Functions: Infinite Precision is finite cost and Tractable on average*, SIAM J. on Comput. **15** (1986) 384–398.

Main outcome of this manuscript:

Theorem

$$\frac{vol\{x \in B(0,r) \ : \ |Q(x)| < \varepsilon\}}{vol[B(0,r)]} \leq C_Q \frac{\varepsilon^{1/d}}{r},$$

where $Q \in \mathbb{R}[X_1, \ldots, X_n]$ is a polynomial of degree at most d and $B(0,r)$ is the ball of radius r centered at the origin.

Transforming this outcome into a recipe for this conference

Put something concerning the growth of the absolute value of multivariate polynomials.

Transforming this outcome into a recipe for this conference

Put something concerning the growth of the absolute value of multivariate polynomials.

Add some average and probability.

Transforming this outcome into a recipe for this conference

Put something concerning the growth of the absolute value of multivariate polynomials.

Add some average and probability.

And, finally, add some algebraic varieties and metrics and see what happens...

Main Topics of the Talk

- On the Expected Growth of Multivariate Polynomials

Main Topics of the Talk

- On the Expected Growth of Multivariate Polynomials
- On the Expected Separations of zeros of a polynomial system (illustrating Mike's double fibration technique).

Main Topics of the Talk

- On the Expected Growth of Multivariate Polynomials
- On the Expected Separations of zeros of a polynomial system (illustrating Mike's double fibration technique).
- On the Expected Distance between two complex projective varieties (same technique).

Main Topics of the Talk

- On the Expected Growth of Multivariate Polynomials
- On the Expected Separations of zeros of a polynomial system (illustrating Mike's double fibration technique).
- On the Expected Distance between two complex projective varieties (same technique).
- On the Expected average height of resultants:

Main Topics of the Talk

- On the Expected Growth of Multivariate Polynomials
- On the Expected Separations of zeros of a polynomial system (illustrating Mike's double fibration technique).
- On the Expected Distance between two complex projective varieties (same technique).
- On the Expected average height of resultants: An Arithmetic Poisson Formula for the Multi-variate Resultant.

Basic Notations (I)

*(d):= $(d_1, \ldots, d_m)$ a list of degrees.

* $\{X_0, \ldots, X_n\}$:= A list of variables

* $\mathcal{H}_{(d)}^{(m)}$:= Lists $(f_1, \ldots, f_m)$ of m complex homogeneous polynomials of respective degrees $deg(f_i) = d_i$.

*$\mathcal{P}_{(d)}^{(m)}$:= Affine polynomials in $\{X_1, \ldots, X_n\}$ with $deg(f_i) \leq d_i$.

*$\mathcal{D}_{(d)}$:=$\prod_{i=1}^{m} d_i$ the Bézout number.

* N := the complex dimension of $\mathcal{H}_{(d)}^{(m)}$ is $N + 1$.

Basic Notations (II)

*$\mathbb{P}_n(\mathbb{C})$:= $\mathbb{P}(\mathbb{C}^{n+1})$, the projective complex space,

*$d_R(x, y)$:= the Riemannian distance between two points $x, y \in \mathbb{P}(\mathbb{C}^{n+1})$ and

$$d_{\mathbb{P}}(x, y) := \sin d_R(x, y),$$

the "projective" distance.

*$V_{\mathbb{P}}(f)$:= for $f \in \mathcal{H}_{(d)}^{(m)}$, the projective variety (in $\mathbb{P}_n(\mathbb{C})$) of the common zeros of polynomials in the list f.

*$V_{\mathbb{A}}(f)$:= for $f \in \mathcal{P}_{(d)}^{(m)}$, the affine variety (in $\mathbb{C}^n$) of the common zeros of polynomials in the list f.

Bombieri-Weil norm

An Hermitian form which is also an expectation:

$$||f||_{\Delta}^2 := \binom{d+n}{n} \frac{1}{vol[S^{2n+1}]} \int_{S^{2n+1}} |f(z)|^2 d\nu_S(z) = \binom{d+n}{n} E_{S^{2n+1}}[|f|^2].$$

Bombieri-Weil norm

An Hermitian form which is also an expectation:

$$||f||_{\Delta}^{2} := \binom{d+n}{n} \frac{1}{vol[S^{2n+1}]} \int_{S^{2n+1}} |f(z)|^2 d\nu_S(z) = \binom{d+n}{n} E_{S^{2n+1}}[|f|^2].$$

* $\mathbb{S}(\mathcal{H}_{(d)}^{(m)}) :=$ the sphere of radius one in $\mathcal{H}_{(d)}^{(m)}$ with respect to Bombieri's norm $|| \cdot ||_{\Delta}$.

Growth of Polynomials

Expected Growth of Polynomials

An almost immediate statement

An Estimate

Assume $deg(f) = d \in 2\mathbb{Z}$ is of even degree. Then,

$$E_{\gamma}[||f(z)||] \leq ||f||_{\Delta} \left(\sum_{k=0}^{d/2} \binom{d/2}{k} \frac{2^{k-n-1}\Gamma(n+k+1)}{\pi^n \Gamma(n+1)} \right),$$

where E_{γ} is the expectation with respect to the Gaussian distribution in $\mathbb{C}^n$.

Let's look for something a little bit sharper (I)

Assume $\mathbb{C}^n$ is endowed with the pull–back distribution induced by the canonical embedding $\varphi_0 : \mathbb{C}^n \longrightarrow \mathbb{P}_n(\mathbb{C})$.

Let's look for something a little bit sharper (I)

Assume $\mathbb{C}^n$ is endowed with the pull–back distribution induced by the canonical embedding $\varphi_0 : \mathbb{C}^n \longrightarrow \mathbb{P}_n(\mathbb{C})$. Change $|f(z)|$ by $\log |f(z)|$,

Let's look for something a little bit sharper (I)

Assume $\mathbb{C}^n$ is endowed with the pull–back distribution induced by the canonical embedding $\varphi_0 : \mathbb{C}^n \longrightarrow \mathbb{P}_n(\mathbb{C})$. Change $|f(z)|$ by $\log |f(z)|$, and change the role of $||f||_\Delta$.

Let's look for something a little bit sharper (I)

Assume $\mathbb{C}^n$ is endowed with the pull–back distribution induced by the canonical embedding $\varphi_0 : \mathbb{C}^n \longrightarrow \mathbb{P}_n(\mathbb{C})$. Change $|f(z)|$ by $\log|f(z)|$, and change the role of $||f||_\Delta$. Then, study:

$$\mathfrak{E} := E_{f \in \mathbb{S}(P_d^{(n)})}[E_{\mathbb{C}^n}[\log|f|]],$$

where $\mathbb{S}(P_d^{(n)})$ is the sphere of radius one with respect to Bombieri–Weil norm.

Let's look for something a little bit sharper (I)

Assume $\mathbb{C}^n$ is endowed with the pull–back distribution induced by the canonical embedding $\varphi_0 : \mathbb{C}^n \longrightarrow \mathbb{P}_n(\mathbb{C})$. Change $|f(z)|$ by $\log |f(z)|$, and change the role of $||f||_\Delta$. Then, study:

$$\mathfrak{E} := E_{f \in \mathbb{S}(P_d^{(n)})}[E_{\mathbb{C}^n}[\log |f|]],$$

where $\mathbb{S}(P_d^{(n)})$ is the sphere of radius one with respect to Bombieri–Weil norm.

This is the average value of something similar to the marking time in [Blum-Shub, 86].

A little bit sharper (II)

Proposition

With these notations we have:

$$\mathfrak{E} := \frac{1}{2}\left(dH_n - H_R\right),$$

where H_r *is the* $r-$*th harmonic number and* $R := \binom{d+n}{n}$

A little bit sharper (II)

Proposition

With these notations we have:

$$\mathfrak{E} := \frac{1}{2}\left(dH_n - H_R\right),$$

where H_r is the $r-$th harmonic number and $R := \binom{d+n}{n}$

Recall that

$$H_r \approx \log(r) + \gamma + O(\frac{1}{r}),$$

where γ is Euler-Mascheroni number and, hence

$$\mathfrak{E} \approx \frac{d}{2}\log\left(\frac{dn}{d+n}\right).$$

An example: A Complementary of [Blum-Shub, 86]

Corollary

$$\mathfrak{E} := E_{f \in \mathbb{S}(P_d^{(n)})}[E_{\mathbb{C}^n}[|f|^{-1}]] \geq e^{-\frac{d}{2}(H_n - H_R/d)},$$

Other "potential" applications (not finished yet)

Motivated by the talk by Diego Armentano and question (d) in [Armentano-Shub, 12]...slighty modified...

Other "potential" applications (not finished yet)

Motivated by the talk by Diego Armentano and question (d) in [Armentano-Shub, 12]...slighty modified... Hints on $I(f)$, for $f \in \mathcal{H}_{(d)}^{(n)}$, and:

$$I(f) := \int_{S^{2n+1}} \frac{e^{\frac{||f(z)||^2}{2}}}{||f(z)||^{2n-1}} dz.$$

Other "potential" applications (not finished yet)

Motivated by the talk by Diego Armentano and question (d) in [Armentano-Shub, 12]...slighty modified... Hints on $I(f)$, for $f \in \mathcal{H}_{(d)}^{(n)}$, and:

$$I(f) := \int_{S^{2n+1}} \frac{e^{\frac{||f(z)||^2}{2}}}{||f(z)||^{2n-1}} dz.$$

A good "hint" could be the Expectation $E_{\mathbb{S}(\mathcal{H}_{(d)})}[I(f)]$ and this is given by the following:

Other "potential" applications (not finished yet)

Motivated by the talk by Diego Armentano and question (d) in [Armentano-Shub, 12]...slighty modified... Hints on $I(f)$, for $f \in \mathcal{H}_{(d)}^{(n)}$, and:

$$I(f) := \int_{S^{2n+1}} \frac{e^{\frac{||f(z)||^2}{2}}}{||f(z)||^{2n-1}} dz.$$

A good "hint" could be the Expectation $E_{\mathbb{S}(\mathcal{H}_{(d)})}[I(f)]$ and this is given by the following:

$$E_{\mathbb{S}(H_d)}[I(f)] = \left(\sum_{k=0}^{\infty} \frac{\nu_{2n+1}}{k! 2^k} \left(\sum_{j=0}^{\infty} \frac{(2k-n+1)^j}{j!} E_{\mathbb{S}(\mathcal{H}_{(d)})}[E_{S^{2n+1}}[\log^j ||f||]] \right) \right)$$

Separation of Solutions

On the average separation of the solutions

A Technique: solution variaties like as desing. à la Room-Kempf (I)

Double fibration (introduced and used by M. Shub and S. Smale in their Bézout series) with deep and interesting consequences, when combined with Federer's co-area formula.

A Technique: solution variaties like as desing. à la Room-Kempf (I)

Double fibration (introduced and used by M. Shub and S. Smale in their Bézout series) with deep and interesting consequences, when combined with Federer's co-area formula.

We consider the (smooth) solution variety:

$$V_{(d)}^{(m)} = \{(f, \zeta) \in \mathbb{P}(\mathcal{H}_{(d)}^{(m)}) \times \mathbb{P}_n(\mathbb{C}) : \zeta \in V_{\mathbb{P}}(f)\} \subseteq \mathbb{P}(\mathcal{H}_{(d)}^{(m)}) \times \mathbb{P}_n(\mathbb{C}).$$

A Technique: solution variaties like as desing. à la Room-Kempf (I)

Double fibration (introduced and used by M. Shub and S. Smale in their Bézout series) with deep and interesting consequences, when combined with Federer's co-area formula.

We consider the (smooth) solution variety:

$$V_{(d)}^{(m)} = \{(f, \zeta) \in \mathbb{P}(\mathcal{H}_{(d)}^{(m)}) \times \mathbb{P}_n(\mathbb{C}) : \zeta \in V_{\mathbb{P}}(f)\} \subseteq \mathbb{P}(\mathcal{H}_{(d)}^{(m)}) \times \mathbb{P}_n(\mathbb{C}).$$

and we consider the two canonical projections:

$$\begin{array}{ccc} & V_{(d)}^{(m)} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathbb{P}(\mathcal{H}_{(d)}^{(m)}) & & \mathbb{P}_n(\mathbb{C}) \end{array}$$

A Technique: solution variaties like as desing. à la Room-Kempf (II)

* $\pi_2^{-1}(x) :=$ is a "linear" (of co–dimension m) in $\mathbb{P}(\mathcal{H}_{(d)}^{(m)})$.

* $\pi_1^{-1}(f) :=$ is the set of common zeros $V_{\mathbb{P}}(f_1, \ldots, f_m)$ and it is "generically" a smooth projective variety of co–dimension m.

* For $m \leq n$, π_1 is onto.

A Technique: solution variaties like as desing. à la Room-Kempf (II)

* $\pi_2^{-1}(x) :=$ is a "linear" (of co–dimension m) in $\mathbb{P}(\mathcal{H}_{(d)}^{(m)})$.
* $\pi_1^{-1}(f) :=$ is the set of common zeros $V_{\mathbb{P}}(f_1, \ldots, f_m)$ and it is "generically" a smooth projective variety of co–dimension m.
* For $m \leq n$, π_1 is onto.

Shub–Smale's idea

Averaging in $\mathbb{P}(\mathcal{H}_{(d)}^{(m)})$ *may be translated to averaging in* $\mathbb{P}_n(\mathbb{C})$ *through this double fibration.*

Separation of Solutions

For a zero–dimensional complete intersection variety $V_{\mathbb{P}}(f) \subseteq \mathbb{P}_n(\mathbb{C})$, the separation among its zeros:

$$sep(f) := \min\{d_{\mathbb{P}}(\zeta, \zeta') \ : \ \zeta, \zeta' \in V_{\mathbb{P}}(f), \zeta \neq \zeta'\}.$$

Lower bounds for these quantity are due to many authors:

- *The Davenport–Mahler-Mignotte lower bound for the univariate case:* $\Omega(2^{-d^2})$
- *Other authors (Dedieu, Emiris, Mourrain, Tsigaridas, ...) have also treated the multivariate* [2] *case:*

$$\Omega(2^{-\mathcal{D}(d)}).$$

[2] In [Castro–Haegele–Morais, p., 01] we also exhibited examples where this lower bounds is achieved

Separation of Solutions: An algorithmic question

* $f \in \mathcal{H}_{(d)}$
* $z_1, z_2 \in \mathbb{P}_n(C)$ that satisfy $\alpha-$Theorem ([Shub–Smale]):

$$\alpha(f, z_1) \leq \alpha_0, \;\; \alpha(f, z_2) \leq \alpha_0.$$

Decide whether:

$$\lim_{k \mapsto \infty} N_f(z_1) = \lim_{k \mapsto \infty} N_f(z_2)?.$$

Separation of Solutions: An algorithm

* $t \in \mathbb{N}$

eval

$$z_1^{(t)} := N_f^t(z_1), \quad z_2^{(t)} := N_f^t(z_2).$$

if $d_{\mathbb{P}}(z_1^{(t)}, z_2^{(t)}) > \frac{2}{2^{2^{t-1}}}$, **then** OUTPUT: They approach DIFFERENT zeros of f

else, OUTPUT: They approach the same zero of f

fi

Separation of Solutions: An algorithm

* $t \in \mathbb{N}$

eval

$$z_1^{(t)} := N_f^t(z_1), \quad z_2^{(t)} := N_f^t(z_2).$$

if $d_{\mathbb{P}}(z_1^{(t)}, z_2^{(t)}) > \frac{2}{2^{2^{t-1}}}$, **then** OUTPUT: They approach DIFFERENT zeros of f

else, OUTPUT: They approach the same zero of f

fi

The algorithm works provided that:

$$sep(f) > \frac{4}{2^{2^{t-1}}}.$$

Expected Minimum Separation (I)

For a zero–dimensional complete Intersection variety $V_{\mathbb{P}}(f) \subseteq \mathbb{P}_n(\mathbb{C})$, the "average" separation:

$$sep_{\text{av}}(f) := \frac{1}{\mathcal{D}_{(d)}(\mathcal{D}_{(d)} - 1)} \sum_{\zeta, \zeta' \in V_{\mathbb{P}}(f), \zeta \neq \zeta'} d_{\mathbb{P}}(\zeta, \zeta').$$

Theorem

Then, the following inequality holds:

$$E_{\mathbb{P}(\mathcal{H}_{(d)})}[sep_{\text{av}}] \geq \frac{1}{2}\sqrt{\frac{1}{d^3(N + 1/2)n}}.$$

Expected Minimum Separation (II)

What about the minimum separation of solutions?

$$sep_{\min}(f) := \min_{\zeta, \zeta' \in V_{\mathbb{P}}(f), \zeta \neq \zeta'} d_{\mathbb{P}}(\zeta, \zeta').$$

Expected Minimum Separation (II)

What about the minimum separation of solutions?

$$sep_{\min}(f) := \min_{\zeta,\zeta' \in V_{\mathbb{P}}(f), \zeta \neq \zeta'} d_{\mathbb{P}}(\zeta, \zeta').$$

Theorem

The following inequality holds:

$$E_{\mathbb{S}(\mathcal{H}_{(d)})}\left[sep_{\min}(f)\right] \geq \frac{1}{4e\mathcal{D}_{(d)} d^{3/2}} (N + 1/2)^{-1/2}.$$

Distance between two Complete Intersections

On the expected distance between two Complete Intersections

Expected Distance between two Complete Intersections

Let us consider two projective complete intersection varieties:

$$V_{\mathbb{P}}(f) := \{z \in \mathbb{P}_n(\mathbb{C}) \ : \ f_i(z) = 0, 1 \leq i \leq m\}.$$

$$V_{\mathbb{P}}(g) := \{z \in \mathbb{P}_n(\mathbb{C}) \ : \ g_j(z) = 0, 1 \leq j \leq s\}.$$

Expected Distance between two Complete Intersections

Let us consider two projective complete intersection varieties:

$$V_{\mathbb{P}}(f) := \{z \in \mathbb{P}_n(\mathbb{C}) \ : \ f_i(z) = 0, 1 \leq i \leq m\}.$$

$$V_{\mathbb{P}}(g) := \{z \in \mathbb{P}_n(\mathbb{C}) \ : \ g_j(z) = 0, 1 \leq j \leq s\}.$$

Average distance between $V_{\mathbb{P}}(f)$ and $V_{\mathbb{P}}(g)$ as

$$D_{av}(V_{\mathbb{P}}(f), V_{\mathbb{P}}(g)) := \frac{1}{vol[V_{\mathbb{P}}(f)]vol[V_{\mathbb{P}}(g)]} \int_{V_{\mathbb{P}}(f)\times V_{\mathbb{P}}(g)} d_{\mathbb{P}}(x,y) dV_{\mathbb{P}}(f) dV_{\mathbb{P}}(g).$$

Expected Distance between two Complete Intersections

Let us consider two projective complete intersection varieties:

$$V_{\mathbb{P}}(f) := \{z \in \mathbb{P}_n(\mathbb{C}) \ : \ f_i(z) = 0, 1 \leq i \leq m\}.$$

$$V_{\mathbb{P}}(g) := \{z \in \mathbb{P}_n(\mathbb{C}) \ : \ g_j(z) = 0, 1 \leq j \leq s\}.$$

Average distance between $V_{\mathbb{P}}(f)$ and $V_{\mathbb{P}}(g)$ as

$$D_{av}(V_{\mathbb{P}}(f), V_{\mathbb{P}}(g)) := \frac{1}{vol[V_{\mathbb{P}}(f)]vol[V_{\mathbb{P}}(g)]} \int_{V_{\mathbb{P}}(f)\times V_{\mathbb{P}}(g)} d_{\mathbb{P}}(x,y)dV_{\mathbb{P}}(f)dV_{\mathbb{P}}(g).$$

Theorem

With these notations, we have:

$$E_{f,g}[D_{av}(V_{\mathbb{P}}(f), V_{\mathbb{P}}(g))] = (1 - \frac{1}{n+2}).$$

Expected Distance between two Complete Intersections (II)

Same notations. Assume $V_{\mathbb{P}}(f)$ is zero–dimensional and $s \geq 1$ (i.e. $V\mathbb{P}(f) \cap V_{\mathbb{P}}(g) = \emptyset$ a.e.).
Distance between $V_{\mathbb{P}}(f)$ and $V_{\mathbb{P}}(g)$ as

$$d_{\mathbb{P}}(V_{\mathbb{P}}(f), V_{\mathbb{P}}(g)) := \min\{d_{\mathbb{P}}(x, y) \ : \ x \in V_{\mathbb{P}}(f), y \in V_{\mathbb{P}}(g)\}.$$

Expected Distance between two Complete Intersections (II)

Same notations. Assume $V_{\mathbb{P}}(f)$ is zero–dimensional and $s \geq 1$ (i.e. $V\mathbb{P}(f) \cap V_{\mathbb{P}}(g) = \emptyset$ a.e.).
Distance between $V_{\mathbb{P}}(f)$ and $V_{\mathbb{P}}(g)$ as

$$d_{\mathbb{P}}(V_{\mathbb{P}}(f), V_{\mathbb{P}}(g)) := \min\{d_{\mathbb{P}}(x,y) \ : \ x \in V_{\mathbb{P}}(f), y \in V_{\mathbb{P}}(g)\}.$$

Theorem

Assume $V_{\mathbb{P}}(g)$ is of co–dimension s and $deg(V_{\mathbb{P}}(g)) := \mathcal{D}' = \prod_{i=1}^{s} d_i'$, we have

$$E_{f,g}[d_{\mathbb{P}}(V_{\mathbb{P}}(f), V_{\mathbb{P}}(g))] \geq \frac{2s-1}{\mathcal{D}_{(d)}\left(1 + 2\left(\prod_{i=1}^{s} \frac{d_i' e^2}{s^2}\right)\right)},$$

Moreover, for $s \geq 3$, this may be rewritten as

$$E_{f,g}[d_{\mathbb{P}}(V_{\mathbb{P}}(f), V_{\mathbb{P}}(g))] \geq \frac{2s-1}{\mathcal{D}_{(d)} + 2deg(V_{\mathbb{P}}(g))\mathcal{D}_{(d)}}.$$

On the Height of the Multi-variate Resultant Variety

On the Height of the Multi-variate Resultant: an Arithmetic Poisson formula

Multi-variate Resultant (I)

We recall the "solution variety" in the case of over–determined systems:

$$V_{(d)}^{(n+1)} = \{(f,\zeta) \in \mathbb{P}(\mathcal{H}_{(d)}^{(n+1)}) \times \mathbb{P}_n(\mathbb{C}) : \zeta \in V_{\mathbb{P}}(f)\} \subseteq \mathbb{P}(\mathcal{H}_{(d)}^{(n+1)}) \times \mathbb{P}_n(\mathbb{C}).$$

and we also consider the two canonical projections:

$$\begin{array}{ccc} & V_{(d)}^{(n+1)} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathbb{P}(\mathcal{H}_{(d)}^{(n+1)}) & & \mathbb{P}_n(\mathbb{C}) \end{array}$$

Multi-variate Resultant (II)

* $\pi_2^{-1}(x) :=$ is a "linear" (of co–dimension $n+1$) in $\mathbb{P}(\mathcal{H}_{(d)}^{(n+1)})$.

* $\pi_1^{-1}(f) :=$ is the set of common zeros $V_{\mathbb{P}}(f_0, \ldots, f_n)$ and it is either $\emptyset$ or "generically" a single point.

* $\pi_1(V_{(d)}^{(n+1)}) := R_{(d)}^{(n+1)}$ is an irreducible complex hyper–surface, usually known as the *multi–variate resultant variety*.

* Multi–variate resultant $Res_{(d)}^{(n+1)} :=$ is the multi–homogeneous irreducible polynomial which defines $R_{(d)}^{(n+1)}$.

Multi-variate Resultant (III)

Multi-variate Resultant and Resultant Variety $R_{(d)}^{(n+1)}$ is a classical object in Elimination Theory (also Computational Algebraic Geometry, MEGA...).

It has been studied by many authors since XIX-th century with different approaches and variations: Bézout, Sylvester, Macaulay, Chow,..., and, more recently, Jouanolou, Chardin, Gelfand, Kapranov, Zelevinsky, Sturmfels, Rojas, Heintz, Giusti, Dickenstein, D'Andrea, Krick, Szanto, Sombra...

Multi-variate Resultant (III)

Multi-variate Resultant and Resultant Variety $R_{(d)}^{(n+1)}$ is a classical object in Elimination Theory (also Computational Algebraic Geometry, MEGA...).

It has been studied by many authors since XIX-th century with different approaches and variations: Bézout, Sylvester, Macaulay, Chow,..., and, more recently, Jouanolou, Chardin, Gelfand, Kapranov, Zelevinsky, Sturmfels, Rojas, Heintz, Giusti, Dickenstein, D'Andrea, Krick, Szanto, Sombra...

The list is too long to be complete...

Height of the Multi-Variate Resultant (I)

* Height of the multivariate resultant is an attempt to measure the length of the integer coefficients...

Height of the Multi-Variate Resultant (I)

* Height of the multivariate resultant is an attempt to measure the length of the integer coefficients...

It has interesting applications both in complexity and arithmetic geometry.

With several variations (Chow forms, elimination polynomials, Arithmetic Nullstellensatz...) it has been studied by many authors: Nesterenko, Philippon, Krick, Sombra, D'Andrea, Rémond, P. ...

Height of the Multi-Variate Resultant (I)

* Height of the multivariate resultant is an attempt to measure the length of the integer coefficients...

It has interesting applications both in complexity and arithmetic geometry.

With several variations (Chow forms, elimination polynomials, Arithmetic Nullstellensatz...) it has been studied by many authors: Nesterenko, Philippon, Krick, Sombra, D'Andrea, Rémond, P. ...

Multi–variate resultants satisfy a Poisson formula which is a helpful statement for the knowledge of its properties.

Here, we are modest: we focus on the arithmetic version of Poisson Formula, whose geometric (degree) property can be stated as follows:

“Geometric” Poisson Formula

Let $(d) := (d_0, d_1, \ldots, d_n)$ be a degree list, and $(d') := (d_1, \ldots, d_n)$.

"Geometric" Poisson Formula

Let $(d) := (d_0, d_1, \ldots, d_n)$ be a degree list, and $(d') := (d_1, \ldots, d_n)$.

Let $Res_{(d)}^{(n+1)}$ be the resultant associated to (d) in $n+1$ variables and $Res_{(d')}^{(n)}$ the corresponding one associated to (d').

"Geometric" Poisson Formula

Let $(d) := (d_0, d_1, \ldots, d_n)$ be a degree list, and $(d') := (d_1, \ldots, d_n)$.

Let $Res_{(d)}^{(n+1)}$ be the resultant associated to (d) in $n+1$ variables and $Res_{(d')}^{(n)}$ the corresponding one associated to (d').

For instance, Poisson's Formula implies:

"Geometric" Poisson Formula

With these notations we have:

$$deg(Res_{(d)}^{(n+1)}) = d_0 deg(Res_{(d')}^{(n)}) + \mathcal{D}_{(d')},$$

where $\mathcal{D}_{(d')} := \prod_{i=1}^{n} d_i$ is the Bézout number.

"Geometric" Poisson Formula

Let $(d) := (d_0, d_1, \ldots, d_n)$ be a degree list, and $(d') := (d_1, \ldots, d_n)$.

Let $Res_{(d)}^{(n+1)}$ be the resultant associated to (d) in $n+1$ variables and $Res_{(d')}^{(n)}$ the corresponding one associated to (d').
For instance, Poisson's Formula implies:

"Geometric" Poisson Formula

With these notations we have:

$$deg(Res_{(d)}^{(n+1)}) = d_0 deg(Res_{(d')}^{(n)}) + \mathcal{D}_{(d')},$$

where $\mathcal{D}_{(d')} := \prod_{i=1}^{n} d_i$ is the Bézout number.

Inductively, we conclude :

$$deg(Res_{(d)}^{(n+1)}) = \sum_{i=0}^{n} \prod_{j \neq i} d_j.$$

Height of the Multi-Variate Resultant (I)

We may define the logarithmic height of the multi–resultant variety either following any of the usual definitions [Philippon, 91], [Bost, Gillet, Soulé, 94], [Rémon,01], [McKinnon, 01], [D'Andrea, Krick, Sombra, 11]....
We just modify them by using the unitarily invariant height $ht_u(R_{(d)}^{(n+1)})$:

Height of the Multi-Variate Resultant (I)

We may define the logarithmic height of the multi–resultant variety either following any of the usual definitions [Philippon, 91], [Bost, Gillet, Soulé, 94], [Rémon,01], [McKinnon, 01], [D'Andrea, Krick, Sombra, 11]....
We just modify them by using the unitarily invariant height $ht_u(R_{(d)}^{(n+1)})$:
The only difference with "usual" notions is that we take into account Bombieri's metric in the logarithmic Mahler's measure (instead of the usual Hermitian product).

(Unitarily invariant) Logarithmic Mahler measure

Define the (unitarily invariant) logarithmic Mahler measure :

$$m_{\mathfrak{S}_{(d)}^{(n+1)}}(Res_{(d)}^{(n+1)}) := E_{\mathfrak{S}_{(d)}^{(n+1)}}\left[\log |Res_{(d)}^{(n+1)}(f_0, \ldots, f_n)|\right]$$

where

$$\mathfrak{S}_{(d)}^{(n+1)} := \prod_{i=0}^{n} \mathbb{S}(H_{d_i}).$$

(Unitarily invariant) Logarithmic Mahler measure

Define the (unitarily invariant) logarithmic Mahler measure :

$$m_{\mathfrak{S}_{(d)}^{(n+1)}}(Res_{(d)}^{(n+1)}) := E_{\mathfrak{S}_{(d)}^{(n+1)}}\left[\log\left|Res_{(d)}^{(n+1)}(f_0,\ldots,f_n)\right|\right]$$

where

$$\mathfrak{S}_{(d)}^{(n+1)} := \prod_{i=0}^{n} \mathbb{S}(H_{d_i}).$$

For technical reasons, define:

$$\mathfrak{R}_{(d)}^{(n+1)} := \frac{ht_u(R_{(d)}^{(n+1)})}{\mathcal{D}_{(d)}},$$

Note that this quantity only depends of $(d) := (d_0, \ldots, d_n)$ the "degree" list.

Arithmetic Poisson Formula

Theorem

With these notations we have:

$$\mathfrak{R}_{(d)}^{(n+1)} = \prod_{i=1}^{n} \left(\frac{n}{d_i + n} \right) \left(\mathfrak{R}_{(d')}^{(n)} + \mathfrak{I}_{(d)} \right) + \frac{\mathfrak{E}}{d_0},$$

Arithmetic Poisson Formula

Theorem

With these notations we have:

$$\mathfrak{R}_{(d)}^{(n+1)} = \prod_{i=1}^{n} \left(\frac{n}{d_i + n} \right) \left(\mathfrak{R}_{(d')}^{(n)} + \mathfrak{I}_{(d)} \right) + \frac{\mathfrak{E}}{d_0},$$

where $(d') := (d_1, \ldots, d_n)$, *and*

$$\frac{\mathfrak{E}}{d_0} = \frac{1}{2}(H_n - \frac{H_R}{d_0})$$

Arithmetic Poisson Formula

Theorem

With these notations we have:

$$\mathfrak{R}_{(d)}^{(n+1)} = \prod_{i=1}^{n} \left(\frac{n}{d_i + n}\right) \left(\mathfrak{R}_{(d')}^{(n)} + \mathfrak{I}_{(d)}\right) + \frac{\mathfrak{E}}{d_0},$$

where $(d') := (d_1, \ldots, d_n)$, *and*

$$\frac{\mathfrak{E}}{d_0} = \frac{1}{2}(H_n - \frac{H_R}{d_0})$$

$$-\frac{1}{4} \leq \mathfrak{I}_{(d)} \leq 0.$$

Some corollaries (I)

Corollary

With the same notations, we have:

$$|\mathfrak{R}_{(d)}^{(n+1)} - \prod_{i=1}^{n} \left(\frac{n}{d_i + n}\right) \mathfrak{R}_{(d')}^{(n)}| \leq \frac{1}{2}\log(\frac{d_0 n}{d_0 + n}) + O(\frac{1}{n}),$$

Some Corollaries (II)

The straightforward inductive argument yields

$$\mathfrak{R}_{(d)}^{(n+1)} \leq \frac{1}{2}\sum_{i=0}^{n} \log\left(\frac{d_i n}{d_i + n}\right) + O(1) \approx \frac{1}{2} n \log(n) + O(1),$$

and

Some Corollaries (II)

The straightforward inductive argument yields

$$\mathfrak{R}_{(d)}^{(n+1)} \leq \frac{1}{2}\sum_{i=0}^{n} \log\left(\frac{d_i n}{d_i + n}\right) + O(1) \approx \frac{1}{2} n\log(n) + O(1),$$

and

$$ht_u(R_{(d)}^{(n+1)}) \leq \frac{\mathcal{D}_{(d)}}{2}\left(\sum_{i=0}^{n} \log\left(\frac{d_i n}{d_i + n}\right) + c\right) \approx \frac{\mathcal{D}_{(d)}}{2}(n\log(n) + c).$$

Controlling the growth (in the sense of [Blum-Shub, 86])

Let us consider now the Gaussian distribution γ_Δ in $\mathcal{H}_{(d)}^{(n+1)}$ induced by the Bombieri's norm.

Corollary

With these notations, we have:

$$\mathrm{Prob}_{\gamma_\Delta}\left[\log \mid Res_{(d)}^{(n+1)}(f_0,\ldots,f_n)\mid \geq \varepsilon^{-1} + \sum_{i=0}^{n}\left(\prod_{j\neq i} d_j\right)\log ||f||_\Delta\right] \leq$$

$$\leq \frac{\mathcal{D}_{(d)}}{2}(n\log(n)+c)\varepsilon,$$

for some constant $c > 0$.

Some corollaries (III)

* van der Waerden's U-resultant χ_U is a classical object in Elimination Theory (some times called Chow form, Elimination Polynomial,...).
* Upper bounds for the complexity of computing $U-$resultants were shown in [Jerónimo-Krick-Sabia-Sombra, 03]

Some corollaries (III)

* van der Waerden's U-resultant χ_U is a classical object in Elimination Theory (some times called Chow form, Elimination Polynomial,...).
* Upper bounds for the complexity of computing $U-$resultants were shown in [Jerónimo-Krick-Sabia-Sombra, 03]
* In [Heintz-Morgenstern, 93]: Computation of the $U-$resultant is **NP**–hard.

Some corollaries (III)

* van der Waerden's U-resultant χ_U is a classical object in Elimination Theory (some times called Chow form, Elimination Polynomial,...).
* Upper bounds for the complexity of computing $U-$resultants were shown in [Jerónimo-Krick-Sabia-Sombra, 03]
* In [Heintz-Morgenstern, 93]: Computation of the $U-$resultant is **NP**–hard.

Modestly, we may immediate obtain:

Some Corollaries (IV)

Let $\mathfrak{E}_U$ be the expected logarithmic Mahler's measure of the $U-$resultant with respect to some projective variety determined by the degree list $(d') := (d_1, \ldots, d_n)$:

Corollary

With these notations, we have

$$E_{\mathfrak{S}^{(n)}_{(d')}}[m(\chi_U)] \leq \left(\prod_{i=1}^{n} d_i\right)(n\log(n) + c),$$

And something diophantine

Finally, you may use the ideas by Hardy, Mordell, Davenport and others[3] on the equidistribution of polynomial systems with Gaussian rational coefficients of bounded height in the projective space $\mathbb{P}(H_d)$ to conclude to conclude that

[3] See also [Castro-Montaña-P.-San Martin, 02] or [P.-San Martin, 04].

And something diophantine

Finally, you may use the ideas by Hardy, Mordell, Davenport and others[3] on the equidistribution of polynomial systems with Gaussian rational coefficients of bounded height in the projective space $\mathbb{P}(H_d)$ to conclude to conclude that

Corollary

The same bound holds for the expected Mahler's measure of the $U-$resultant for random systems with Gaussian rational coefficients of bounded height and uniform distribution.

[3] See also [Castro-Montaña-P.-San Martin, 02] or [P.-San Martin, 04].

Forthcoming Tasks

These bounds are not satisfactory yet.

Forthcoming Tasks

These bounds are not satisfactory yet. There must be improvements and more precise estimates...ongoing research...

Forthcoming Tasks

These bounds are not satisfactory yet. There must be improvements and more precise estimates...ongoing research...

* Continue with studies on average properties of MAAG.
* Compare unitarily invariant height to other notions of height......

Forthcoming Tasks

These bounds are not satisfactory yet. There must be improvements and more precise estimates...ongoing research...

* Continue with studies on average properties of MAAG.
* Compare unitarily invariant height to other notions of height......
* Continue exploration of the over–determined case...the main problem.

Forthcoming Tasks

Happy May'68, Mike!

Forthcoming Tasks

Happy May'68, Mike!

and thanks to all of you for your patience!