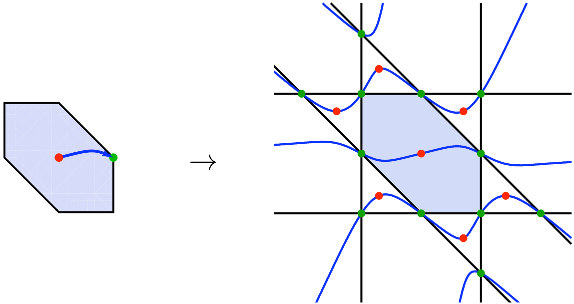
The Central Curve in Linear Programming

Cynthia Vinzant, U. Michigan



joint work with Jesús De Loera and Bernd Sturmfels

The Central Path of a Linear Program

Linear Program: $\text{Maximize}_{\mathbf{x}\in\mathbb{R}^n}$ $\mathbf{c}\cdot\mathbf{x}$ s.t. $A\cdot\mathbf{x}=\mathbf{b}$ and $\mathbf{x}\geq 0$.

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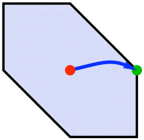

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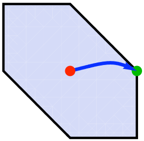

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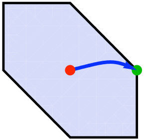

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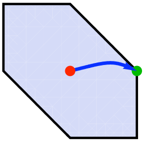

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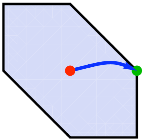

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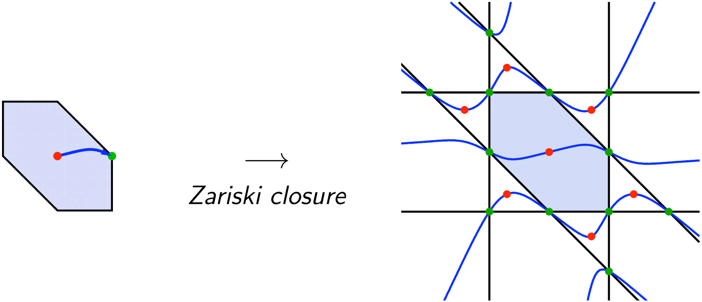
Bounds on curvature of the path $\rightarrow$ bounds on # Newton steps

We can use concepts from algebraic geometry and matroid theory to bound the total curvature of the central path.

The Central Curve

The central curve $\mathcal{C}$ is the Zariski closure of the central path. It contains the central paths of all polyhedra in the hyperplane arrangement $\{x_i = 0\}_{i=1,\ldots,n} \subset \{A \cdot \mathbf{x} = \mathbf{b}\}$.



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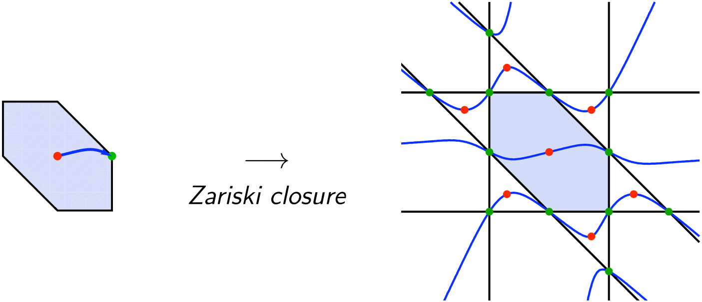

Goal: Study the nice algebraic geometry of this curve and its applications to the linear program

History and Contributions

Motivating Question: What is the maximum total curvature of the central path given the size of the matrix A?

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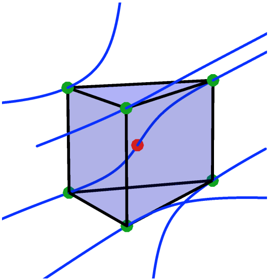
Our contribution is to use results from algebraic geometry and matroid theory to find defining equations of the central curve and refine bounds on its degree and total curvature.

Outline

- Algebraic conditions for optimality
- Degree of the curve (and other combinatorial data)
- Total curvature and the Gauss map
- Defining equations
- The primal-dual picture



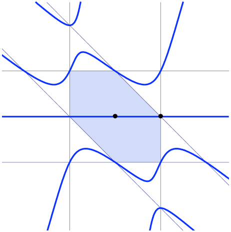
Some details

Here we assume that . . .

1) A is a $d \times n$ matrix of rank-d (possibly very special), and

2) $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^d$ are generic.

(This ensures that the central curve is irreducible and nonsingular.)



Algebraic Conditions for Optimality

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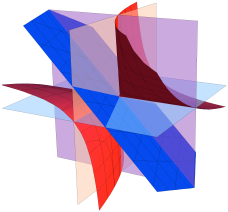

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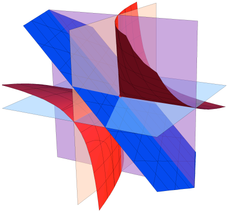

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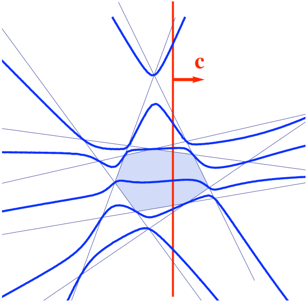
Proposition. The central curve equals the intersection of the central sheet $\mathcal{L}_{A,\mathbf{c}}^{-1}$ with the affine space $\{A\cdot\mathbf{x} = \mathbf{b}\}$.

Level sets of the cost function

Consider intersecting the central curve $\mathcal{C}$ with the level set $\{\mathbf{c} \cdot \mathbf{x} = c_0\}$.



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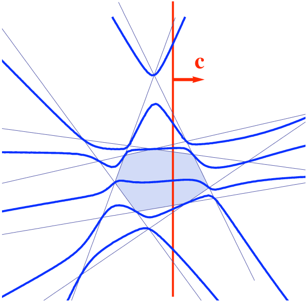

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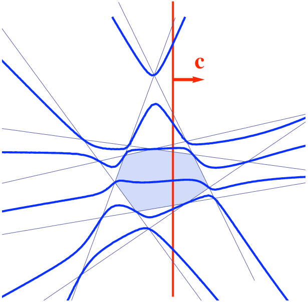

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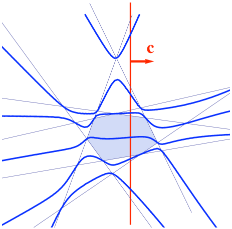

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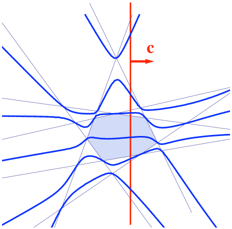

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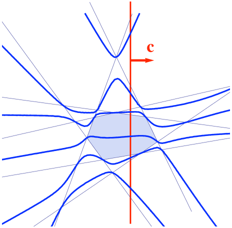

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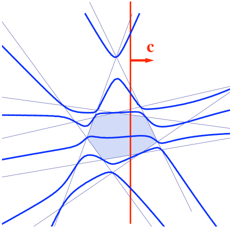
For matroid enthusiasts,
this number is the absolute value
of the Möbius invariant of $\binom{A}{\mathbf{c}}$.



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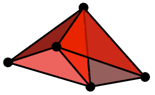

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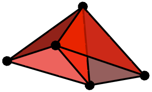

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$$\Rightarrow \quad \deg(\mathcal{C}) = \sum_{i=0}^{d} h_i \quad \text{and} \quad \text{genus}(\mathcal{C}) = 1 - \sum_{j=0}^{d} (1-j) h_j \ .$$

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Classic differential geometry: The total curvature of any real algebraic curve $\mathcal{C}$ in $\mathbb{R}^m$ is the arc length of its image under the Gauss map $\gamma : \mathcal{C} \to \mathbb{S}^{m-1}$. This quantity is bounded above by π times the degree of the projective Gauss curve in $\mathbb{P}^{m-1}$. That is,

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Theorem: The degree of the projective Gauss curve of the central curve $\mathcal{C}$ satisfies a bound in terms of matroid invariants:

$$\deg(\gamma(\mathcal{C})) \quad \leq \quad 2 \cdot \sum_{i=1}^{d} i \cdot h_i \quad \leq \quad 2 \cdot (n-d-1) \cdot \binom{n-1}{d-1}.$$

Equations

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where v runs over the vectors in kernel$\binom{A}{\mathbf{c}}$ of minimal support. These correspond to the circuits of the matroid $M(\mathcal{L}_{A,\mathbf{c}})$.

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Circuit 123

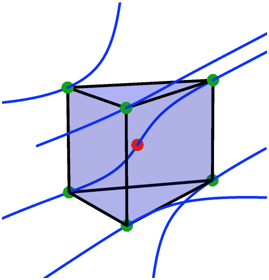
$$\rightarrow \quad v = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$$\rightarrow \quad -2x_2x_3 + 1x_1x_3 + 1x_1x_2.$$

Example

$(n = 5,\ d = 2)$

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 1 & 2 & 0 & 4 & 0 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$



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Polynomials defining $\mathcal{C}$:

$-2x_2x_3 + x_1x_3 + x_1x_2,$
$4x_2x_4x_5 - 4x_1x_4x_5 + x_1x_2x_5 - x_1x_2x_4,$
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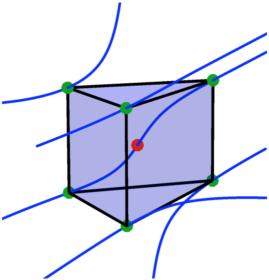
$x_1 + x_2 + x_3 - 3$
$x_4 + x_5 - 2$



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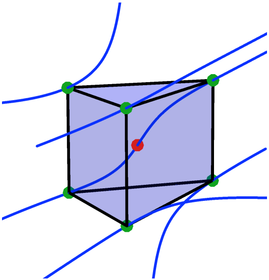

$h = (1, 2, 2) \ \Rightarrow\ \deg(\mathcal{C}) = 5, \quad \text{total curvature}(\mathcal{C}) \leq 12\pi$

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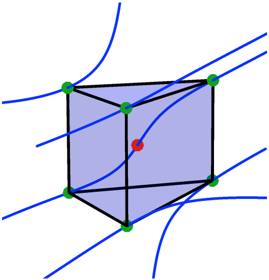

$h = (1, 2, 2) \ \Rightarrow\ \deg(\mathcal{C}) = 5, \quad \text{total curvature}(\mathcal{C}) \leq 12\pi \ (\leq 16\pi)$

Duality

Dual LP: Minimize$_{\mathbf{s}\in\mathbb{R}^n}$ $\mathbf{v}^T\mathbf{s}$ s.t. $B\cdot\mathbf{s}=\mathbf{w}$, $\mathbf{s}\geq 0$,

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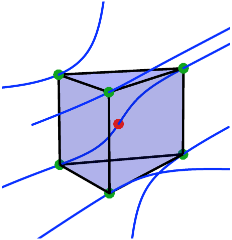

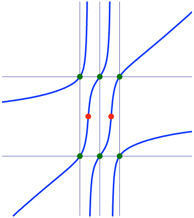

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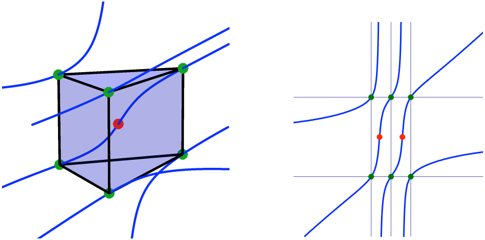

The primal-dual central path is cut out by the system of polynomial equations

$$A \cdot \mathbf{x} = \mathbf{b}, \quad B \cdot \mathbf{s} = \mathbf{w}, \quad \text{and} \quad x_1 s_1 = \ldots = x_n s_n = \lambda.$$

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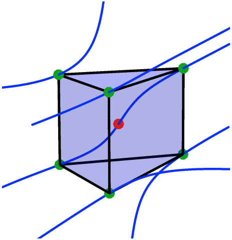

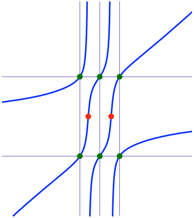

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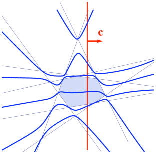
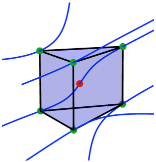
Examine $\lambda\to 0$ and $\lambda\to\infty$.

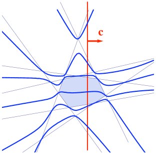
Further Questions

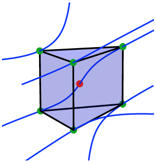




Further Questions

◦ What can be said about non-generic behavior?





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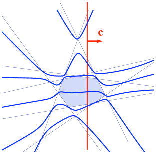

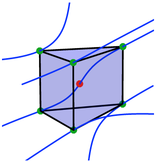

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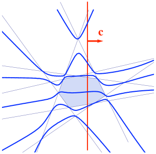

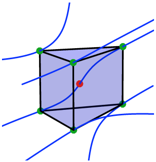

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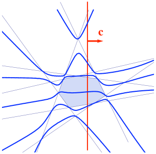

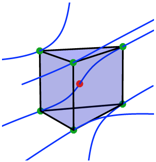
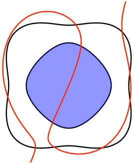

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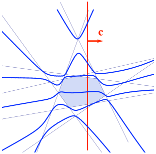

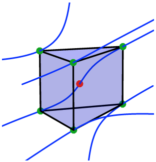
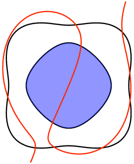

Thanks and Happy Birthday to Mike!