A mathematician's foray into signal processing

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From Complexity to Dynamics: A conference celebrating the work of Mike Shub

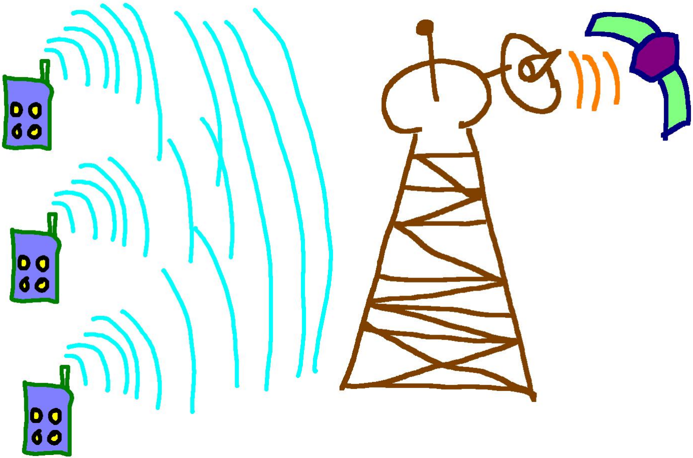
Credits

This work has been greatly inspired by Mike's thoughts and works

Coautors: Óscar González and Rafael Santamaría

How is it possible?

20 people can use their mobiles at the same time in the same room



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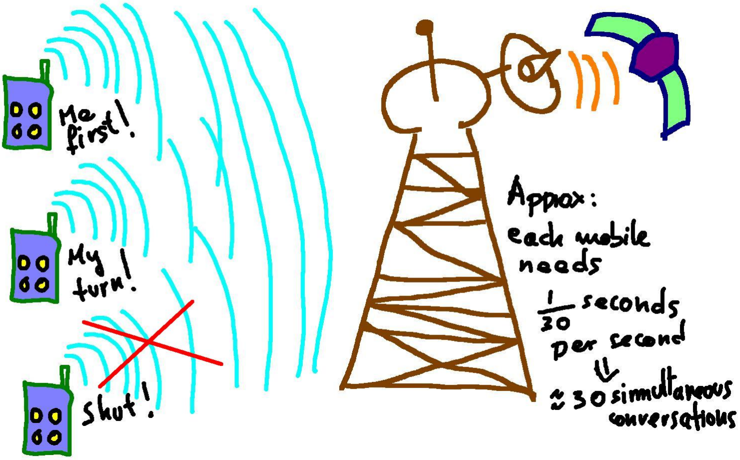

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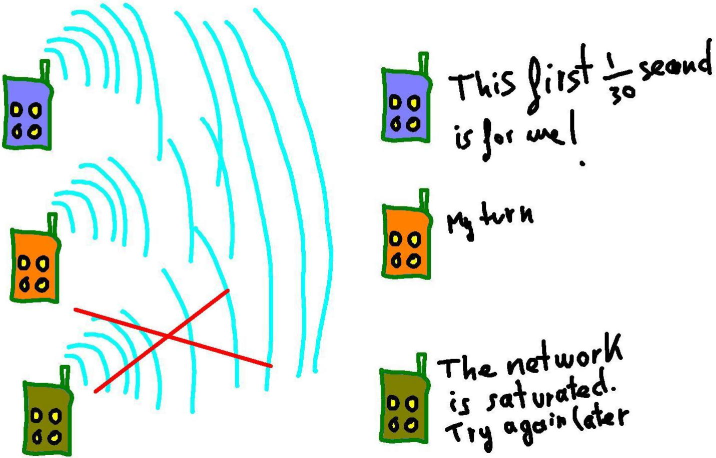

Why $0, 1$ sequences are waves?

And one reason for engineers to know complex numbers

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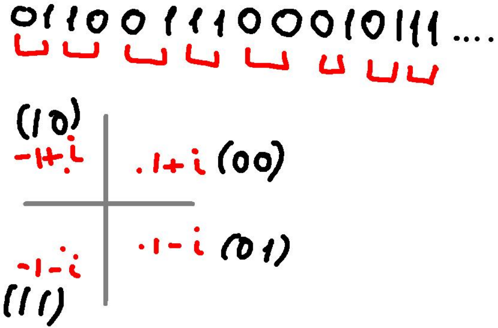

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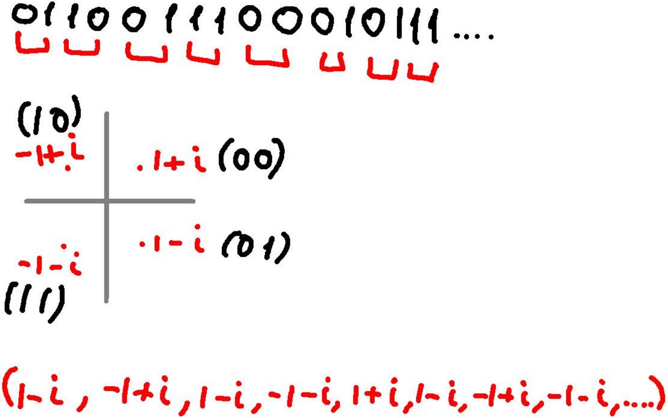

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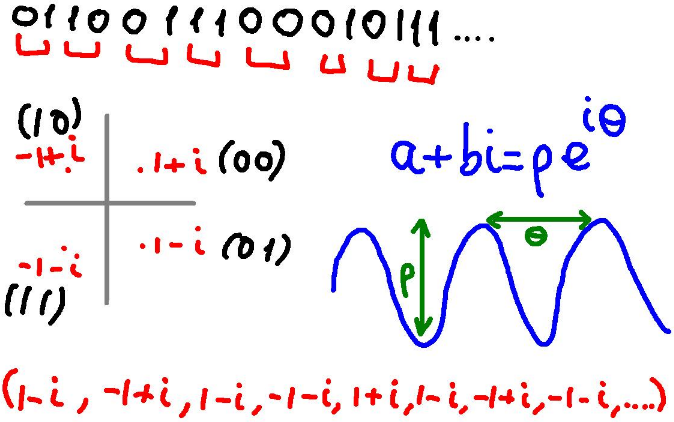

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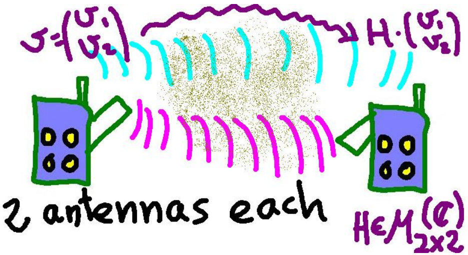
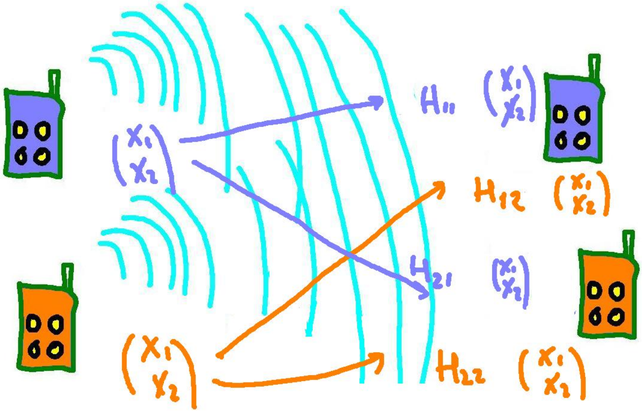


Interference Alignment: an idea of Jafar's and Khandani's research groups

Each phone must do some linear algebra



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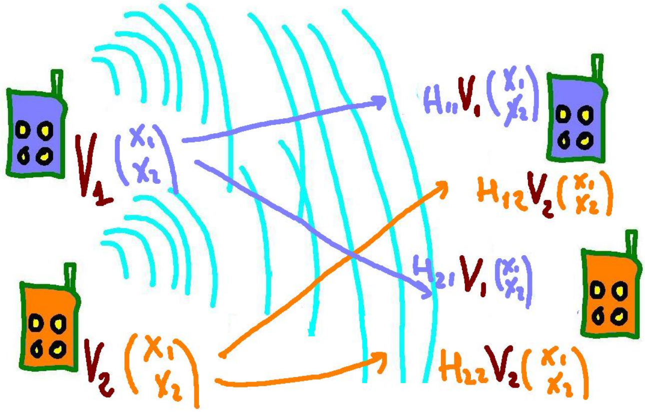

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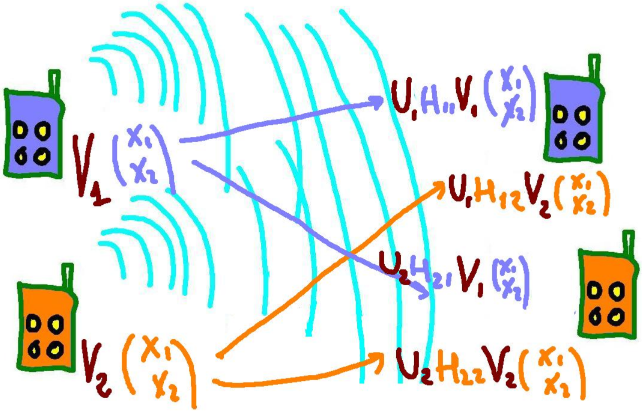

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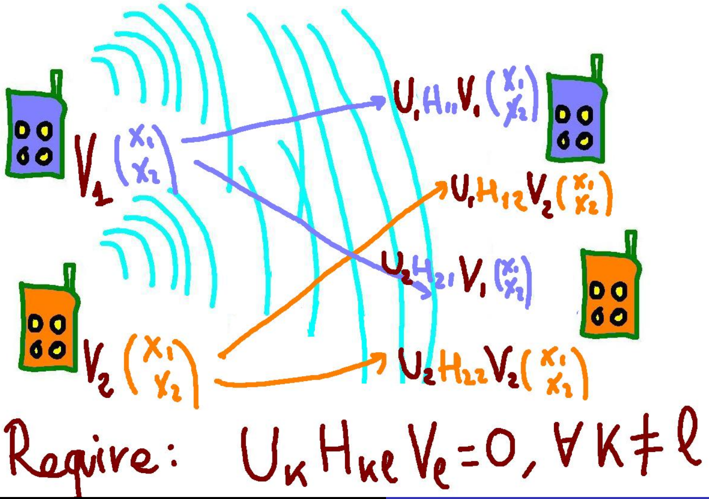

The full problem

After engineering considerations have been taken into

Let K be the number of transmitters/receivers. Let

$$\Phi = \{(k,\ell) : \text{transmitter } \ell \text{ interfers receiver } k\} \subseteq \{1,\ldots,K\}^2.$$

Let transmitter ℓ have M_ℓ antennas, receiver k have N_k antennas. Let $d_j \leq \min\{M_j, N_j\}$, $1 \leq j \leq K$, and let

$$H_{k\ell} \in \mathcal{M}_{N_k \times M_\ell}(\mathbb{C})$$

be fixed (known). Do there exist $U_k \in \mathcal{M}_{M_k \times d_k}(\mathbb{C})$, $1 \leq k \leq K$ and $V_\ell \in \mathcal{M}_{N_\ell \times d_\ell}(\mathbb{C})$, $1 \leq \ell \leq K$ such that

$$U_k^T H_{k\ell} V_\ell = 0 \in \mathcal{M}_{d_k \times d_\ell}(\mathbb{C}), \quad k \neq \ell?$$

Equivalently, compute the maximal d_j that you can use (degrees of Freedom=what SP guys want). **This problem has been open since 2006. About 60 research papers.**

Seen Mike's Complexity papers? you've seen this before

So our question is: is $\pi_1^{-1}(H_{kl}) = \emptyset$? For which choices of $(H_{kl})_{(k,l)\in\Phi}$?

$$H_{k,l} \in \mathbb{P}(\mathcal{M}_{N_k \times M_l}(\mathbb{C}))$$

$$U_k \in \mathbb{G}_{N_k \times d_k}, \quad V_l \in \mathbb{G}_{M_l \times d_l}$$

$$\mathcal{V} = \{(H_{kl}, U_k, V_l) : U_k^T H_{kl} V_l = 0\}$$

$$\pi_1 : \mathcal{V} \to \mathcal{H}, \qquad \pi_2 : \mathcal{V} \to \mathcal{S}$$

$$\mathcal{H} = \prod_{k,l} \mathbb{P}(\mathcal{M}_{N_k \times M_l}) \qquad \mathcal{S} = \prod \mathbb{G}_{N_k \times d_k} \times \prod \mathbb{G}_{M_l \times d_l}$$

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This "double fibration" scheme is a whole business in complexity theory and numerical analysis. See for example the works of Shub, Smale and many others by other authors like Armentano, B., Boito, Burgisser, Cucker, Dedieu, Kim, Leykin, Malajovich, Marsten, Pardo, Renegar, Rojas, Shutherland... and others.

Compute some dimensions

The only non–elementary task follows from the preimage theorem

$\mathcal{V}$ is a manifold, and

$$\dim_{\mathbb{C}} \mathcal{H} = \sum_{(k,l)\in\Phi} (N_k M_l - 1).$$

$$\dim_{\mathbb{C}} \mathcal{S} = \sum_{1\leq j\leq K} (d_j(N_j + M_j - 2d_j)).$$

$$\dim_{\mathbb{C}} \mathcal{V} = \left(\sum_{(k,l)\in\Phi} N_k M_l - d_k d_l \right) + \left(\sum_{k\in\Phi_R} N_k d_k - d_k^2 \right) + \left(\sum_{l\in\Phi_T} M_l d_l - d_l^2 \right) - \sharp(\Phi).$$

Are you paying attention?

The problem is therefore solved:

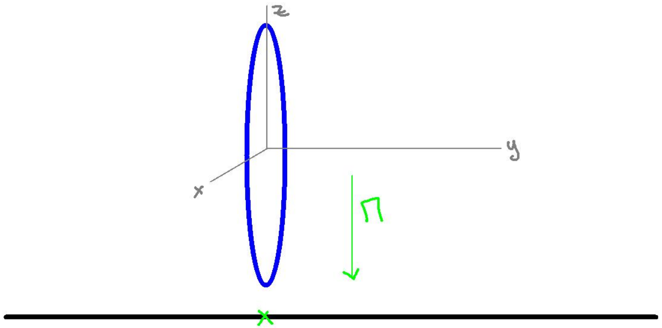
- If $\dim \mathcal{H} > \dim \mathcal{V}$ there is no hope that the problem can be solved for generic $(H_{kl}) \in \mathcal{H}$.
- If $\dim \mathcal{H} \leq \dim \mathcal{V}$ the problem can be solved for generic $(H_{kl}) \in \mathcal{H}$, because we are in complex and algebraic situations.

... So 60 papers can be summarized with a dimension count argument.
The simple case that every transmitter has M, every receiver has N antennas and d degrees of freedom are reached, this dimension count reads:

$$(K+1)d \leq M+N, K \text{ the maximum number of users}$$

A projection between equal dimensions whose image is a zero measure set

But this WON'T happen in real–life problems like the one here, right?



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A general mathematical truth

and two examples

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$$\begin{pmatrix}\text{hyperbolicity}\\ \text{accesibility}\end{pmatrix} \to \text{ ergodic}$$

Theorem (Ehresmann 1951)

Let X, Y be smooth manifolds with Y connected. Let $U \subseteq X$ be a nonempty open subset of X, and let $\pi : U \to Y$ satisfy:

One of the most beautiful theorems I have seen

For the bored listener: which is yours?

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- *π is a submersion.*
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Corollary

If additionally we assume $\dim(X) = \dim(Y)$ then π is a covering map. In particular, the number of preimages of every $y \in Y$ is finite and constant.

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Let us recall the hypotheses of Ehresmann's theorem.We did not check that $U \neq \emptyset$. **And this is exactly what happens in these "singularly projected" cases: U is empty sometimes.**

A test for feasibility

There may be not formula for deciding feasibility. But there is a test:

- Choose some $(H, U, V) \in \mathcal{V}$.
- Compute the rank of $D\pi_1(H, U, V)$.
- If the rank is maximal, answer **the problem is feasible**. Otherwise, answer **the problem is infeasible**.

Because of Sard's Theorem, if (H, U, V) are chosen "generically", they will be a regular point (if there is some regular point) so this test checks if the set of regular points is empty or not. Just as we wanted.

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The second step above is just LA. The first one can be changed to:

$$U_j = V_j = \begin{pmatrix} I_d \\ 0 \end{pmatrix}, \ 1 \leq j \leq K, \quad H_{kl} = \begin{pmatrix} 0 & A_{kl} \\ B_{kl} & 0 \end{pmatrix},$$

where A_{kl} and B_{kl} are chosen with complex coefficients following the normal distribution.

A discrete algorithm in **BPP**

One can describe a discrete algorithm using a classical result by Milnor on the number of connected components of algebraically closed sets, and a result relating the height of numbers appearing in a set to the number of connected components of the set (first such a result due to Koiran):

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Just a comment on the reactions to this work

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- This paper uses too high mathematics and thus has to be rejected because it cannot be understood.
- This is a great paper and must be accepted.

Happily, the third referee's opinion was the prevalent one in the editorial board.

Querido amigo:

Felicidades!!!

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